A+

Calculus A

Chapter 1: Functions and Models

Review

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Chapter 1

a) find the exact value / Prove

b) logarithmic function

c) find the inverse
- → domain
- → range
- → one-to-one

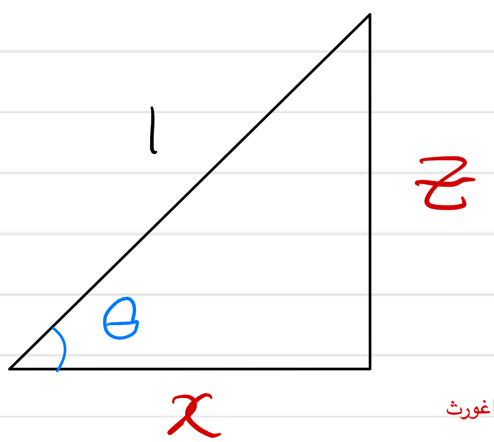
Prove that

a) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

b) find $\sin(\cos^{-1}(1/2))$

Let $\theta = \cos^{-1} x$

$\therefore \cos\theta = x$



بينا الضلع المفقود (z) عن طريق فيثاغورث

$\because 1^2 = x^2 + z^2 \Rightarrow z = \sqrt{1-x^2}$

$\therefore \sin(\cos^{-1} x) = \sin(\theta)$

$\sin(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$ ✓

b) $\sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$

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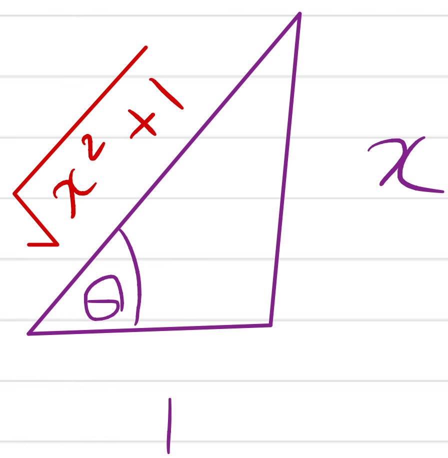
7. $(5 + 5 = 10 \text{ pts})$

(a) Show that $\sin\left(\tan^{-1} x\right) = \dfrac{x}{\sqrt{1+x^2}}$.

(b) Find the exact value of $\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$.

Let $\theta = \tan^{-1} x$

$\therefore \tan\theta = x$



$\therefore \sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

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b) $\sin\left(\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\right) = \sin\left(\dfrac{\pi}{6}\right)$

$= \dfrac{1}{2}$

70–72 Simplify the expression.

70. $\tan(\sin^{-1}x)$ **71.** $\sin(\tan^{-1}x)$

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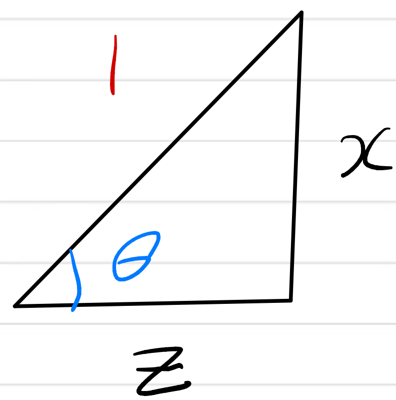
70) Let $\theta = \sin^{-1} x$

$\therefore \sin\theta = x$

using Pythagorean theorem

$z = \sqrt{1^2 - x^2}$



$\therefore \tan(\sin^{-1}x) = \tan\theta$

$\therefore \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

---

71) $\dfrac{x}{\sqrt{1+x^2}}$

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4. [10 pts.] Find the exact value of each expression

(b) $\log_{10} 4 + 2\log_{10} 5$.

(b) $\boxed{\log_{10} 4 + 2\log_{10} 5 = \log_{10} 4 + \log_{10} 5^2 = \log_{10} 4 + \log_{10} 25 = \log_{10} 100 = 2}$.

Q2. [5+5+10=20 pts.] Let $f(x) = 2^x + \ln(x)$ and $g(x) = \frac{x-1}{x+1}$.

(b) If $f(1) = 2$, find $f^{-1}(2)$.

(c) Find $g^{-1}(x)$.

(b) Since $f(1) = 2 \therefore$ $\boxed{f^{-1}(2) = 1}$.

(c) Find $g^{-1}(x)$.

We first write $y = \frac{x-1}{x+1}$. Then we solve this equation for x as a function of y:

$$y(x+1) = x - 1$$
$$y + 1 = x(1 - y)$$
$$x = \frac{y+1}{1-y}.$$

Finally, we interchange x and y : $y = \frac{x+1}{1-x}$. Therefore the inverse function is $\boxed{g^{-1}(x) = \frac{x+1}{1-x}}$.

2. [5 + 5 = 10 pts.] The graph of f is given below.

(a) Explain why f is one-to-one.

(b) Find $f^{-1}(2)$.

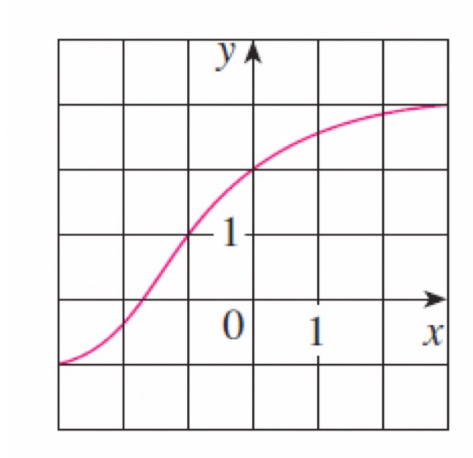

Figure 2: The graph of $y = f(x)$.

(a) Any horizontal line intersects the graph of f at most once. Therefore, the function is one-to-one by the horizontal line test.

(b) $\boxed{f^{-1}(2) = 0 \text{ since } f(0) = 2}$.

---

2. [5 + 5 + 5 = 15 pts.] Let $f(x) = \dfrac{x+1}{2x+1}$.

