



**Kuwait University**

# **Calculus 1 – The Chain Rule**

## **(Section 3.4)**

**For Contact and Support:**



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بص

### BASIC PROPERTIES

$$(cf(x))' = c(f'(x))$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$\frac{d}{dx}(c) = 0$$

### PRODUCT RULE

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

### QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

### POWER RULE

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

### CHAIN RULE

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

### COMMON DERIVATIVES

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(e^x) = e^x$$

## Section 3-9 : Chain Rule

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We've taken a lot of derivatives over the course of the last few sections. However, if you look back they have all been functions similar to the following kinds of functions.

$$R(z) = \sqrt{z} \quad f(t) = t^{50} \quad y = \tan(x) \quad h(w) = e^w \quad g(x) = \ln x$$

These are all fairly simple functions in that wherever the variable appears it is by itself. What about functions like the following,

$$R(z) = \sqrt{5z-8} \quad f(t) = (2t^3 + \cos(t))^{50} \quad y = \tan(\sqrt[3]{3x^2 + \tan(5x)})$$
$$h(w) = e^{w^4 - 3w^2 + 9} \quad g(x) = \ln(x^{-4} + x^4)$$

None of our rules will work on these functions and yet some of these functions are closer to the derivatives that we're liable to run into than the functions in the first set.

Let's take the first one for example. Back in the [section](#) on the definition of the derivative we actually used the definition to compute this derivative. In that section we found that,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

If we were to just use the power rule on this we would get,

$$\frac{1}{2}(5z-8)^{\frac{1}{2}} = \frac{1}{2\sqrt{5z-8}} \quad \times$$

which is not the derivative that we computed using the definition. It is close, but it's not the same. So, the power rule alone simply won't work to get the derivative here.

Let's keep looking at this function and note that if we define,

$$f(z) = \sqrt{z} \quad g(z) = 5z - 8$$

then we can write the function as a composition.

$$R(z) = (f \circ g)(z) = f(g(z)) = \sqrt{5z-8}$$

and it turns out that it's actually fairly simple to differentiate a function composition using the **Chain Rule**. There are two forms of the chain rule. Here they are.

### Chain Rule

Suppose that we have two functions  $f(x)$  and  $g(x)$  and they are both differentiable.

1. If we define  $F(x) = (f \circ g)(x)$  then the derivative of  $F(x)$  is,

$$F'(x) = f'(g(x)) g'(x)$$

2. If we have  $y = f(u)$  and  $u = g(x)$  then the derivative of  $y$  is,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

**Example 1** Use the Chain Rule to differentiate  $R(z) = \sqrt{5z-8}$ .

**Solution**

We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$\begin{aligned} f(z) &= \sqrt{z} & g(z) &= 5z - 8 \\ f'(z) &= \frac{1}{2\sqrt{z}} & g'(z) &= 5 \end{aligned}$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z)) g'(z) \\ &= f'(5z-8) g'(z) \\ &= \frac{1}{2} (5z-8)^{-\frac{1}{2}} (5) \\ &= \frac{1}{2\sqrt{5z-8}} (5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

And this is what we got using the definition of the derivative.

In general, we don't really do all the composition stuff in using the Chain Rule. That can get a little complicated and in fact obscures the fact that there is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition.

Let's take the function from the previous example and rewrite it slightly.

$$\begin{aligned} R(z) &= \underbrace{(5z-8)}_{\text{inside function}} \underbrace{\frac{1}{2}}_{\text{outside function}} \\ R'(z) &= \frac{1}{2} \underbrace{(5z-8)^{-\frac{1}{2}}}_{\substack{\text{inside function} \\ \text{left alone}}} \underbrace{(5)}_{\substack{\text{derivative of} \\ \text{inside function}}} \end{aligned}$$

In general, this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(g(x))}_{\substack{\text{inside function} \\ \text{left alone}}} \underbrace{g'(x)}_{\substack{\text{times derivative} \\ \text{of inside function}}}$$

**Example 2** Differentiate each of the following.

(a)  $f(x) = \sin(3x^2 + x)$

(b)  $f(t) = (2t^3 + \cos(t))^{50}$

(c)  $h(w) = e^{w^4 - 3w^2 + 9}$

(e)  $y = \sec(1 - 5x)$

(f)  $P(t) = \cos^4(t) + \cos(t^4)$

**Example 3** Differentiate each of the following.

(a)  $f(x) = [g(x)]^n$

(b)  $f(x) = e^{g(x)}$

✓ (c)  $f(x) = \ln(g(x))$

(a) The outside function is the exponent and the inside is  $g(x)$ .

$$f'(x) = n[g(x)]^{n-1} g'(x)$$

(b) The outside function is the exponential function and the inside is  $g(x)$ .

$$f'(x) = g'(x)e^{g(x)}$$

**Example 4** Differentiate each of the following.

(b)  $f(z) = \sin(ze^z)$

(c)  $y = \frac{(x^3 + 4)^5}{(1 - 2x^2)^3}$

(d)  $h(t) = \left(\frac{2t+3}{6-t^2}\right)^3$

باقي الحل في الصفحات التالية

$$(c) \ y = \frac{(x^3 + 4)^5}{(1 - 2x^2)^3}$$

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$$\text{(d) } h(t) = \left( \frac{2t+3}{6-t^2} \right)^3$$

$$24. f(t) = 2^{t^3}$$

**Example 5** Differentiate each of the following.

(a) 
$$h(z) = \frac{2}{(4z + e^{-9z})^{10}}$$

(b) 
$$f(y) = \sqrt{2y + (3y + 4y^2)^3}$$

(d) 
$$g(t) = \sin^3(e^{1-t} + 3\sin(6t))$$

$$(b) f(y) = \sqrt{2y + (3y + 4y^2)^3}$$

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$$(d) g(t) = \sin^3(e^{1-t} + 3\sin(6t))$$

We'll need to be a little careful with this one.

$$\begin{aligned} g'(t) &= 3\sin^2(e^{1-t} + 3\sin(6t)) \frac{d}{dt} \sin(e^{1-t} + 3\sin(6t)) \\ &= 3\sin^2(e^{1-t} + 3\sin(6t)) \cos(e^{1-t} + 3\sin(6t)) \frac{d}{dt} (e^{1-t} + 3\sin(6t)) \\ &= 3\sin^2(e^{1-t} + 3\sin(6t)) \cos(e^{1-t} + 3\sin(6t)) (e^{1-t}(-1) + 3\cos(6t)(6)) \\ &= 3(-e^{1-t} + 18\cos(6t)) \sin^2(e^{1-t} + 3\sin(6t)) \cos(e^{1-t} + 3\sin(6t)) \end{aligned}$$

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## CHAIN RULE

دیا

$$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$$

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

$$\frac{d}{dx}(a^{f(x)}) = a^{f(x)} \ln a \cdot f'(x)$$

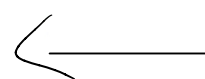
$$\frac{d}{dx}(\sin[f(x)]) = f'(x) \cos[f(x)]$$

$$\frac{d}{dx}(\cos[f(x)]) = -f'(x) \sin[f(x)]$$

$$\frac{d}{dx}(\tan[f(x)]) = f'(x) \sec^2[f(x)]$$

$$\frac{d}{dx}(\sec[f(x)]) = f'(x) \sec[f(x)] \tan[f(x)]$$

- If  $f(x) = \sqrt{u}$ , then  $f'(x) = \frac{u'}{2\sqrt{u}}$



أو تسبقه  
كادي

- If  $f(x) = \sqrt[3]{u} = u^{1/3}$  then  $f'(x) = \frac{1}{3} u^{-2/3} \cdot u'$

## 3.4 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate  $F'(x)$ .

**The Chain Rule** If  $g$  is differentiable at  $x$  and  $f$  is differentiable at  $g(x)$ , then the composite function  $F = f \circ g$  defined by  $F(x) = f(g(x))$  is differentiable at  $x$  and  $F'$  is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if  $y = f(u)$  and  $u = g(x)$  are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The Chain Rule can be written either in the prime notation

$$\boxed{2} \quad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if  $y = f(u)$  and  $u = g(x)$ , in Leibniz notation:

$$\boxed{3} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

### DOUBLE-ANGLE FORMULAS

**Formula for sine:**  $\sin 2x = 2 \sin x \cos x$

**Formulas for cosine:**  $\cos 2x = \cos^2 x - \sin^2 x$   
 $= 1 - 2 \sin^2 x$   
 $= 2 \cos^2 x - 1$

**Formula for tangent:**  $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

To do the chain rule:

1. Differentiate the outer function, keeping the inner function the same.
  2. Multiply this by the derivative of the inner function.
- 

$$1) f(x) = (4x - 3)^5$$

$$2) g(x) = \cos(2x)$$

$$3) h(x) = (\sin 2x)^4$$

**EXAMPLE 1** Find  $F'(x)$  if  $F(x) = \sqrt{x^2 + 1}$ .

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$$\frac{d}{dx} \sin(x^2)$$

**EXAMPLE 2** Differentiate (a)  $y = \sin(x^2)$  and (b)  $y = \sin^2 x$ .

**4 The Power Rule Combined with the Chain Rule** If  $n$  is any real number and  $u = g(x)$  is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

Alternatively, 
$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

**EXAMPLE 3** Differentiate  $y = (x^3 - 1)^{100}$ .

**EXAMPLE 4** Find  $f'(x)$  if  $f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$ .

**EXAMPLE 5** Find the derivative of the function

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**EXAMPLE 6** Differentiate  $y = (2x + 1)^5(x^3 - x + 1)^4$ .

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**EXAMPLE 7** Differentiate  $y = e^{\sin x}$ .

**7-46** Find the derivative of the function.

1.  $y = \sqrt[3]{1 + 4x}$

**2.**  $y = (2x^3 + 5)^4$

**3.**  $y = \tan \pi x$

**4.**  $y = \sin(\cot x)$



**7-46** Find the derivative of the function.

9.  $f(x) = \sqrt{5x + 1}$

$f(x) = \sqrt[3]{5x + 1}$



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**11.**  $f(\theta) = \cos(\theta^2)$

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**12.**  $g(\theta) = \cos^2 \theta$

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**22.**  $y = \left(x + \frac{1}{x}\right)^5$

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**25.**  $g(u) = \left(\frac{u^3 - 1}{u^3 + 1}\right)^8$

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**28.**  $f(z) = e^{z/(z-1)}$







$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{du} \frac{du}{dx} \frac{dx}{dt}$$

**EXAMPLE 8** If  $f(x) = \sin(\cos(\tan x))$ , then

**EXAMPLE 9** Differentiate  $y = e^{\sec 3\theta}$ .

45.  $y = \cos \sqrt{\sin(\tan \pi x)}$



II. If  $h(x) = g(f(x))$ , where  $f(2) = 3$ ,  $f'(2) = 4$ ,  $g(3) = 6$ , and  $g'(3) = -5$ , then  $h'(2) =$

a) 12.

b) 24.

c) . ✓

d) None of the above.



The Chain Rule can be written either in the prime notation

2

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

II. If  $f(x) = 5x$  and  $g(x) = 2 \cos x$ , then  $(f \circ g)'(\pi) =$

a)  $-5$ .

b)  $0$ .

c)  $10$ .

d)  $-10$ .

e) None of the above.