(a) Show that f is one-to-one.

---

if $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1+1}{2x_1+1} = \dfrac{x_2+1}{2x_2+1}$

$(x_1+1)(2x_2+1) = (2x_1+1)(x_2+1)$

$2x_1x_2 + x_1 + 2x_2 + 1 = 2x_1x_2 + 2x_1 + x_2 + 1$

$x_1 + 2x_2 = 2x_1 + x_2 \Rightarrow 2x_2 - x_2 = 2x_1 - x_1$

$x_1 = x_2$ $\therefore$ f is one to one

3. [10 pts.] Let $f(x) = \ln(2^x - 1)$. Find $f^{-1}(x)$.

Let $y = \ln(2^x - 1)$

$$e^y = e^{\ln(2^x - 1)}$$

$$e^y = 2^x - 1$$

$$e^y + 1 = 2^x$$

$$\log_2(e^y + 1) = \log_2(2)^x$$

$$\log_2(e^y + 1) = x$$

$$\therefore f^{-1}(x) = \log_2(e^x + 1)$$

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$$f(x) = e^{\frac{x+e^3}{2}}, \text{ find } f^{-1}(e^3)$$

$$y = e^{\frac{x+e^3}{2}}$$

$$\ln y = \ln e^{\frac{x+e^3}{2}}$$

$$\ln y = \frac{x+e^3}{2}$$

$$2\ln y = x + e^3$$

$$2\ln y - e^3 = x$$

$$f^{-1}(x) = 2\ln x - e^3$$

$$f^{-1}(e^3) = 2\ln e^3 - e^3$$

$$= 3 \times 2 - e^3$$

$$f^{-1}(e^3) = 6 - e^3$$

find the domain:-

$$f(x) = \frac{\ln(e^x - 2)}{x^2 - 9}$$

$D_{\ln(e^x-2)}$: $e^x - 2 > 0$

$\Rightarrow e^x > 2 \quad \Rightarrow \ln e^x > \ln 2$

$x > \ln 2$

D_{x^2-9} : $x^2 - 9 \neq 0$

$\Rightarrow (x-3)(x+3) \neq 0$

$\therefore \mathbb{R} / \{-3, 3\}$

$\therefore D_f : (\ln 2, 3) \cup (3, \infty)$

let $f(x) = 4 + e^{(x-3)}$

find the domain, $f^{-1}(x)$, range of f

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* D_f : $\mathbb{R} = (-\infty, \infty)$

* for the inverse function :-

$y = 4 + e^{(x-3)} \Rightarrow y - 4 = e^{x-3}$

$\ln(y-4) = \ln e^{x-3}$

$\ln(y-4) = x - 3$

$\ln(y-4) + 3 = x$

$\therefore f^{-1}(x) = \ln(x-4) + 3$

$D_{f^{-1}(x)}$: $(4, \infty)$ = Range of $f(x)$

Let $f(x) = \dfrac{2e^x}{e^x+1}$

1) Domain

2) $f^{-1}(x)$

---

a) Domain $(-\infty, \infty)$

b) $\ln\left(\dfrac{x}{2-x}\right)$

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$$f(x) = 1 + e^{(x+1)}$$

1) Domain of $f(x)$
2) inverse of $f(x)$
3) Range of $f(x)$

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1) $(-\infty, \infty)$

2) $f^{-1}(x) = \ln(x-1) - 1$

3) $(1, \infty)$

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3. [10 pts.] Let $f(x) = \ln\left(\dfrac{x}{2x-1}\right)$. Find $f^{-1}(x)$.

Let $y = \ln\left(\dfrac{x}{2x-1}\right)$

$$e^{y} = e^{\ln\left(\frac{x}{2x-1}\right)}$$

$$e^{y} = \frac{x}{2x-1}$$

$$e^{y}(2x-1) = x$$

$$2xe^{y} - e^{y} = x$$

$$2xe^{y} - x = e^{y}$$

$$x(2e^{y}-1) = e^{y}$$

$$x = \frac{e^{y}}{2e^{y}-1}$$

$$\therefore f^{-1}(x) = \frac{e^{x}}{2e^{x}-1}$$

2. [15 pts.] Find the domain, range and a formula for the inverse of the function $f(x) = \ln(e^{2x} - 2)$.

for domain:-

$e^{2x} - 2 > 0 \Rightarrow e^{2x} > 2$

$\ln e^{2x} > \ln 2 \Rightarrow 2x > \ln 2$

$x > \frac{\ln 2}{2}$

$\therefore$ The domain is $\left(\frac{\ln 2}{2}, \infty\right)$

for the range & inverse:-

$y = \ln(e^{2x} - 2) \Rightarrow e^{y} = e^{\ln(e^{2x}-2)}$

دربالك تييب الln من الحين لازم نودي ال-٢ طرف الثاني

$e^{y} = e^{2x} - 2 \Rightarrow e^{y} + 2 = e^{2x}$

$\ln(e^{y}+2) = \ln e^{2x} \Rightarrow \ln(e^{y}+2) = 2x$

$x = \frac{\ln(e^{y}+2)}{2} \Rightarrow f^{-1}(x) = \frac{\ln(e^{x}+2)}{2}$

Domain of f^{-1} = Range of f = $e^{x} + 2 > 0 \quad \therefore \mathbb{R}$

$e^{x} > 0$; عدد موجب

لانه دائما العدد الي داخل الln رح يكون أكبر من الصفر لو عوضت بأي قيمة مكان الx عشان جذي الrange هو R

2. [10 pts.] Let $f(x) = \dfrac{2^x - 3}{2^x - 1}$. Find $f^{-1}(x)$.

2. [10 pts.] Let $f(x) = \dfrac{2^x - 3}{2^x - 1}$. Find $f^{-1}(x)$.

(1) Let $y = \dfrac{2^x - 3}{2^x - 1}$.

(2) Solve for x in terms of y: $\boxed{x = \log_2\left(\dfrac{y-3}{y-1}\right)}$.