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**Calculus 1 – Implicit  
Differentiation**

**(Section 3.5)**

**For Contact and Support:**



## 3.5 Implicit Differentiation

$$\frac{dy}{dx}$$

نستخدم طريقة الاشتقاق الضمني ( implicit Differentiation ) لايجاد المشتقة  
لما يكون صعب عليي أفصل ال  $y$  عن  $x$  مثال :

examples:-

$$- x^2 + y^2 = 25$$

$$- 2xy + \sin x = 6x \sin(x) - 1$$

$$- \sin(x+y) = x$$

$$1) x^2 \xrightarrow{\frac{d}{dx}} 2x$$

طرق الاشتقاق

$$2) y^2 \xrightarrow{\frac{d}{dx}} 2y y' \quad y' = \frac{dy}{dx}$$

$$3) y e^{\sin x} \xrightarrow{\frac{d}{dx}} y' e^{\sin x} + y e^{\sin x} \cdot \cos x$$

طريقة حل الاشتقاق الضمني :

$$\frac{dy}{dx} = y'$$

١- نشتق الطرفين بالنسبة للمتغير  $x$

٢- نخلي  $\frac{dy}{dx}$  بطرف بروحها

### EXAMPLE 1

(a) If  $x^2 + y^2 = 25$ , find  $\frac{dy}{dx}$ .

(b) Find an equation of the tangent to the circle  $x^2 + y^2 = 25$  at the point (3, 4).

نفس السؤال بس حلها نفس طريقة سكشن 3.4 chain rule .. بس دربالك مو كل مرة تقدر تحل بال chain rule على طول...

ملاحظة : عادت لما تشوف ال  $y$  مو بطرف بروحها بسؤال .. فمعناته رح تحل بالاشتقاق الضمني

### EXAMPLE 1

(a) If  $x^2 + y^2 = 25$ , find  $\frac{dy}{dx}$ .

(b) Find an equation of the tangent to the circle  $x^2 + y^2 = 25$  at the point (3, 4).

## EXAMPLE 2

(a) Find  $y'$  if  $x^3 + y^3 = 6xy$ .

(b) Find the tangent to the folium of Descartes  $x^3 + y^3 = 6xy$  at the point  $(3, 3)$ .

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**EXAMPLE 3** Find  $y'$  if  $\sin(x + y) = y^2 \cos x$ .

**EXAMPLE 4** Find  $y''$  if  $x^4 + y^4 = 16$ .

**5–20** Find  $dy/dx$

**7.**  $x^4 + x^2y^2 + y^3 = 5$

**5.**  $x^2 - 4xy + y^2 = 4$



**10.**  $x e^y = x - y$

**11.**  $y \cos x = x^2 + y^2$

14.  $e^y \sin x = x + xy$

4. [10 pts.] Find  $\frac{dy}{dx}$  by **implicit differentiation**:  $e^y \sin x = x + xy$ .







## Derivatives of Inverse Trigonometric Functions

### Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

**EXAMPLE 5** Differentiate (a)  $y = \frac{1}{\sin^{-1}x}$  and (b)  $f(x) = x \arctan \sqrt{x}$ .

**49–60** Find the derivative of the function. Simplify when possible.

**49.**  $y = (\tan^{-1} x)^2$

**50.**  $y = \tan^{-1}(x^2)$

51.  $y = \sin^{-1}(2x + 1)$

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Below the equation, there are 20 horizontal lines for writing.

$$52. g(x) = \arccos \sqrt{x}$$

54.  $y = \tan^{-1}(x - \sqrt{1 + x^2})$



57.  $y = x \sin^{-1} x + \sqrt{1 - x^2}$  \_\_\_\_\_  
\_\_\_\_\_

III. If  $\sin^{-1} x + e^y = e$ , then  $\frac{dy}{dx} \Big|_{(0,1)} =$  \_\_\_\_\_  
\_\_\_\_\_

III. If  $f(x) + x \sin(f(x)) = x^3 + \frac{\pi}{2}$ , then  $f'(0)$  is equal to

a) 2.

b) -1.

c) 0.

d) 1.

e) None of the above.

IV. If  $\sin(x) + y^3 = x^2 + 1$ , then  $\frac{dy}{dx} \Big|_{(0,1)} =$

a)  $-1/3$ .

b)  $1/3$ .

c) 0.

d) 1. e) None of the above.

I. If  $f(x) = \sin^{-1}(2x)$ , then  $f'(x) =$

(A)  $\frac{1}{\sqrt{1-4x^2}}$ .

(B)  $\frac{-1}{\sqrt{1-4x^2}}$ .

(C)  $\frac{2}{\sqrt{1-4x^2}}$ .

(D)  $\frac{-2}{\sqrt{1-4x^2}}$ .

(E) None of the above.



Show that or prove

$$3) \frac{d}{dx} (\sin^{-1} 2x) = \frac{2}{\sqrt{1-4x^2}}$$

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**Calculus 1 – Derivatives of  
Logarithmic  
(Section 3.6)**

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### 3.6 Derivatives of Logarithmic Functions

$$\frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx} (\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

$$\frac{d}{dx} [\ln g(x)] = \frac{g'(x)}{g(x)}$$

**EXAMPLE 1** Differentiate  $y = \ln(x^3 + 1)$ .

في بعض المسائل رح نستخدم خواص الـ  $\ln$  أو  $\log$  أكثر ما نقدر عشان  
تسهل علينا عملية الاشتقاق

### LAWS OF LOGARITHMS

Let  $a$  be a positive number, with  $a \neq 1$ . Let  $A$ ,  $B$ , and  $C$  be any real numbers with  $A > 0$  and  $B > 0$ .

#### Law

1.  $\log_a(AB) = \log_a A + \log_a B$

2.  $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$

3.  $\log_a(A^C) = C \log_a A$

#### Description

The logarithm of a product of numbers is the sum of the logarithms of the numbers.

The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.

The logarithm of a power of a number is the exponent times the logarithm of the number.

•  $\ln(AB) = \ln A + \ln B$

•  $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

•  $\ln A^x = x \ln A$

↳ ex:  $\ln x^4 = 4 \ln x$ ,  $\ln x^{\sin x} = (\sin x) \ln x$

### PROPERTIES OF NATURAL LOGARITHMS

#### Property

1.  $\ln 1 = 0$

2.  $\ln e = 1$

3.  $\ln e^x = x$

4.  $e^{\ln x} = x$

#### Reason

We must raise  $e$  to the power 0 to get 1.

We must raise  $e$  to the power 1 to get  $e$ .

We must raise  $e$  to the power  $x$  to get  $e^x$ .

$\ln x$  is the power to which  $e$  must be raised to get  $x$ .

**EXAMPLE 2** Find  $\frac{d}{dx} \ln(\sin x)$ .

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**EXAMPLE 3** Differentiate  $f(x) = \sqrt{\ln x}$ .

**EXAMPLE 4** Differentiate  $f(x) = \log_{10}(2 + \sin x)$ .

**EXAMPLE 5** Find  $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}}$ .

**EXAMPLE 6** Find  $f'(x)$  if  $f(x) = \ln |x|$ .

**2-22** Differentiate the function.

**2.**  $f(x) = x \ln x - x$

**3.**  $f(x) = \sin(\ln x)$

**7.**  $f(x) = \log_{10}(1 + \cos x)$

I. Let  $f(x) = \log_{10}(x^2 + 1)$ . Then:

(A)  $f'(x) = \frac{2x}{x^2 + 1}$ .

(B)  $f'(x) = \frac{2x}{(x^2 + 1) \ln 10}$ .

(C)  $f'(x) = \frac{1}{x^2 + 1}$ .

(D)  $f'(x) = \frac{1}{(x^2 + 1) \ln 10}$ .

(E) None of the above.

$$\frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}$$

$$9. g(x) = \ln(xe^{-2x})$$

$$11. F(t) = (\ln t)^2 \sin t$$

**12.**  $h(x) = \ln(x + \sqrt{x^2 - 1})$





**16.**  $y = \ln |1 + t - t^3|$

**17.**  $T(z) = 2^z \log_2 z$





**21.**  $y = \tan[\ln(ax + b)]$

**5.** [10 pts.] Find  $y''$  if  $y = \ln(1 + \ln x)$ .

**23–26** Find  $y'$  and  $y''$ .

**23.**  $y = \sqrt{x} \ln x$

## ■ Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

ex:  $x^{\sin x}$ ,  $x^{\sqrt{x}}$ ,  $\sqrt{x}^{\ln x}$  ... : نستخدم الطريقة هذي لما

- ١- دالة أس دالة
- ٢- دالة معقدة علينا

**EXAMPLE 7** Differentiate  $y = \frac{x^{3/4} \sqrt{x^2 + 1}}{(3x + 2)^5}$ .

طريقة حل اشتقاق دالة أس دالة:

$$\frac{d}{dx} [f(x)]^{g(x)}$$

$$1) y = f(x)^{g(x)}$$

$$2) \ln y = \ln f(x)^{g(x)}$$

أخذ  $\ln$

$$3) \ln y = g(x) \ln f(x)$$

خواص  $\ln$

$$4) \frac{1}{y} y' = g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x)$$

اشتقوا الطرفين

$$5) y' = y \left( g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x) \right)$$

أبسط  
إذافي  
مجال التبسيط

بعدين أعوض بقيمة  $y$

⊗ You should distinguish carefully between the Power Rule  $[(x^n)' = nx^{n-1}]$ , where the base is variable and the exponent is constant, and the rule for differentiating exponential functions  $[(b^x)' = b^x \ln b]$ , where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

- |                                  |   |
|----------------------------------|---|
| Constant base, constant exponent | 1. $\frac{d}{dx}(b^n) = 0$ (b and n are constants) $\frac{d}{dx} 2^\pi = 0$ , $\frac{d}{dx} e^4 = 0$    |
| Variable base, constant exponent | 2. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$ $\frac{d}{dx} (2x+3)^3 = 3(2x+3)^2 (2)$                  |
| Constant base, variable exponent | 3. $\frac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$ $\frac{d}{dx} 3^{\sin x} = 3^{\sin x} \ln(3) \cos x$ |
| Variable base, variable exponent | 4. To find $(d/dx)[f(x)]^{g(x)}$ , logarithmic differentiation can be used, as in the next example.     |

---

**EXAMPLE 8** Differentiate  $y = x^{\sqrt{x}}$ .

---



3. [10 pts.] Let  $f(x) = (3 + 4x)^{\cos x}$ . Find  $f'(x)$ .

---

1. [10 pts.] Let  $y = (x^2 + 1)^{\cos x}$ . Use logarithmic differentiation to find  $\frac{dy}{dx}$  at  $x = 0$ .

---

2. [10 pts.] Find an equation of the tangent line to the curve  $y = x^{\ln(x)}$  at the point  $(1, 1)$ .

---

3. [10 pts.] Let  $y = (x^2 + 1)^x$ . Find  $\frac{dy}{dx}$ .

---



$$40. y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1}$$

$$41. y = \sqrt{\frac{x - 1}{x^4 + 1}}$$







3. [10 pts.] Let  $y = (\sqrt{x})^{\sin x}$ . Find  $\frac{dy}{dx}$ .

---

III. If  $f(x) = x^3 e^{x^2-1}$ , then  $f'(1)$ .

- a) 5.
- b) 0.
- c) 1.
- d) 2.
- e) None of the above.



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**Calculus 1 – Related Rates**  
**(Section 3.9)**

**For Contact and Support:**



**YouTube: Precalculusq8**

**Twitter: Precalculusq8**

S=Surface Area = A = Area.

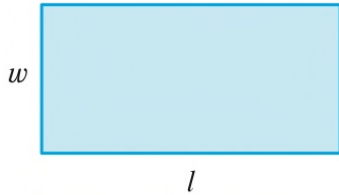
P = Perimeter = Circumference = C.