(3) Interchange x and y: $y = \log_2\left(\dfrac{x-3}{x-1}\right)$. Therefore, $\boxed{f^{-1}(x) = \log_2\left(\dfrac{x-3}{x-1}\right)}$.

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3. [10 + 5 = 15 pts.] Let $f(x) = \ln\left(\dfrac{2}{x} + 3\right)$.

(a) Find the inverse function of f.

(b) Find the range of f.

We let $y = \ln\left(\dfrac{2}{x} + 3\right)$, then $e^y = \dfrac{2}{x} + 3$. It is clear that $\dfrac{2}{e^y - 3} = x$. Therefore, $f^{-1}(x) = \dfrac{2}{e^x - 3}$. Now the range of the function f is the domain of the inverse function that is $\mathbb{R} \setminus \{\ln 3\}$.

$$y = \ln\left(\frac{2}{x} + 3\right) \Rightarrow e^y = e^{\ln\left(\frac{2}{x}+3\right)}$$

$$e^y = \frac{2}{x} + 3 \Rightarrow e^y - 3 = \frac{2}{x}$$

$$x(e^y - 3) = 2 \Rightarrow x = \frac{2}{e^y - 3} \Rightarrow f^{-1}(x) = \frac{2}{e^x - 3}$$

$$D_{f^{-1}(x)} = \text{Range of } f(x) = e^x - 3 \neq 0 \Rightarrow e^x \neq 3$$

$$\Rightarrow x \neq \ln 3 \quad \therefore \mathbb{R}/\{\ln 3\}$$

Sections 1.2, 1.4-1.5

Question 1. For each of the following functions, find (1) $\mathcal{D}_f$, (2) $f^{-1}(x)$ and (3) $\mathcal{R}_f$:

(a) $f(x) = \ln(2x+3)$.

(b) $f(x) = \ln(2e^x - 1)$.

(c) $f(x) = 4 + e^{x-3}$.

(d) $f(x) = \frac{x-1}{x+1}$

a) $D_f : (-\frac{3}{2}, \infty)$ $f^{-1}(x) : \frac{e^x - 3}{2}$ $R_f : (-\infty, \infty)$

b) $D_f : (\ln(\frac{1}{2}), \infty)$ $f^{-1}(x) : \ln(\frac{e^x + 1}{2})$ $R_f : (-\infty, \infty)$

c) $D_f : (-\infty, \infty)$ $f^{-1}(x) : \ln(x-4) + 3$ $R_f : (4, \infty)$

d) $D_f : \mathbb{R} / \{-1\}$ $f^{-1}(x) : \frac{x+1}{1-x}$ $R_f : \mathbb{R} / \{1\}$

or $(-\infty, -1) \cup (-1, \infty)$

Calculus A

Chapter 2: Limits & Derivatives

Sections: 2.2 The Limit of a Function

Review

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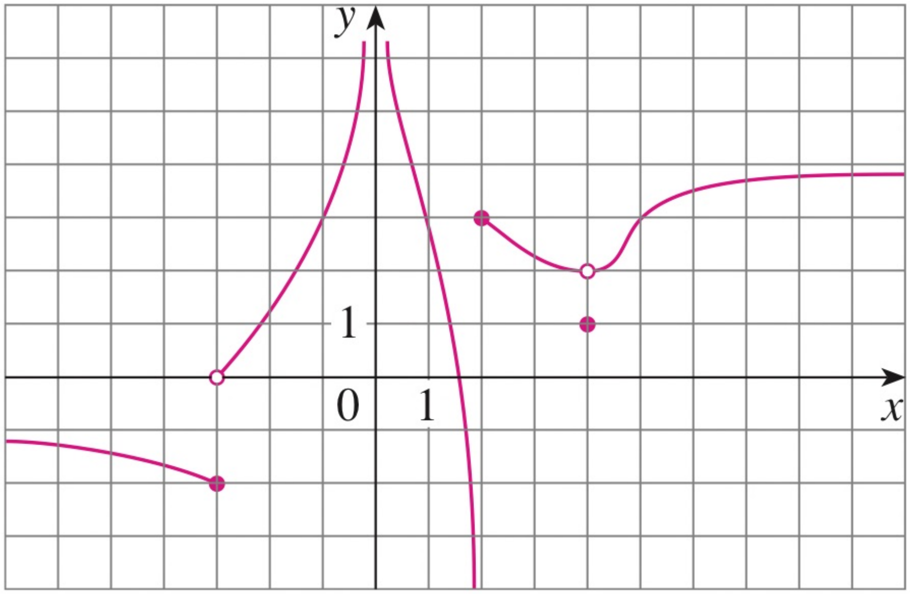
Chapter 2

* use the graph to show the limit
* evaluate the limit
* find the values that make function continuous
* IVT theorem
* find the HA & VA
* use the definition of derivative
* find & classy discontinuous

1. The graph of f is given.



(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x \to 2^+} f(x)$ (ii) $\lim_{x \to -3^+} f(x)$ (iii) $\lim_{x \to -3} f(x)$

(iv) $\lim_{x \to 4} f(x)$ (v) $\lim_{x \to 0} f(x)$ (vi) $\lim_{x \to 2^-} f(x)$

(c) State the equations of the vertical asymptotes.