Volume = V

### Rectangle

$$A = lw$$

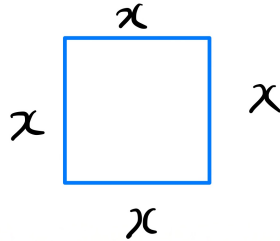
$$P = 2l + 2w$$



### Square

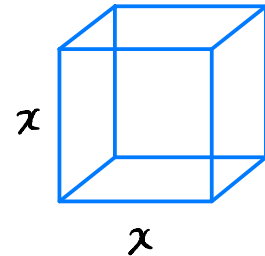
$$A = x^2$$

$$P = 4x$$



### Cube

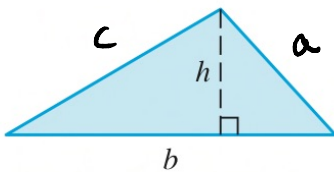
$$V = x^3$$



### Triangle

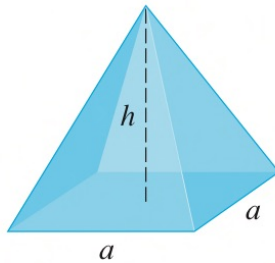
$$A = \frac{1}{2}bh$$

$$P = a + b + c$$



### Pyramid

$$V = \frac{1}{3}ha^2$$



$$S = 6x^2$$

### RECTANGULAR SOLID

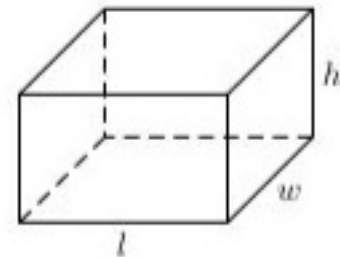
$l$  = length,  $w$  = width,

$h$  = height

Volume:  $V = lwh$

Surface Area:

$S = 2lw + 2lh + 2wh$

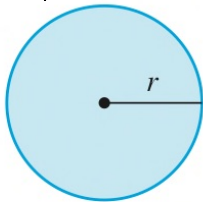


### Circle

$$A = \pi r^2$$

$$C = 2\pi r$$

$$D = 2r$$

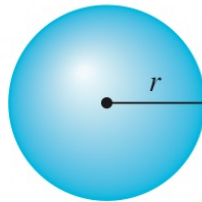


### Sphere

$$V = \frac{4}{3}\pi r^3$$

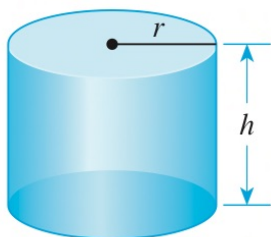
$$A = 4\pi r^2$$

$$D = 2r$$



### Cylinder

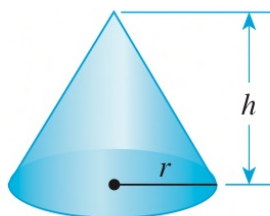
$$V = \pi r^2 h$$



$$S = 2\pi r h + 2\pi r^2$$

### Cone

$$V = \frac{1}{3}\pi r^2 h$$



$$S = \pi r \sqrt{r^2 + h^2}$$

Name	unit	الترجمة
Side	m, ft, cm ----	ضلع
Length	m, ft, cm --	طول
Height	m, ft, cm ...	ارتفاع
Base	m, ft, cm --	قاعدة
Radius	m, ft, cm ..	نصف القطر
Diameter	m, ft, cm ...	قطر
Area	m <sup>2</sup> , ft <sup>2</sup> , cm <sup>2</sup> ...	مساحة
Surface Area	m <sup>2</sup> , ft <sup>2</sup> , cm <sup>2</sup> ...	مساحة السطح
Volume	m <sup>3</sup> , ft <sup>3</sup> , cm <sup>3</sup> ...	الحجم
Rate	تغير الشيء بالنسبة للوقت	معدل
Increasing	+	في ازدياد
Decreasing	—	في تناقص

ex: Area increasing by 3m<sup>2</sup>/s

$$\therefore \frac{dA}{dt} = 3 \text{ m}^2/\text{s}$$

إذا شغنا بالسؤال مسافة أو ضلع أو حجم أو نصف القطر أو ارتفاع أو طول إلخ.. مقسوم على الزمن... يعني هذي المشتقة لهذا الشيء

@Precalculusq8

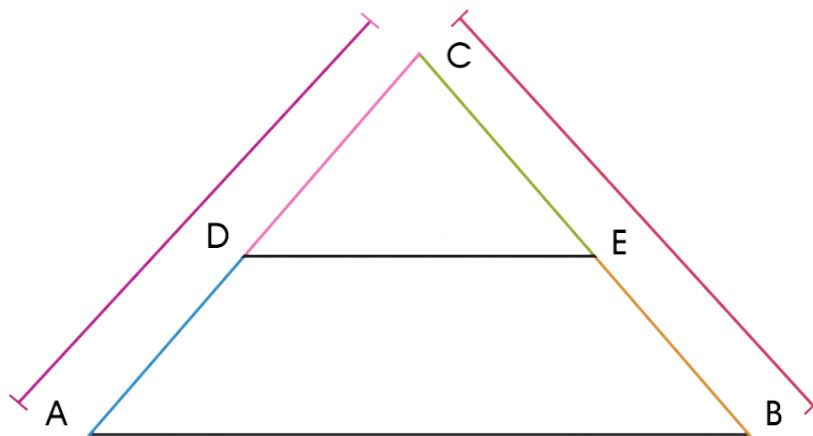
## PART II: Related Rates

لِحْنِي  
المِشْتَقَّة

*Related rates* problems can be identified by their request for finding **how quickly some quantity is changing** when you are given how quickly another **variable is changing**. There exist a few classic *types* of related rates problems with which you should familiarize yourself.

1. The Falling Ladder (and other Pythagorean Problems)
2. The Leaky Container
3. The Lamppost and the Shadow
4. The Change in Angle Problem

Triangle similarity theorem



$$\frac{DC}{DA} = \frac{EC}{EB} \quad \text{or} \quad \frac{DC}{AC} = \frac{EC}{BC}$$

@Precalculusq8

## STEPS:

1. As you read the problem pull out essential information & make a diagram if possible.

اكتب المعطيات كلها والمطلوب من السؤال

2. Write down any known rate of change & the rate of change you are looking for, e.g.

$$\frac{dV}{dt} = 3 \quad \& \quad \frac{dr}{dt} = ?$$

3. Be careful with signs...if the amount is decreasing, the rate of change is negative.

4. Pay attention to whether quantities are fixed or varying. For example, if a ladder is 12 meters long you can just call it 12. And if a radius is changing a changing rate, just call it  $r$ . You will plug in values for varying quantities at the end.

القانون المناسب الي يربط بين المعطيات والمطلوب

6. Set up an equation involving the appropriate quantities.

7. Differentiate with respect to  $t$  using implicit differentiation.

8. Plug in known items (you may need to find some quantities using geometry).

9. Solve for the item you are looking for, most often this will be a rate of change.

10. Express your final answer in a full sentence with units that answers the question asked.

$$\frac{d}{dt} = -$$

أي كمية يعطيني إياها قاعده تتناقص مع الوقت

رح أحط ماينس جدامه

مثال :

السرعة تتناقص ، تقلص الحجم ، الخ...

1. If  $V$  is the volume of a cube with edge length  $x$  and the cube expands as time passes, find  $dV/dt$  in terms of  $dx/dt$ .

- 
2. (a) If  $A$  is the area of a circle with radius  $r$  and the circle expands as time passes, find  $dA/dt$  in terms of  $dr/dt$ .
- (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?



## 3.9 Related Rates

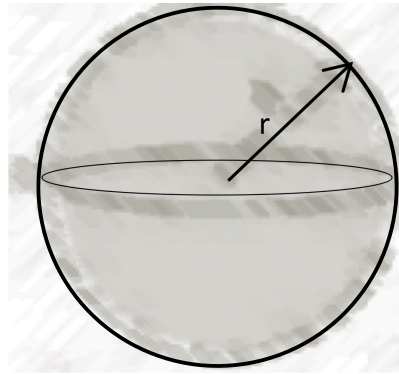
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**EXAMPLE 1** Air is being pumped into a spherical balloon so that its volume increases at a rate of  $100 \text{ cm}^3/\text{s}$ . How fast is the radius of the balloon increasing when the diameter is 50 cm?

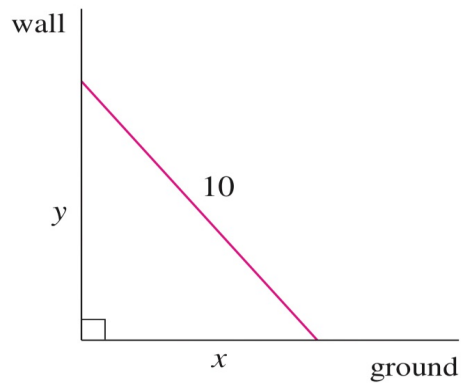
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**Example 2: “The Leaky Container”**

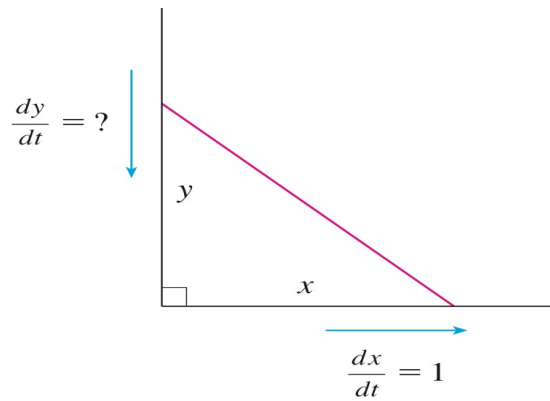
Gas is escaping from a spherical balloon at the rate of 2 cubic feet per minute. How fast is the surface area shrinking when the radius of the balloon is 12 feet?



**EXAMPLE 2** A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



**FIGURE 1**



**FIGURE 2**

### Example 1: “The Falling Ladder”

A ladder is sliding down along a vertical wall. If the ladder is 10 meters long and the top is slipping at the constant rate of 10 m/s, how fast is the bottom of the ladder moving along the ground when the bottom is 6 meters from the wall?

---

**EXAMPLE 4** Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

**EXAMPLE 5** A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

---









9. Suppose  $y = \sqrt{2x + 1}$ , where  $x$  and  $y$  are functions of  $t$ . \_\_\_\_\_

(a) If  $dx/dt = 3$ , find  $dy/dt$  when  $x = 4$ . \_\_\_\_\_

(b) If  $dy/dt = 5$ , find  $dx/dt$  when  $x = 12$ . \_\_\_\_\_





3. [10 pts.] The height of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of  $2 \text{ cm}^2/\text{min}$ . At what rate is the base of the triangle changing when the height is 10 cm and the area is  $100 \text{ cm}^2$ ?

2. [10 pts.] Two people start running from the same point  $P$ . One runs east at 6 km/h and the other runs north at 8 km/h. How fast is the distance between the two people changing after 0.5 hour?

3. [10 pts.] A rectangle has a fixed area of  $24 \text{ cm}^2$ . The length is decreasing at a rate of  $2 \text{ cm}/\text{min}$ . At what rate the width is increasing when the length is  $8 \text{ cm}$ .

---

3. [10 pts.] A cylindrical tank of diameter 6 m is being filled with water at a rate of  $2 \text{ m}^3/\text{hr}$ , how fast is the water level rising?