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1. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2
(v) ∞ (vi) $-\infty$

(c) $x = 0, x = 2$

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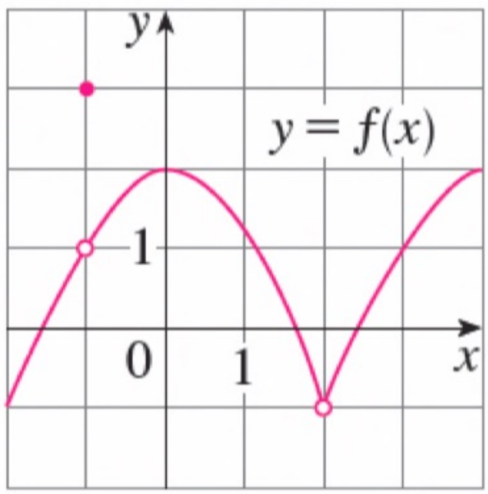
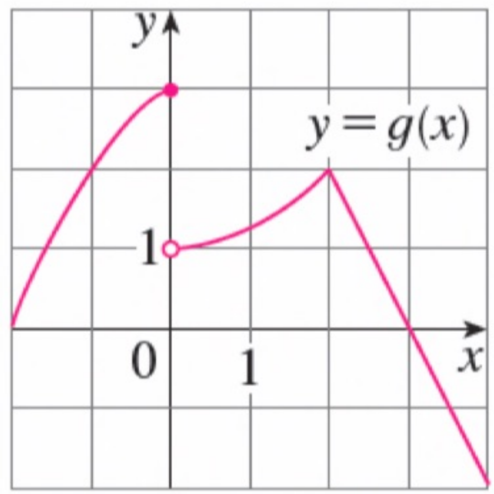
1. [5 + 5 + 10 = 20 pts.] Use the given graphs of f and g to find each of the following limits, if it exists.





(a) $\lim_{x \to 2} [f(x) + g(x)] =$

(b) $\lim_{x \to 3} \frac{g(x)}{f(x)} =$

(c) $\lim_{x \to 0} [f(x)g(x)] =$

(a) $\lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = -1 + 2 = 1.$

(b) $\lim_{x \to 3} \frac{g(x)}{f(x)} = \frac{\lim_{x \to 3} g(x)}{\lim_{x \to 3} f(x)} = \frac{0}{1} = 0.$

(c) $\lim_{x \to 0} [f(x)g(x)] =$ DNE. (2 pts)

Since $\lim_{x \to 0^+} [f(x)g(x)] = \lim_{x \to 0^+} f(x) \lim_{x \to 0^+} g(x) = (2)(1) = 2$ (4 pts)
and $\lim_{x \to 0^-} [f(x)g(x)] = \lim_{x \to 0^-} f(x) \lim_{x \to 0^-} g(x) = (2)(3) = 6.$ (4 pts)

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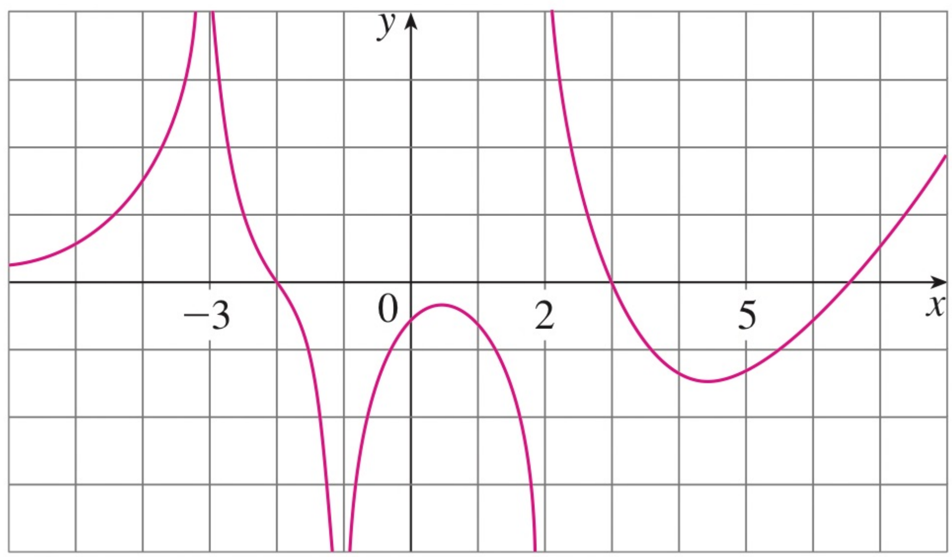
8. For the function A whose graph is shown, state the following.

(a) $\lim_{x \to -3} A(x)$ (b) $\lim_{x \to 2^-} A(x)$

(c) $\lim_{x \to 2^+} A(x)$ (d) $\lim_{x \to -1} A(x)$

(e) The equations of the vertical asymptotes



a) $\lim_{x \to -3} A(x) = +\infty$

b) $\lim_{x \to 2^-} A(x) = -\infty$

c) $\lim_{x \to 2^+} A(x) = +\infty$

d) $\lim_{x \to -1} A(x) = -\infty$

e) $x = -3$, $x = -1$, $x = 2$

3–20 Find the limit.

3. $\lim_{x \to 1} e^{x^3 - x}$

$$\lim_{x \to 1} e^{x^3 - x} = e^{(1)^3 - 1} = e^0 = 1$$

4. $\lim_{x \to 3} \frac{x^2 - 9}{x^2 + 2x - 3}$

$$\lim_{x \to 3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{3^2 - 9}{3^2 + 2(3) - 3} = \frac{0}{12} = 0$$

5. $\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3}$

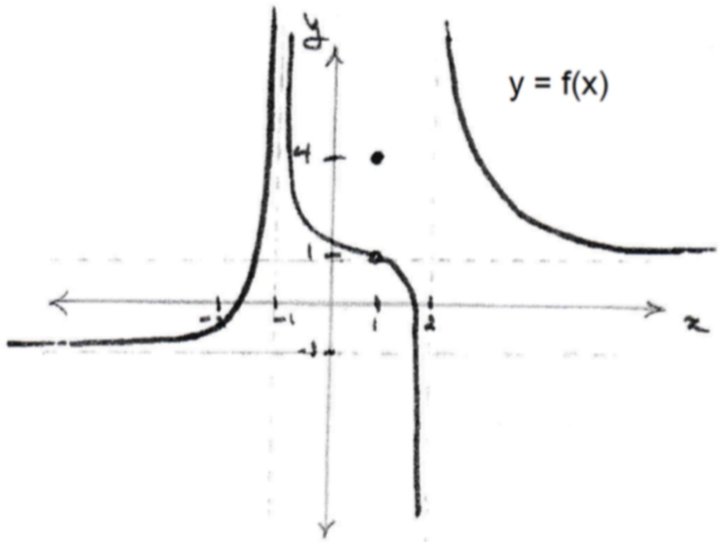
$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{(-3)^2 - 9}{(-3)^2 + 2(-3) - 3} = \frac{9 - 9}{9 - 6 - 3} = \frac{0}{0}$$