3. [10 pts.] A ladder 10 m long rests on horizontal ground and leans against a vertical wall. If the top of the ladder is being pulled up the wall at a rate of  $1 \text{ m}/\text{min}$ , how fast is the bottom of the ladder approaching the wall when the bottom of the ladder is 8 m away from the wall?

5. [10 pts.] The radius of a circle is decreasing at a rate of  $2 \text{ cm}/\text{min}$ . At what rate is the area of the circle changing when the area is  $25\pi \text{ cm}^2$ ?

6. [10 pts.] Air is being pumped into a spherical balloon so that its volume increases at a rate of  $100 \text{ cm}^3/\text{s}$ . How fast is the radius of the balloon increasing when the radius is  $25 \text{ cm}$ ?

طيارة ورقية

30. A kite 100 ft above the ground moves horizontally at a speed of  $8 \text{ ft/s}$ . At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



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# **Calculus 1 – Liner Approximation and Differential**

**(Section 3.10)**

**For Contact and Support:**



**YouTube: Precalculusq8**

**Twitter: Precalculusq8**

### أنواع الأسئلة في سكتشن 3.10

1) Find the linearization

$$L(x) = f(a) + f'(a)(x - a)$$

2) Use a linear approximation (or differentials) to estimate

$$f(x) \approx f(a) + f'(a)(x - a)$$

↑ يعني تقريبا

ex:  $\sqrt{0.9}$  ,  $x = 0.9$   $\therefore a = 1$  إذا مو عطيك قيمة الـ  $a$  بالسؤال .. أنت رح تيبب قيمة الـ  $a$  الي هي رح تكون العدد الاقرب للسؤال شلون تختار  $a$

$e^{0.1}$  ,  $x = 0.1$   $\therefore a = 0$

$\sqrt{99.8}$  ,  $x = 99.8$   $\therefore a = 100$

١- تكون قيمة قريبة من  $x$   
٢- تكون عارف قيمة الـ  $f(a)$

3) Find the differential of each function.

$$dy = f'(x) dx \quad \Delta y = f(x + \Delta x) - f(x)$$

## 3.10 Linear Approximations and Differentials

is called the **linear approximation** or **tangent line approximation** of  $f$  at  $a$ . The linear function whose graph is this tangent line, that is,

2

$$L(x) = f(a) + f'(a)(x - a)$$

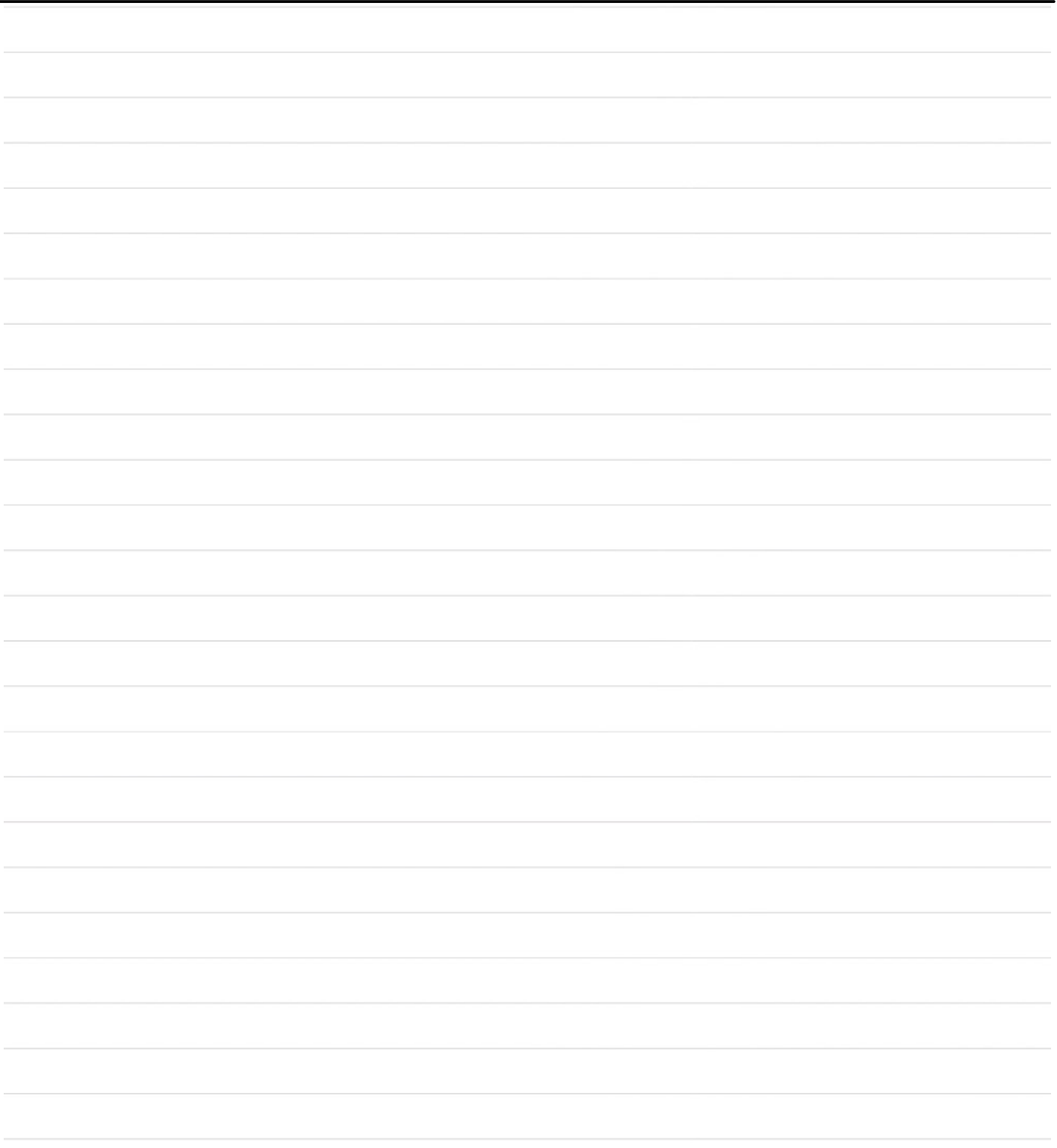
is called the **linearization** of  $f$  at  $a$ .

**EXAMPLE 1** Find the linearization of the function  $f(x) = \sqrt{x + 3}$  at  $a = 1$  and use it to approximate the numbers  $\sqrt{3.98}$  and  $\sqrt{4.05}$ . Are these approximations overestimates or underestimates?

**1-4** Find the linearization  $L(x)$  of the function at  $a$ .

**1.**  $f(x) = x^3 - x^2 + 3, \quad a = -2$

---



2.  $f(x) = \sin x, \quad a = \pi/6$

---

3.  $f(x) = \sqrt{x}, \quad a = 4$

**11-14** Find the differential of each function

**11.** (a)  $y = xe^{-4x}$

(b)  $y = \sqrt{1 - t^4}$

**11–14** Find the differential of each function

12. (a)  $y = \frac{1 + 2u}{1 + 3u}$

(b)  $y = \theta^2 \sin 2\theta$

(b)  $y = \theta^2 \sin 2\theta$

**11–14** Find the differential of each function

**13.** (a)  $y = \tan \sqrt{t}$

(b)  $y = \frac{1 - v^2}{1 + v^2}$

**11–14** Find the differential of each function

**14.** (a)  $y = \ln(\sin \theta)$

(b)  $y = \frac{e^x}{1 - e^x}$

(b)  $y = \frac{e^x}{1 - e^x}$



**19–22** Compute  $\Delta y$  and  $dy$  for the given values of  $x$  and  $dx = \Delta x$ .

**21.**  $y = \sqrt{x - 2}$ ,  $x = 3$ ,  $\Delta x = 0.8$

\_\_\_\_\_

\_\_\_\_\_

\_\_\_\_\_

- |







1) Find the linear approximation

$$\sin(46^\circ)$$

2) estimate  $\sqrt{99.8}$

Use linear approximation to estimate

$$\ln(1.013)$$

7. [10 pts.] Use **linear approximation** to estimate  $2e^{0.1} + (0.1)^2$ .



6. [10 pts.] Let  $f(x) = \frac{x - 1}{x^2 - 3}$

(a) Find the linearization of  $f$  at  $x = 1$ .

(b) Use part (a) to estimate  $f(1.02)$ .

4. [10 pts.] Use linear approximation to estimate  $f(2.1)$ , where  $f(x) = \sqrt{x^3 + 1}$ .

5. [10 pts.] Find the linearization of the function  $f(x) = x \ln x$  at  $a = 1$  and use it to approximate  $f(1.1)$ .

3. [10 pts.] Let  $f(x) = \tan^{-1}(x) - \frac{\pi}{4}$ . Use linear approximation to estimate  $f(1.05)$ .

4. [10 pts.] Let  $f(x) = e^x \cos x$ . Use linear approximation to estimate  $f(0.1)$ .



**Kuwait University**

# **Calculus 1 – Hyperbolic Functions**

## **(Section 3.11)**

**For Contact and Support:**



**YouTube: Precalculusq8**

**Twitter: Precalculusq8**

## 3.11 Hyperbolic Functions

ع

### Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

انتبه هذي مو المشتقة ..  
هذيل بينهم يساوي!!!!

ع

### Hyperbolic Identities

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

**EXAMPLE 1** Prove (a)  $\cosh^2 x - \sinh^2 x = 1$  and (b)  $1 - \tanh^2 x = \operatorname{sech}^2 x$ .

---

---

$$\frac{d}{dx} (\sinh x) = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

### 1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx} (\sinh x) = \cosh x$$

$$\frac{d}{dx} (\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx} (\cosh x) = \sinh x$$

$$\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx} (\operatorname{coth} x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx} (\cosh \sqrt{x}) = \sinh \sqrt{x} \cdot \frac{d}{dx} \sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

**1–6** Find the numerical value of each expression.

**1.** (a)  $\sinh 0$

---

(b)  $\cosh 0$

---

(a)  $\operatorname{sech} 0$

**1-6** Find the numerical value of each expression. \_\_\_\_\_

\_\_\_\_\_

**2.** (a)  $\tanh 0$  \_\_\_\_\_

\_\_\_\_\_

---

(b)  $\tanh 1$  \_\_\_\_\_

\_\_\_\_\_

**1–6** Find the numerical value of each expression. \_\_\_\_\_

**3.** (a)  $\cosh(\ln 5)$  \_\_\_\_\_ (b)  $\cosh 5$  \_\_\_\_\_

(b)  $\cosh 5$  \_\_\_\_\_



**7-19** Prove the identity.

**7.**  $\sinh(-x) = -\sinh x$

(This shows that  $\sinh$  is an odd function.)

**8.**  $\cosh(-x) = \cosh x$

(This shows that  $\cosh$  is an even function.)