قيمة غير معرفة

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \to -3} \frac{(x-3)(x+3)}{(x+3)(x-1)} = \lim_{x \to -3} \frac{x-3}{x-1}$$

$$= \frac{-3 - 3}{-3 - 1} = \frac{-6}{-4} = \frac{3}{2}$$

Question 4. The graph of the function $f(x)$ is shown below. Use the graph to evaluate the following limits:



(a) $\lim_{x \to 1} f(x) =$ 1

(b) $\lim_{x \to 2^-} f(x) =$ $-\infty$

(c) $\lim_{x \to 2^+} f(x) =$ ∞

(d) $\lim_{x \to -1} f(x) =$ ∞

Calculus A

Chapter 2: Limits & Derivatives

Sections: 2.3 Calculating Limits Using the Limits Laws

Review

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6. [8+7 = 15 pts.] Evaluate each of the following limits, if it exists.

(a) $\lim_{x\to 2} \dfrac{x^2-4}{\sqrt{x+2}-2}$.

$$= \frac{2^2-4}{\sqrt{2+2}-2} = \frac{4-4}{2-2} = \frac{0}{0}$$

$$\lim_{x\to 2} \frac{x^2-4}{\sqrt{x+2}-2} * \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$$

$$\lim_{x\to 2} \frac{(x^2-4)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}$$

$$\lim_{x\to 2} \frac{(x^2-4)(\sqrt{x+2}+2)}{x+2-4} = \lim_{x\to 2} \frac{(x^2-4)(\sqrt{x+2}+2)}{x-2}$$

$$\lim_{x\to 2} \frac{(x-2)(x+2)(\sqrt{x+2}+2)}{x-2} = \lim_{x\to 2} (x+2)(\sqrt{x+2}+2)$$

$$= (2+2)(\sqrt{2+2}+2) = 4(2+2) = 4(4) = 16$$

(a) $\lim_{x\to 2} \dfrac{x^2-4}{\sqrt{x+2}-2} = \lim_{x\to 2} \dfrac{(x^2-4)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}$

$= \lim_{x\to 2} \dfrac{(x-2)(x+2)(\sqrt{x+2}+2)}{x-2} = \lim_{x\to 2} (x+2)(\sqrt{x+2}+2) = 16$.

III. $\lim_{x \to 2} \frac{x-2}{\sqrt{2x-3}-1} =$

a) 0.

b) 1.

c) 2.

d) Does not exist.

e) None of the above.

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2. [10 + 10 = 20 pts.] Evaluate each of the following limits, if it exists.

(a) $\lim\limits_{x \to 0^+} \tan^{-1}\left(\frac{\pi}{x}\right)$

(a) $\lim\limits_{x \to 0^+} \tan^{-1}\left(\frac{\pi}{x}\right) = \boxed{\frac{\pi}{2}}$.

(b) $\lim\limits_{x \to 1} \sin^{-1}\left(\frac{x-1}{x^2-1}\right) =$

(b) $\lim\limits_{x \to 1} \sin^{-1}\left(\frac{x-1}{x^2-1}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$.

[Since $\sin^{-1}$ is continuous on its domain and $\lim\limits_{x \to 1}\left(\frac{x-1}{x^2-1}\right) = \lim\limits_{x \to 1}\left(\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}\right) = \lim\limits_{x \to 1}\left(\frac{1}{x+1}\right) = 1/2$].

6. [8+7 = 15 pts.] Evaluate each of the following limits, if it exists.

(b) $\lim_{x\to 0} x^2 e^{\sin\left(\frac{\pi}{x}\right)}$.

$= 0\, e^{\sin\left(\frac{\pi}{0}\right)} = 0\, e^{\sin \infty}$ $\therefore$ squeeze theorem

$-1 \leqslant \sin\left(\frac{\pi}{x}\right) \leqslant 1$

$e^{-1} \leqslant e^{\sin\left(\frac{\pi}{x}\right)} \leqslant e$

$x^2 e^{-1} \leqslant x^2 e^{\sin\left(\frac{\pi}{x}\right)} \leqslant x^2 e$

$\lim_{x\to 0} x^2 e^{-1} \leqslant \lim_{x\to 0} x^2 e^{\sin\left(\frac{\pi}{x}\right)} \leqslant \lim_{x\to 0} x^2 e$

$\because \lim_{x\to 0} x^2 e^{-1} = 0 * \frac{1}{e} = 0$

$\because \lim_{x\to 0} x^2 e = 0 * e = 0$

$\therefore$ By S.T $\lim_{x\to 0} x^2 e^{\sin\left(\frac{\pi}{x}\right)} = 0$

Since $-1 \leqslant \sin\left(\frac{\pi}{x}\right) \leqslant 1$, it follows that $e^{-1} \leqslant e^{\sin\left(\frac{\pi}{x}\right)} \leqslant e$. Hence, $x^2 e^{-1} \leqslant x^2 e^{\sin\left(\frac{\pi}{x}\right)} \leqslant x^2 e$.

Since $\lim_{x\to 0} x^2 e^{-1} = \lim_{x\to 0} x^2 e = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} x^2 e^{\sin\left(\frac{\pi}{x}\right)} = 0$.

حل النمو ذجي

3–20 Find the limit.

7. $\lim_{h\to 0} \frac{(h-1)^3+1}{h}$ $= \frac{0}{0}$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ تذكر

$\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \lim_{h\to 0} \frac{h^3 - 3h^2(1) + 3(1)^2h - (1)^3 + 1}{h}$

$\lim_{h\to 0} = \frac{h^3 - 3h^2 + 3h - 1 + 1}{h} = \frac{h^3 - 3h^2 + 3h}{h}$

$\lim_{h\to 0} \frac{h(h^2 - 3h + 3)}{h} = \lim_{h\to 0} h^2 - 3h + 3$

$= 0 - 3(0) + 3 = 3$

3–20 Find the limit.

20. $\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$ $=\frac{1}{0}+\frac{1}{0}=\infty+\infty$

$$\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}\right)$$

$$\lim_{x\to 1}\left(\frac{1(x-2)}{(x-1)(x-2)}+\frac{1}{(x-1)(x-2)}\right)$$

$$\lim_{x\to 1}\left(\frac{(x-2)}{(x-1)(x-2)}+\frac{1}{(x-1)(x-2)}\right)$$

لاحظ بين x-2 والواحد جمع مو تنسى عبالك ضرب

$$\lim_{x\to 1}\left(\frac{(x-2)+1}{(x-1)(x-2)}\right)=\lim_{x\to 1}\left(\frac{x-1}{(x-1)(x-2)}\right)$$

$$\lim_{x\to 1}\frac{1}{x-2}=\frac{1}{1-2}=\frac{1}{-1}=-1$$

10. $\lim_{v \to 4^+} \frac{4 - v}{|4 - v|}$

$$\frac{4 - 4}{|4 - 4|} = \frac{0}{0}$$

$$|4 - v| \begin{cases} 4 - v, & 4 \geq v \\ -(4 - v), & 4 < v \end{cases}$$

$4 < v = v > 4$ لأنّ بيجيها من جهة اليمين

نفس الشي .. لا تتخربط إذا تغيّرت أماكنهم .. بالنهاية أثنينهم يقولولي إن الـv أكبر من 4

$$\therefore \lim_{x \to 4^+} \frac{4 - v}{|4 - v|} = \lim_{x \to 4^+} \frac{4 - v}{-(4 - v)}$$

$$\lim_{x \to 4^+} \frac{1}{-1} = -1$$

5. [10 pts.] Evaluate the given limit, if it exists.

(b) $\lim_{x\to 1} |x-1| \sin\left(\frac{1}{x^2-1}\right)$

(b) We have $-1 \leqslant \sin\left(\frac{1}{x^2-1}\right) \leqslant 1$. Thus, we obtain $-|x-1| \leqslant \sin\left(\frac{1}{x^2-1}\right) \leqslant |x-1|$. It is easy to see that $\lim_{x\to 1} -|x-1| = 0 = \lim_{x\to 1} |x-1|$. Therefore, by Squeeze Theorem $\lim_{x\to 1} |x-1| \sin\left(\frac{1}{x^2-1}\right) = 0$.

3–20 Find the limit.

8. $\lim_{t \to 2} \dfrac{t^2 - 4}{t^3 - 8}$ $= \dfrac{4-4}{8-8} = \dfrac{0}{0}$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$ تذكر

$$\lim_{t \to 2} \frac{t^2-4}{t^3-8} = \lim_{t \to 2} \frac{(t-2)(t+2)}{(t-2)(t^2+2t+4)}$$

$$= \lim_{t \to 2} \frac{t+2}{t^2+2t+4} = \frac{2+2}{2^2+2(2)+4}$$

$$= \frac{4}{12} = \frac{1}{3}$$

Sections 2.2, 2.3

Question 1. Evaluate each of the following limits, if it exists.

(a) $\lim_{x \to -2} \dfrac{3x^2 + 5x - 2}{x - 2}$.

(b) $\lim_{x \to -2} \dfrac{3x^2 + 5x - 2}{x^2 - 3x - 10}$.

(c) $\lim_{t \to 2} \dfrac{t^2 - 4}{t^3 - 8}$.

(d) $\lim_{t \to 2} \dfrac{t^4 - 4}{t^2 - 4}$.

(e) $\lim_{x \to 1} \cos^{-1}\left(\dfrac{x - 1}{x^2 + 1}\right)$.

(f) $\lim_{x \to 0} \dfrac{4^x - 1}{2^x - 1}$.

(g) $\lim_{x \to 0} \dfrac{e^{2x} - 1}{e^x - 1}$.

(h) $\lim_{h \to 0} \dfrac{(h - 4)^2 - 16}{h}$.

Question 2. Evaluate the following limits:

(a) $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2 + x}\right)$

(b) $\lim_{x \to 4} \dfrac{2 - \sqrt{x}}{4x - x^2}$

(c) $\lim_{x \to 2} \dfrac{\sqrt{4x + 1} - 3}{x - 2}$

Question 3. Evaluate the following limits:

(a) $\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right)$

(b) $\lim_{x \to 0} x e^{\sin\left(\frac{1}{x}\right)}$

(c) $\lim_{x \to 1} (x - 1)^4 \cos\left(\dfrac{2}{x - 1}\right)$

(d) $\lim_{x \to 1} (x - 1)^2 \sin\left(\dfrac{\pi}{x - 1}\right)$

Sections 2.2, 2.3

Question 1. Evaluate each of the following limits, if it exists.

(a) $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x - 2}$.

(b) $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x^2 - 3x - 10}$.

(c) $\lim_{t \to 2} \frac{t^2 - 4}{t^3 - 8}$.

a) $\frac{3(-2)^2 + 5(-2) - 2}{-2 - 2} = \frac{12 - 10 - 2}{-4} = \frac{0}{-4} = 0$

b) $\frac{3(-2)^2 + 5(-2) - 2}{(-2)^2 - 3(-2) - 10} = \frac{12 - 10 - 2}{4 + 6 - 10} = \frac{0}{0}$

$\lim_{x \to -2} \frac{(3x-1)(x+2)}{(x-5)(x+2)} = \lim_{x \to -2} \frac{3x-1}{x-5} = \frac{-7}{-7} = 1$

c) $\frac{4 - 4}{8 - 8} = \frac{0}{0}$

$\lim_{t \to 2} \frac{t^2 - 4}{t^3 - 8} = \lim_{t \to 2} \frac{(t-2)(t+2)}{(t-2)(t^2+2t+4)}$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$ ﴾ت﴿

$= \lim_{t \to 2} \frac{t+2}{t^2+2t+4} = \frac{2+2}{2^2 + 2(2) + 4} = \frac{4}{12} = \frac{1}{3}$

(d) $\lim_{t\to 2} \frac{t^4-4}{t^2-4}$.