**7-19** Prove the identity.

**9.**  $\cosh x + \sinh x = e^x$

**10.**  $\cosh x - \sinh x = e^{-x}$



**30–45** Find the derivative. Simplify where possible. \_\_\_\_\_  
\_\_\_\_\_

**30.**  $f(x) = e^x \cosh x$  \_\_\_\_\_  
\_\_\_\_\_

**31.**  $f(x) = \tanh \sqrt{x}$  \_\_\_\_\_  
\_\_\_\_\_

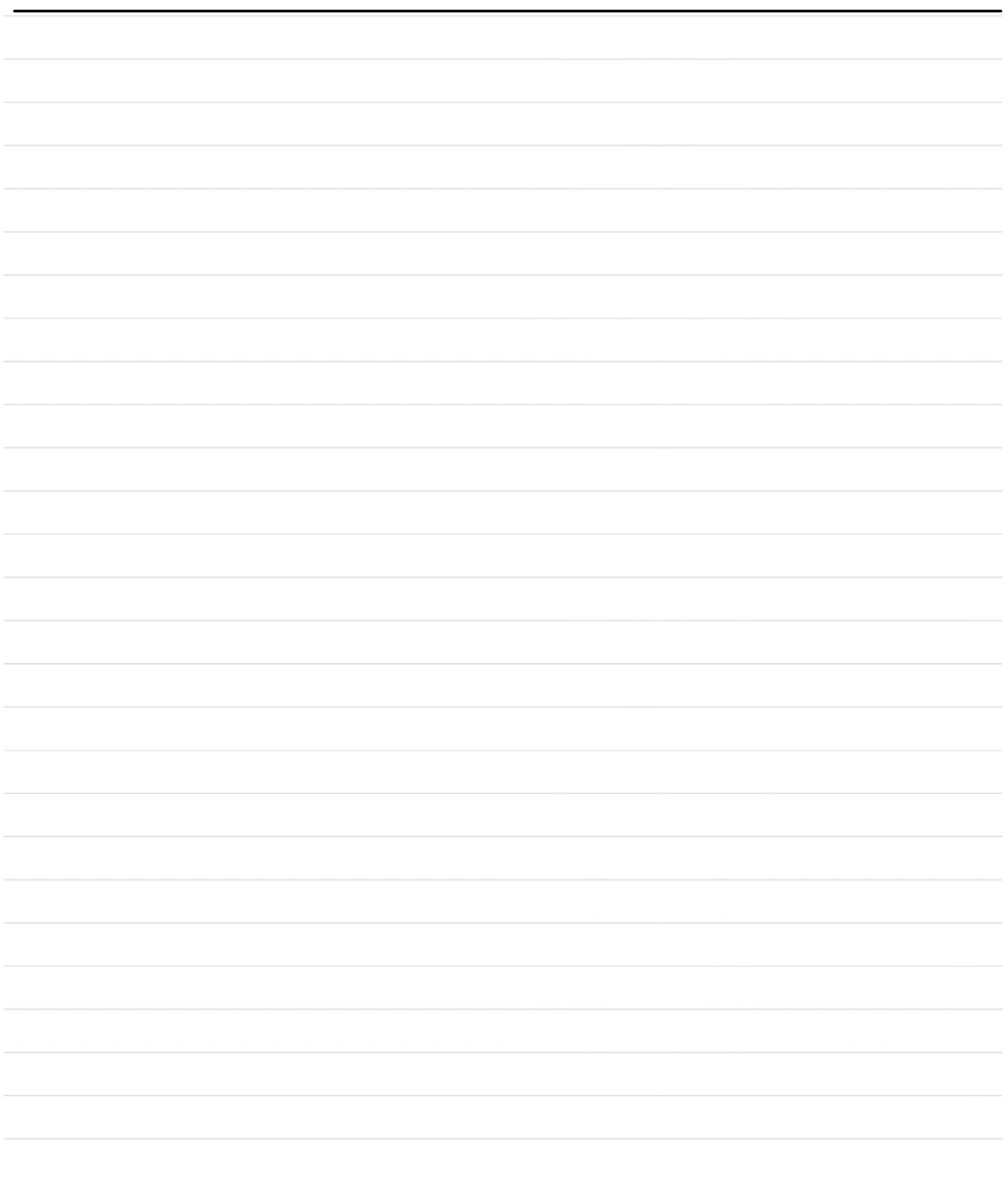
**30–45** Find the derivative. Simplify where possible.

**32.**  $g(x) = \sinh^2 x$

**33.**  $h(x) = \sinh(x^2)$

35.  $G(t) = \sinh(\ln t)$

---

A series of horizontal lines for writing, starting from the first line below the equation and extending to the bottom of the page.





IV.  $\lim_{x \rightarrow 0} \frac{\cosh x - 1}{x} =$

a)  $\boxed{0}$ .

b) 1.

c)  $e$ .

d) None of the above.

1. [10 pts.] Let  $f(x) = x \sinh(x^2 - 1)$ . Find  $f'(1)$ .

2. [10 pts.] Find the slope of the tangent line at the point  $(0, 0)$  to the curve with equation

$$e^{xy} + \sin(x^2) = \cosh(x) + y.$$

IV. The derivative of  $\cosh^2 x - \sinh^2 x$  is

- (A) 0
- (B)  $4 \cosh x \sinh x$ .
- (C)  $-2 \cosh x \sinh x$ .
- (D)  $2 \cosh x \sinh x$ .
- (E) None of the above.

II. The value of  $\lim_{x \rightarrow \infty} \tanh x$  is

a) does not exist.

b) 0.

c)  $-\pi^2$ .

d) 1.

e) None of the above.





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# **Calculus 1 – Maximum and Minimum Values**

**(Section 4.1)**

**For Contact and Support:**



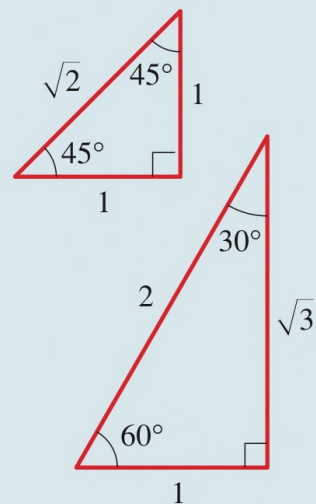
**YouTube: Precalculusq8**

**Twitter: Precalculusq8**

## SPECIAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

The following values of the trigonometric functions are obtained from the special triangles.

$\theta$ in degrees	$\theta$ in radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
0	0	0	1	0	—	1	—
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	—	1	—	0



Remark:-

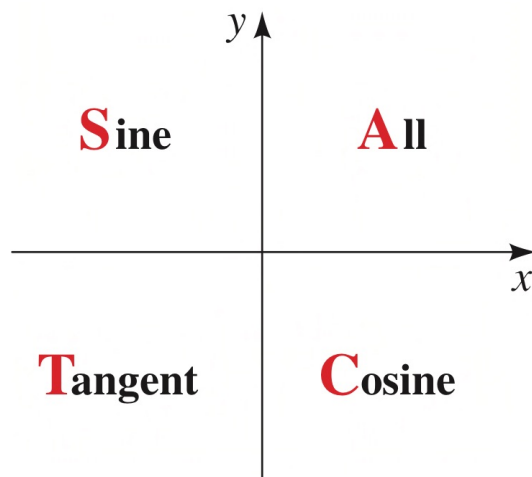
$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

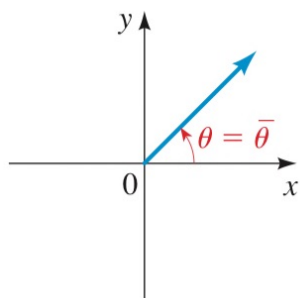
## SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

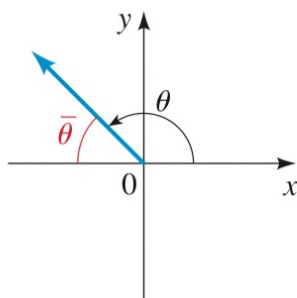
Remark:-



You can remember this as “All Students Take Calculus.”

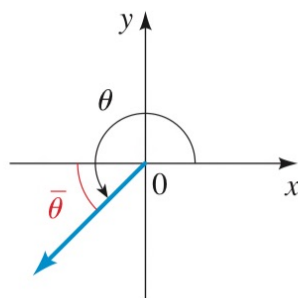


$$\bar{\theta} = \theta$$



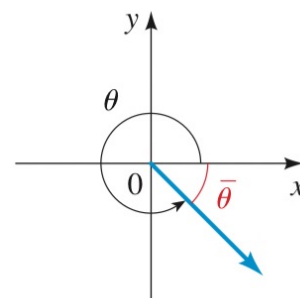
$$\bar{\theta} = \pi - \theta$$

$$\theta = \pi - \bar{\theta}$$



$$\bar{\theta} = \theta - \pi$$

$$\theta = \pi + \bar{\theta}$$



$$\bar{\theta} = 2\pi - \theta$$

$$\theta = 2\pi - \bar{\theta}$$

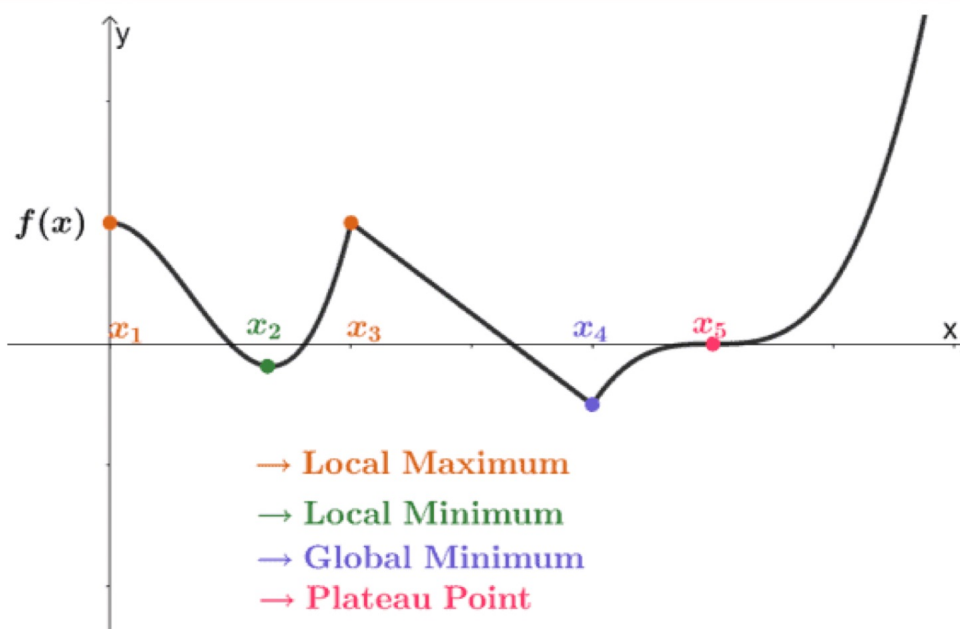
## What are critical numbers?

Critical numbers or critical points are values of  $x$  where the **first derivative of a function is either equal to zero or undefined**. Let's say we have  $x = c$ , the critical numbers of the function,  $f(x)$ , will satisfy either of the following:

$$f'(c) = 0$$

$$f'(c) = \text{DNE (Does Not Exist)}$$

This means that  $x = c$  is a critical number when a **tangent line passing through  $x = c$  is either a horizontal or vertical line**.