(e) $\lim_{x\to 1} \cos^{-1}\left(\frac{x-1}{x^2+1}\right)$.

(f) $\lim_{x\to 0} \frac{4^x-1}{2^x-1}$.

(g) $\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1}$.

(h) $\lim_{h\to 0} \frac{(h-4)^2-16}{h}$.

---

d) $= \frac{2^4-4}{4-4} = \frac{16-4}{4-4} = \frac{12}{0} = \infty$

e) $\cos^{-1}\left(\frac{1-1}{1+1}\right) = \cos^{-1}\left(\frac{0}{2}\right) = \cos^{-1} 0 = \frac{\pi}{2}$

f) $\frac{4^0-1}{2^0-1} = \frac{1-1}{1-1} = \frac{0}{0}$

$\lim_{x\to 0} \frac{(2^x-1)(2^x+1)}{2^x-1} = \lim_{x\to 0} 2^x+1 = 2$

g) $\frac{e^{2(0)}-1}{e^0-1} = \frac{1-1}{1-1} = \frac{0}{0}$

$\lim_{x\to 0} \frac{(e^x-1)(e^x+1)}{(e^x-1)} = \lim_{x\to 0} e^x+1 = 2$

h) $\frac{0}{0} \Rightarrow \lim_{h\to 0} \frac{h^2-8h+16-16}{h} = \lim_{h\to 0} \frac{h(h-8)}{h}$

$\lim_{h\to 0}(h-8) = -8$

Question 2. Evaluate the following limits:

(a) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$

(b) $\lim_{x\to 4}\frac{2-\sqrt{x}}{4x-x^2}$

(c) $\lim_{x\to 2}\frac{\sqrt{4x+1}-3}{x-2}$

---

a) $\Rightarrow \frac{0}{0} \Rightarrow \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x(x+1)}\right) = \lim_{x\to 0}\left(\frac{1(x+1)}{x(x+1)}-\frac{1}{x(x+1)}\right)$

$\lim_{x\to 0}\frac{x+1-1}{x(x+1)} = \lim_{x\to 0}\frac{x}{x(x+1)} = \lim_{x\to 0}\frac{1}{x+1}$

$\frac{1}{0+1} = 1$

b) $\frac{0}{0} \Rightarrow \lim_{x\to 4}\frac{2-\sqrt{x}}{4x-x^2} * \frac{2+\sqrt{x}}{2+\sqrt{x}} = \lim_{x\to 4}\frac{4-x}{(4x-x^2)(2+\sqrt{x})} = \frac{4-x}{x(4-x)(2+\sqrt{x})}$

$\lim_{x\to 4}\frac{1}{x(2+\sqrt{x})} = \frac{1}{4(2+\sqrt{4})} = \frac{1}{4(4)} = \frac{1}{16}$

c) $\lim_{x\to 2}\frac{\sqrt{4x+1}-3}{x-2} * \frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3}$

$\lim_{x\to 2}\frac{4x+1-9}{(x-2)(\sqrt{4x+1}+3)} = \frac{4x-8}{(x-2)(\sqrt{4x+1}+3)} = \frac{4(x-2)}{(x-2)(\sqrt{4x+1}+3)}$

$\lim_{x\to 2}\frac{4}{(\sqrt{4x+1}+3)} = \frac{4}{\sqrt{9}+3} = \frac{4}{3+3} = \frac{4}{6} = \frac{2}{3}$

Question 3. Evaluate the following limits:

(a) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

(b) $\lim_{x\to 0} x e^{\sin\left(\frac{1}{x}\right)}$ $\Rightarrow$ $xe^{-1} \leq xe^{\sin(\frac{1}{x})} \leq xe = 0$

(c) $\lim_{x\to 1} (x-1)^4 \cos\left(\frac{2}{x-1}\right)$ $= 0$ by S.T

(d) $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{\pi}{x-1}\right)$ $= 0$ by S.T

حاول تحلهم

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a) $0 \cos\left(\frac{1}{0}\right) = 0 \cos \infty$ $\therefore$ S.T

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad * x^2$$

$$-x^2 \leq \cos\left(\frac{1}{x}\right) \leq x^2$$

$$\because \lim_{x\to 0} -x^2 = 0$$

$$\because \lim_{x\to 0} x^2 = 0$$

$$\therefore \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

Sections 2.2, 2.3

Question 1. Evaluate each of the following limits, if it exists.

(a) $\lim_{x\to -2} \frac{3x^2+5x-2}{x-2}$.

(b) $\lim_{x\to -2} \frac{3x^2+5x-2}{x^2-3x-10}$.

(c) $\lim_{t\to 2} \frac{t^2-4}{t^3-8}$.

---

a) $\frac{3(-2)^2+5(-2)-2}{-2-2} = \frac{12-10-2}{-4} = \frac{0}{-4} = 0$

b) $\frac{3(-2)^2+5(-2)-2}{(-2)^2-3(-2)-10} = \frac{12-10-2}{4+6-10} = \frac{0}{0}$

$$\lim_{x\to -2} \frac{(3x-1)(x+2)}{(x-5)(x+2)} = \lim_{x\to -2} \frac{3x-1}{x-5} = \frac{-7}{-7} = 1$$

c) $\frac{4-4}{8-8} = \frac{0}{0}$

$$\lim_{t\to 2} \frac{t^2-4}{t^3-8} = \lim_{t\to 2} \frac{(t-2)(t+2)}{(t-2)(t^2+2t+4)}$$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$= \lim_{t\to 2} \frac{t+2}{t^2+2t+4} = \frac{2+2}{2^2+2(2)+4} = \frac{4}{12} = \frac{1}{3}$$

(d) $\lim_{t\to 2} \dfrac{t^4-4}{t^2-4}$.