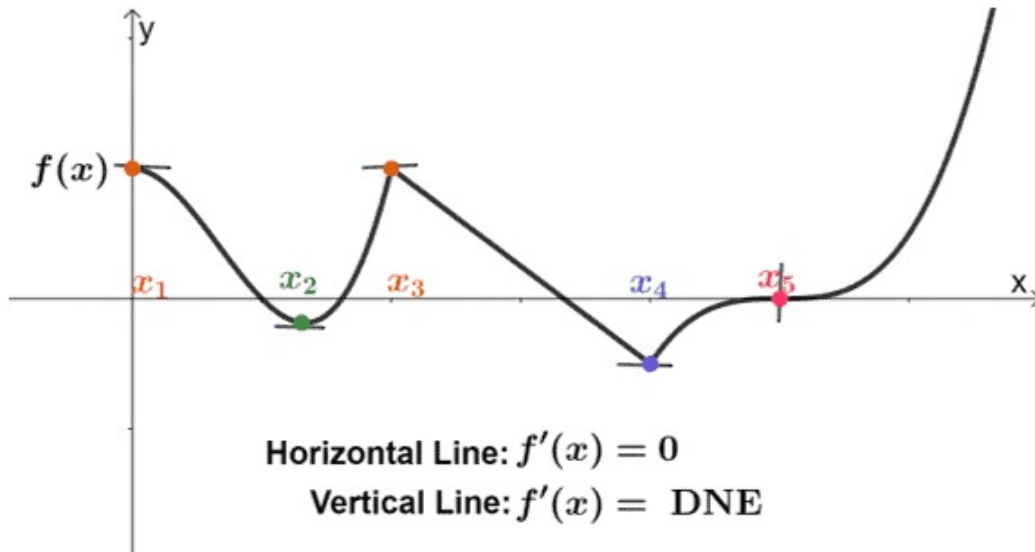


## How to find critical numbers?

We can use the definition of critical numbers to determine the critical numbers – by finding the values of  $x$  where  $f'(x) = 0$  or  $f'(x) = \text{DNE}$ . These are some guide points to help you find a function's critical numbers:

- Use the derivative rules to find  $f'(x)$ 's expression.
- Find the values of  $x$  where  $f'(x) = 0$ .
- Include the values of  $x$  where  $f'(x)$  is undefined.

**Only include critical numbers that are within the function's domain.** This means that values of  $x$  where  $f'(x)$  is undefined will only count when the point is within the domain of  $f(x)$ .



- When the tangent line at  $x = c$  is vertical,  $f'(c) = 0$  and we have critical point at  $x = c$ .
- Similarly, when the tangent line passing through  $x = c$  is horizontal,  $f'(c)$  is undefined and  $x = c$  is also a critical point.

### Example 1

قراءة

What are the critical numbers of the function,  $f(x) = 2x^3 - 8x^2 + 2x - 1$ ?

$$f(x) = 2x^3 - 8x^2 + 2x - 1 \quad D: \mathbb{R}$$

يعني الدالة معرفة بكل مكان فما أقدر أطلع الcritical من هالطريقة

$$f'(x) = 6x^2 - 16x + 2, \text{ Since } f'(x) \text{ is Defined everywhere}$$

We can only find the critical number by  $f'(x) = 0$

$$f'(x) = 0$$

$$6x^2 - 16x + 2 = 0$$

قسمت على ٢ المعادلة كاملة لأنه أقدر اصغر الأعداد

$$3x^2 - 8x + 1 = 0$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(1)}}{2(3)} = \frac{4 \pm \sqrt{13}}{3}$$

This means that  $f(x)$  has critical numbers at  $x = \frac{4 \pm \sqrt{13}}{3}$ .

## Example 2

What are the critical numbers of the function,  $g(x) = \frac{x^2-6}{x-4}$ ?

Domain of  $g(x) = \mathbb{R} / \{4\}$

$$\begin{aligned}g'(x) &= \frac{2x(x-4) - (x^2-6)}{(x-4)^2} = \frac{2x^2 - 8x - x^2 + 6}{(x-4)^2} \\ &= \frac{x^2 - 8x + 6}{(x-4)^2}\end{aligned}$$

We can find the critical numbers of  $g(x)$  by equating  $g'(x)$  to zero or by finding undefined values within its domain. For  $g'(x)$  to be undefined, the denominator must be zero.

$$\begin{aligned}(x-4)^2 &= 0 \\ x &= 4 \quad \notin \mathbb{R} / \{4\}\end{aligned}$$

It may be tempting to include  $x = 4$  as one of  $g(x)$ 's critical numbers, but keep in mind  $x = 4$  is not part of  $g(x)$ 's domain. Hence, we disregard it.

$$\begin{aligned}g'(x) &= 0 \\ x^2 - 8x + 6 &= 0 \quad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(6)}}{2(1)} = \frac{8 \pm \sqrt{64 - 24}}{2} \\ &= \frac{8 \pm 2\sqrt{10}}{2} = 4 \pm \sqrt{10}\end{aligned}$$

This means that  $g(x)$  has critical numbers at  $x = 4 - \sqrt{10}$  and  $x = 4 + \sqrt{10}$ .

**6 Definition** A **critical number** of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.

عشان إذا تبي تعرف ال critical number حق أي function :

1)  $f'(x) = 0$  - 1 نشوف المشتقة ونساويها بالصفر

2- نشوف المشتقة لما تكون غير معرفة ( يعني أساوي المقام بصفر )

2)  $f'(x) = DNE$

ملاحظة رقم 2 أحتاج إنني أوجد المقامات إذا كانت مو موحد

**EXAMPLE 7** Find the critical numbers of  $f(x) = x^{3/5}(4 - x)$ .



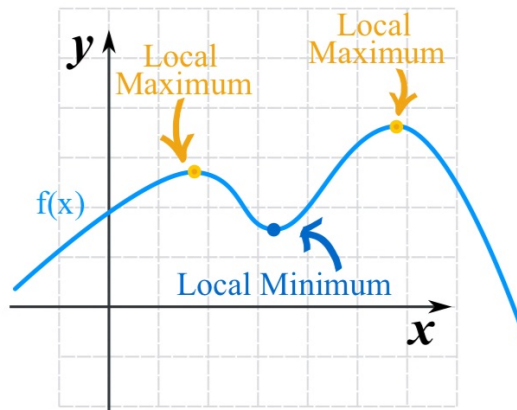




## Local Maximum and Minimum

Functions can have "hills and valleys": places where they reach a minimum or maximum value.

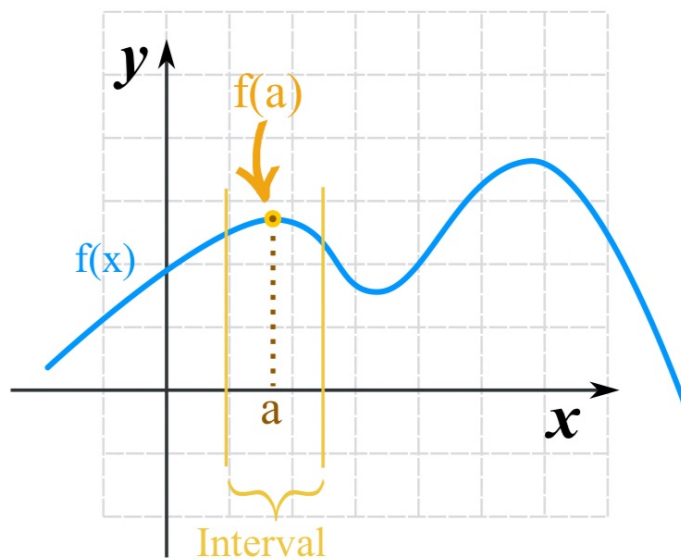
It may not be the minimum or maximum for the **whole function**, but **locally** it is.



We can see where they are,  
but how do we define them?

## Local Maximum

**First** we need to choose an interval:



Then we can say that a local **maximum** is the point where:

$$f(a) \geq f(x) \text{ for all } x \text{ in the interval}$$

In other words, there is no height greater than  $f(a)$ .

Note:  $a$  should be **inside** the interval, not at one end or the other.

## Local Minimum

Likewise, a local **minimum** is:

$$f(a) \leq f(x) \text{ for all } x \text{ in the interval}$$

The plural of Maximum is **Maxima**

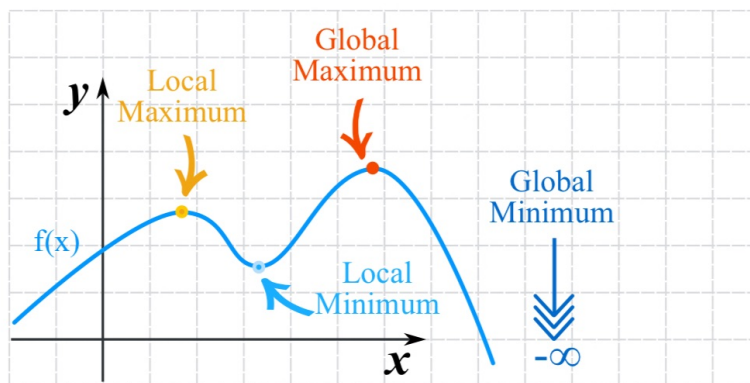
The plural of Minimum is **Minima**

Maxima and Minima are collectively called **Extrema**

## Global (or Absolute) Maximum and Minimum

The maximum or minimum over the **entire function** is called an "Absolute" or "Global" maximum or minimum.

There is only one global maximum (and one global minimum) but there can be more than one local maximum or minimum.



**Assuming** this function continues downwards to left or right:

- The Global Maximum is about 3.7
- The Global Minimum is  $-\infty$

**The Closed Interval Method** To find the *absolute* maximum and minimum values of a continuous function  $f$  on a closed interval  $[a, b]$ :

1. Find the values of  $f$  at the critical numbers of  $f$  in  $(a, b)$ .
2. Find the values of  $f$  at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

1) function is continuous

أكتب أسم الدالة و إن  
الدالة continuous  
على الفترة المطلوبة

2) find the critical numbers  $\in$  Domain

تأكد ال critical values الي  
طلعتهم ينتمون لل domain  
الدالة الاصلية

3) find endpoints

أطراف الفترة و عوضها بال  $f(x)$  مع  
critical numbers ال

4) The largest value  
is the absolute maximum

بعد ما تعوض في الفترة  
المعطاة مع قيم  
ال critical شوف أكبر  
نتيجة تطلع لك وقول عنها  
abs maximum

The smallest value  
is the absolute minimum

بعد ما تعوض  
شوف أصغر  
نتيجة تطلع لك  
وسمها abs  
minimum

@Precalculusq8



**The Closed Interval Method** To find the *absolute* maximum and minimum values of a continuous function  $f$  on a closed interval  $[a, b]$ :

1. Find the values of  $f$  at the critical numbers of  $f$  in  $(a, b)$ .
2. Find the values of  $f$  at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

**EXAMPLE 8** Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \quad -\frac{1}{2} \leq x \leq 4$$





Find the absolute maximum and minimum values of the function

59.  $f(x) = x^{-2} \ln x, \left[\frac{1}{2}, 4\right]$

2) The values of  $f$  at the endpoints

قراءة

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-2} \left(\ln \frac{1}{2}\right) = 4 \ln \frac{1}{2} = 4 \ln 2^{-1} = -4 \ln 2$$

$$f(4) = (4)^{-2} (\ln 4) = \frac{1}{16} (\ln 4)$$

$$f(\sqrt{e}) = \left(\frac{1}{\sqrt{e}}\right)^{-2} (\ln e^{1/2}) = \left(\frac{1}{e}\right) \left(\frac{1}{2} \ln e\right) = \frac{1}{2e}$$

The absolute maximum value of  $f$ ,  
The absolute minimum value of  $f$ ,

ما أدري لانه لازم آلة  
حاسبة .. المهم الفكرة  
والخطوات

**47–62** Find the absolute maximum and absolute minimum values of  $f$  on the given interval.

**62.**  $f(x) = x - 2 \tan^{-1}x, [0, 4]$

b)  $f'(x)$  exists for all  $x$ ,  $\therefore$  The only

قراءة

critical number is  $x = 1$

$$2) f(0) = 0 - 2 \tan^{-1} 0 = 0$$

$$f(4) = 4 - 2 \tan^{-1} 4$$

$$f(1) = 1 - 2 \tan^{-1} 1 = 1 - 2 \frac{\pi}{4} = 1 - \frac{\pi}{2}$$

The absolute maximum value of  $f$ , ما أدري لانه لازم آلة حاسبة.. المهم الفكرة والخطوات

The absolute minimum value of  $f$ ,

Find the absolute maximum and minimum values of the function

$$f(x) = \sin x - \cos x \quad \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right]$$



8. [10 pts.] Find the **absolute maximum** and **absolute minimum** values of  $f(x) = \cos^2 x + \sin x$  on the interval  $\left[0, \frac{\pi}{2}\right]$ .

5. [10 pts.] Find the absolute maximum and minimum values of the function  $f(x) = 3x - 5x^{3/5}$  on the interval  $[-1, 32]$ .

2.

II. Let  $f(x) = x^2 + 1$ . On the interval  $[-1, 1]$ :

- (A) 1 is the absolute minimum value of  $f$  and 2 is the absolute maximum value of  $f$ .
- (B)  $f(0.5)$  is a local minimum value of  $f$ .
- (C)  $f$  has no absolute extreme values.
- (D)  $f(0.5)$  is a local maximum value of  $f$ .
- (E) None of the above.

4. [10 pts.] Find the absolute maximum and absolute minimum values of  $f(x) = x^2e^{-2x}$  on the interval  $[-1, 1]$ .

III. The function  $f$  defined by  $f(x) = \frac{1}{x}$  has a critical point at:

- a)  $x = 0$ .      b)  $x = 1$ .      c)  $x = -1$ .      d)  $x = 2$ .      e) None of the above.

7. [10 pts.] Find the absolute maximum and absolute minimum values of

$$f(x) = xe^{-x} \text{ on } [0, 2].$$



**Kuwait University**

**Calculus 1 – The Mean Value  
Theorem  
(Section 4.2)**

**For Contact and Support:**



**YouTube: Precalculusq8**

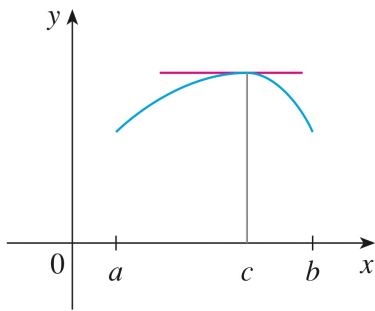
**Twitter: Precalculusq8**

## حفظ نص النظرية !!!

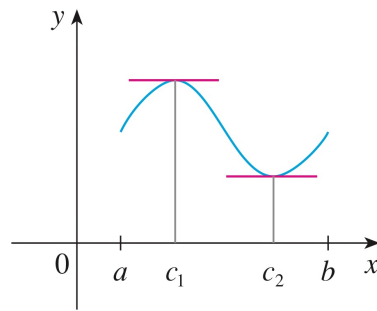
**Rolle's Theorem** Let  $f$  be a function that satisfies the following three hypotheses:

1.  $f$  is continuous on the closed interval  $[a, b]$ .
2.  $f$  is differentiable on the open interval  $(a, b)$ .
3.  $f(a) = f(b)$

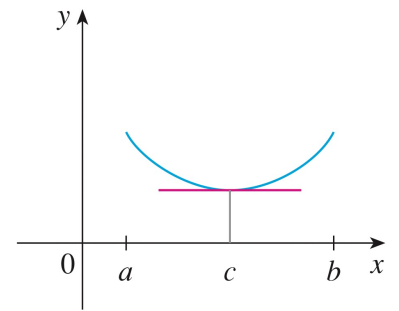
Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .



(b)



(c)



(d)

## حفظ نص النظرية !!!

**The Mean Value Theorem** Let  $f$  be a function that satisfies the following hypotheses:

1.  $f$  is continuous on the closed interval  $[a, b]$ .
2.  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a number  $c$  in  $(a, b)$  such that

1

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2

$$f(b) - f(a) = f'(c)(b - a)$$

كيف

# Remark

**7 Theorem** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

$$-1 \leq \sin x \leq 1$$
$$-1 \leq \cos x \leq 1$$

Contradict = تناقض  
Applicable = قابلة للتطبيق  
Hypothesis = فرضيات  
Root = حل للمعادلة

or critical number

- To prove the function has exactly one root :

→ section 2.5

1 - IVT theorem (to show that a root exist)

2 - Rolles theorem and argue by contradiction

and we suppose it had two roots

a and b  $f(a) = f(b) = 0$ , function

cont. & diff. Then by Rolles theorem, There is number c between a and b such that

$$f'(c) = 0$$

الخطوة الثانية يعني رح نفرض أن هناك حلين للمعادلة ، ورح نثبت أن فرضيتنا غلط باستخدام النظرية Rolles ورح يصير تناقض آخر شي إن ما يصير لها حلين .. ورح نشوف في حل المسائل

- To prove the function at most one real solution

• only using Rolles theorem argue and contradiction

يعني جنبه أهو يدري أن في root  
بس يبي يتأكد إن مافي غير root  
واحد .. فعشان جزي ماله داعي  
أثبت عن طريق نظرية IVT أن في  
at least root يعني على الأقل  
في root

Remark:-

**4 Theorem** If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

**10 The Intermediate Value Theorem** Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .

تذكر طريقة حل  
I.V.T

53.  $x^4 + x - 3 = 0, (1, 2)$

Let  $f(x) = x^4 + x - 3 = 0$

لازم نتأكد أن المعادلة  
صفرية .. إذا ما كانت  
صفرية ... خلفها صفرية

Domain of  $f$  is  $\mathbb{R}$  and continuous

$f(1) = (1)^4 + 1 - 3 = -1 < 0$

$f(2) = (2)^4 + 2 - 3 = 15 > 0$

$f$  is cont. on  $(1, 2)$  Then

by I.V.T, There exists a  $c$  in  $(1, 2)$   
such that  $f(c) = 0 \therefore c$  is a root



9. [10 pts.] Verify that the function  $f(x) = x + \frac{1}{x}$  satisfies the hypotheses of **Rolle's Theorem** on the interval  $\left[\frac{1}{2}, 2\right]$ . Then find all numbers  $c$ , if any, that satisfy the conclusion of Rolle's Theorem.

6. [5+5=10 pts.] Let  $f(x) = \ln(x^2 + 1)$ .

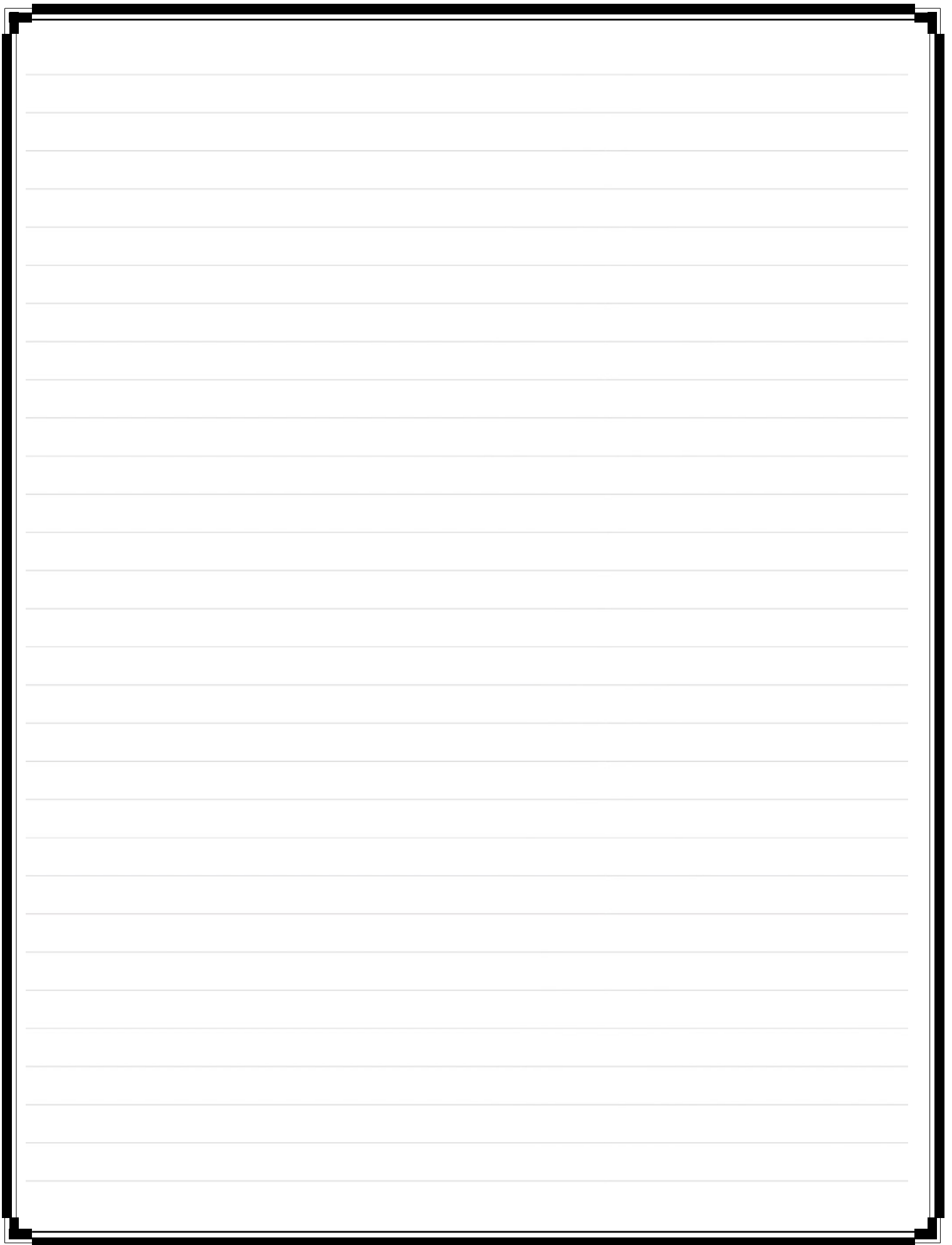
(a) Verify that  $f$  satisfies the conditions of Rolle's Theorem on  $[-1, 1]$ .

(b) Find all  $c \in (-1, 1)$  that satisfy the conclusion of Rolle's Theorem.

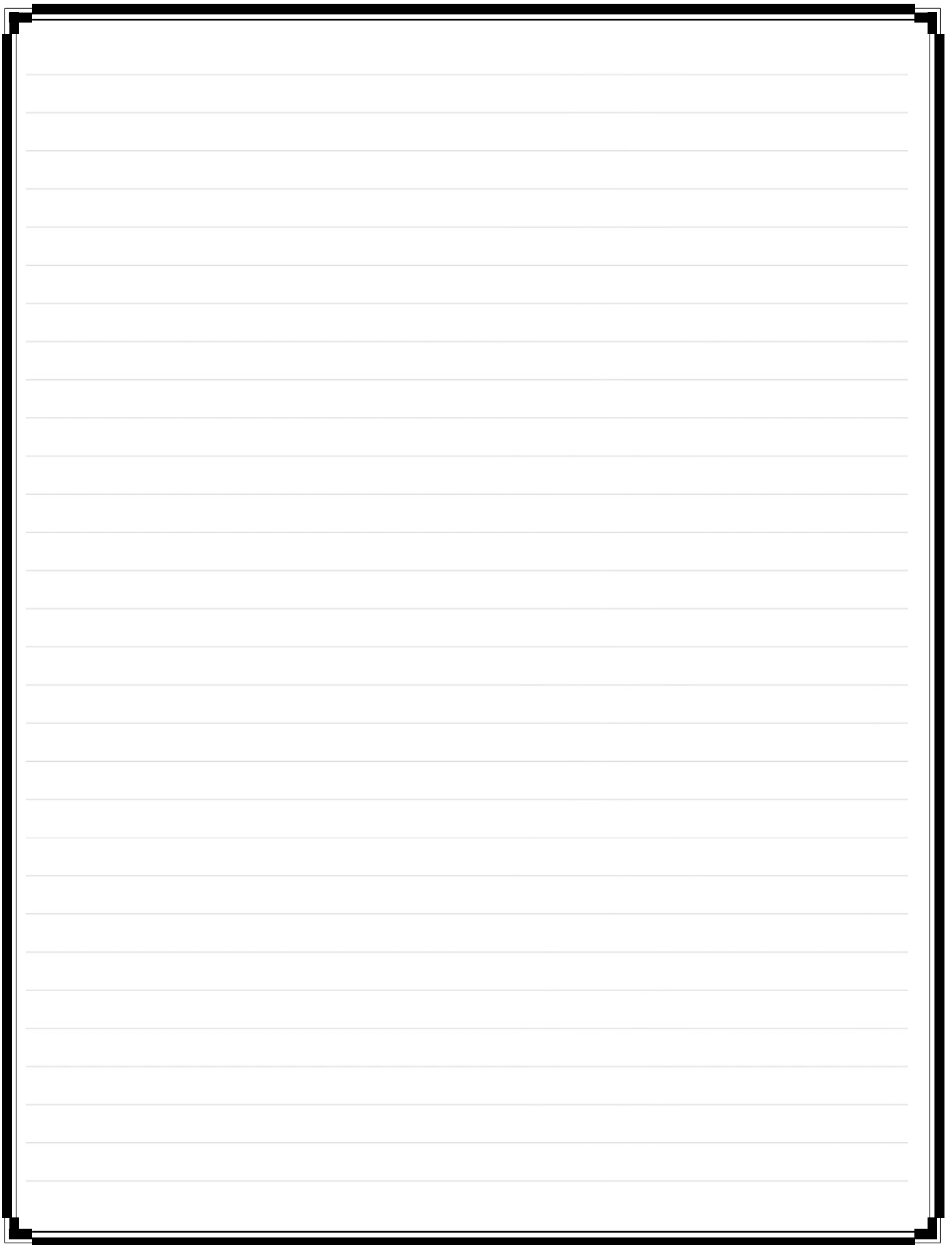


**EXAMPLE 2** Prove that the equation  $x^3 + x - 1 = 0$  has exactly one real root.

---





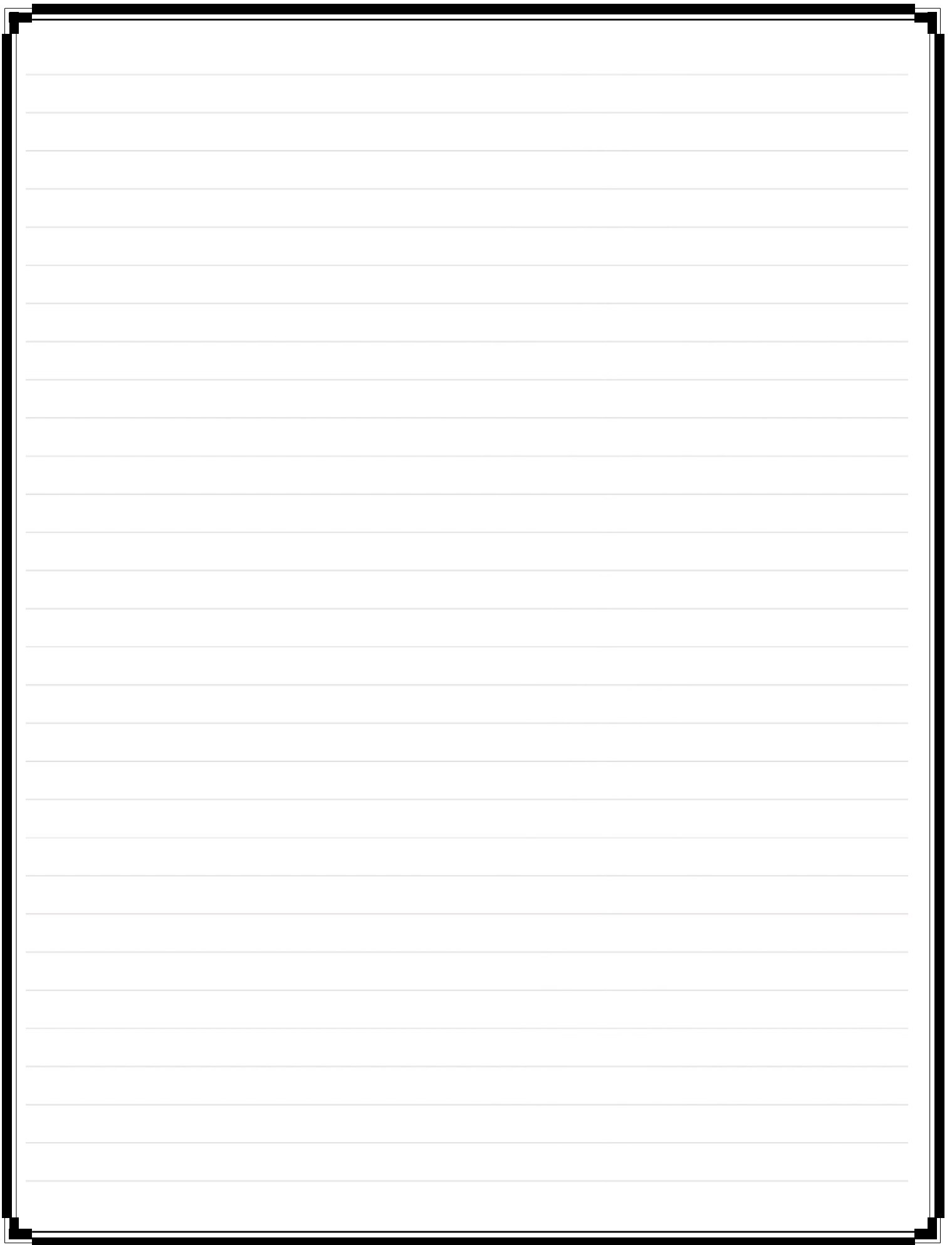


**19–20** Show that the equation has exactly one real root.

**19.**  $2x + \cos x = 0$

---









**Q4.** [10 pts.] Show that, if  $a > 0$ , then the equation:  $x^3 + ax - 1 = 0$  has at most one real root.

**8.** [10 pts.] Verify that  $f(x) = 2x^2 + 3x + 4$  satisfies the hypotheses of Rolle's Theorem on the interval  $\left[-2, \frac{1}{2}\right]$ . Then find all numbers  $c$  that satisfy the conclusion of Rolle's Theorem.

**The Mean Value Theorem** Let  $f$  be a function that satisfies the following hypotheses:

1.  $f$  is continuous on the closed interval  $[a, b]$ .
2.  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a number  $c$  in  $(a, b)$  such that

1 
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 
$$f(b) - f(a) = f'(c)(b - a)$$

3 
$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1. Since  $f'(c)$  is the slope of the tangent line at the point  $(c, f(c))$ , the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point  $P(c, f(c))$  on the graph where the slope of the tangent line is the same as the slope of the secant line  $AB$ . In other words, there is a point  $P$  where the tangent line is parallel to the secant line  $AB$ . (Imagine a line far away that stays parallel to  $AB$  while moving toward  $AB$  until it touches the graph for the first time.)

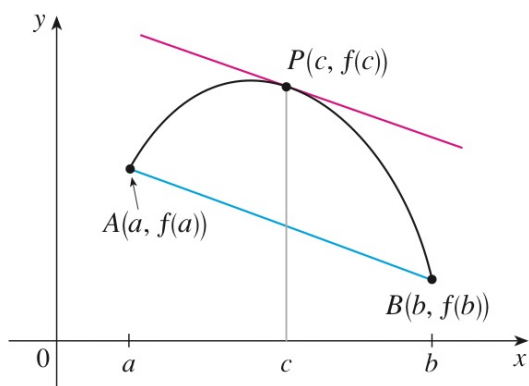


FIGURE 3

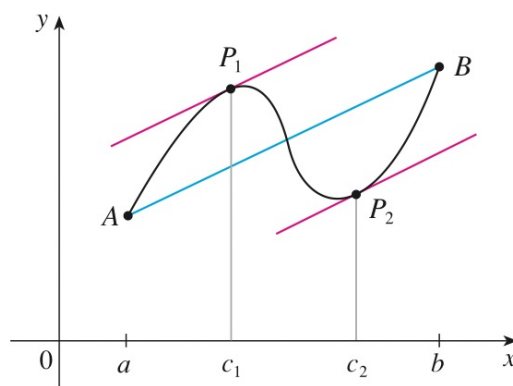


FIGURE 4

**EXAMPLE 3** To illustrate the Mean Value Theorem with a specific function, let's consider  $f(x) = x^3 - x$ ,  $a = 0$ ,  $b = 2$ . Since  $f$  is a polynomial, it is continuous and differentiable for all  $x$ , so it is certainly continuous on  $[0, 2]$  and differentiable on  $(0, 2)$ . Therefore, by the Mean Value Theorem, there is a number  $c$  in  $(0, 2)$  such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now  $f(2) = 6$ ,  $f(0) = 0$ , and  $f'(x) = 3x^2 - 1$ , so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives  $c^2 = \frac{4}{3}$ , that is,  $c = \pm 2/\sqrt{3}$ . But  $c$  must lie in  $(0, 2)$ , so  $c = 2/\sqrt{3}$ .

**EXAMPLE 5** Suppose that  $f(0) = -3$  and  $f'(x) \leq 5$  for all values of  $x$ . How large can  $f(2)$  possibly be?

**11–14** Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers  $c$  that satisfy the conclusion of the Mean Value Theorem.

**11.**  $f(x) = 2x^2 - 3x + 1, [0, 2]$

---

17. Let  $f(x) = (x - 3)^{-2}$ . Show that there is no value of  $c$  in  $(1, 4)$  such that  $f(4) - f(1) = f'(c)(4 - 1)$ . Why does this not contradict the Mean Value Theorem?

---







4. [10 pts.] Verify that  $f(x) = \sqrt{4 - x^2}$  satisfies the hypotheses of the Mean Value Theorem on the interval  $[-2, 0]$ . Then, find all numbers  $c$  that satisfy the conclusion of the Theorem.

#### PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $ x  < c$	$-c < x < c$	
2. $ x  \leq c$	$-c \leq x \leq c$	
3. $ x  > c$	$x < -c$ or $c < x$	
4. $ x  \geq c$	$x \leq -c$ or $c \leq x$	

5. [10 pts.] Given a function  $f(x)$  such that  $f(6) = 5$  and  $f'(x) \leq -5$  for all  $x \in [6, 10]$ . How large can  $f(10)$  possibly be?
- الفترة



$$\text{Let } f'(x) = \frac{1}{3+2x^2} \quad \forall x \in \mathbb{R}$$

and  $f(1) = 0$ , show that

$$\frac{1}{11} < f(2) < \frac{1}{5}$$

---

Use MVT to show that for  $x > 0$

$$\tan^{-1} x < x$$

---