(e) $\lim_{x\to 1} \cos^{-1}\left(\dfrac{x-1}{x^2+1}\right)$.

(f) $\lim_{x\to 0} \dfrac{4^x-1}{2^x-1}$.

(g) $\lim_{x\to 0} \dfrac{e^{2x}-1}{e^x-1}$.

(h) $\lim_{h\to 0} \dfrac{(h-4)^2-16}{h}$.

---

d) $= \dfrac{2^4-4}{4-4} = \dfrac{16-4}{4-4} = \dfrac{12}{0} = \infty$

e) $\cos^{-1}\left(\dfrac{1-1}{1+1}\right) = \cos^{-1}\left(\dfrac{0}{2}\right) = \cos^{-1} 0 = \dfrac{\pi}{2}$

f) $\dfrac{4^0-1}{2^0-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$

$\lim_{x\to 0} \dfrac{(2^x-1)(2^x+1)}{2^x-1} = \lim_{x\to 0} 2^x+1 = 2$

g) $\dfrac{e^{2(0)}-1}{e^0-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$

$\lim_{x\to 0} \dfrac{(e^x-1)(e^x+1)}{(e^x-1)} = \lim_{x\to 0} e^x+1 = 2$

h) $\dfrac{0}{0} \Rightarrow \lim_{h\to 0} \dfrac{h^2-8h+16-16}{h} = \lim_{h\to 0} \dfrac{h(h-8)}{h}$

$\lim_{h\to 0}(h-8) = -8$

Question 2. Evaluate the following limits:

(a) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2 + x} \right)$

(b) $\lim_{x \to 4} \frac{2 - \sqrt{x}}{4x - x^2}$

(c) $\lim_{x \to 2} \frac{\sqrt{4x+1} - 3}{x - 2}$

a) $\Rightarrow \frac{0}{0} \Rightarrow \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x(x+1)} \right) = \lim_{x \to 0} \left(\frac{1(x+1)}{x(x+1)} - \frac{1}{x(x+1)} \right)$

$\lim_{x \to 0} \frac{x+1-1}{x(x+1)} = \lim_{x \to 0} \frac{x}{x(x+1)} = \lim_{x \to 0} \frac{1}{x+1}$

$\frac{1}{0+1} = 1$

b) $\frac{0}{0} \Rightarrow \lim_{x \to 4} \frac{2-\sqrt{x}}{4x-x^2} * \frac{2+\sqrt{x}}{2+\sqrt{x}} = \lim_{x \to 4} \frac{4-x}{(4x-x^2)(2+\sqrt{x})} = \frac{4-x}{x(4-x)(2+\sqrt{x})}$

$\lim_{x \to 4} \frac{1}{x(2+\sqrt{x})} = \frac{1}{4(2+\sqrt{4})} = \frac{1}{4(4)} = \frac{1}{16}$

c) $\lim_{x \to 2} \frac{\sqrt{4x+1} - 3}{x - 2} * \frac{\sqrt{4x+1} + 3}{\sqrt{4x+1} + 3}$

$\lim_{x \to 2} \frac{4x+1-9}{(x-2)(\sqrt{4x+1}+3)} = \frac{4x-8}{(x-2)(\sqrt{4x+1}+3)} = \frac{4(x-2)}{(x-2)(\sqrt{4x+1}+3)}$

$\lim_{x \to 2} \frac{4}{(\sqrt{4x+1}+3)} = \frac{4}{\sqrt{9}+3} = \frac{4}{3+3} = \frac{4}{6} = \frac{2}{3}$

Question 3. Evaluate the following limits:

(a) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

(b) $\lim_{x\to 0} xe^{\sin\left(\frac{1}{x}\right)}$ $\Rightarrow$ $xe^{-1} \leq xe^{\sin(\frac{1}{x})} \leq xe = 0$

(c) $\lim_{x\to 1} (x-1)^4 \cos\left(\frac{2}{x-1}\right)$ $= 0$ by S.T

(d) $\lim_{x\to 1} (x-1)^2 \sin\left(\frac{\pi}{x-1}\right)$ $= 0$ by S.T

حاول تحلهم

a) $0 \cos\left(\frac{1}{0}\right) = 0 \cos \infty$ $\therefore$ S.T

$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$ $\quad * x^2$

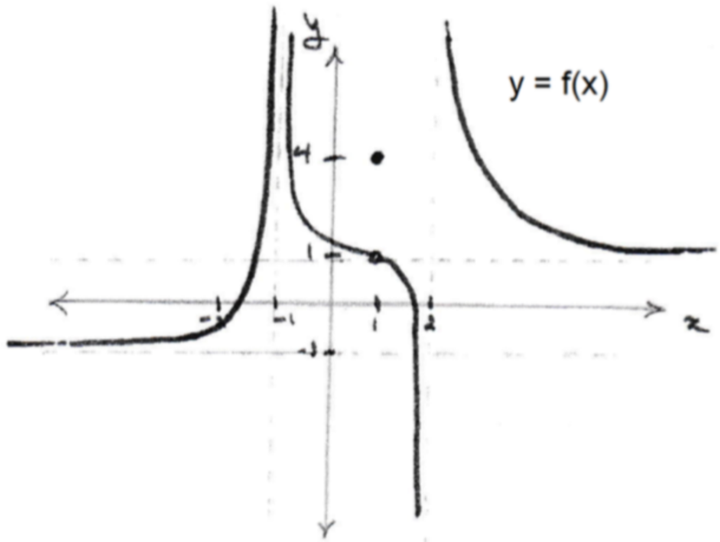
$-x^2 \leq \cos\left(\frac{1}{x}\right) \leq x^2$

$\because \lim_{x\to 0} -x^2 = 0$

$\because \lim_{x\to 0} x^2 = 0$

$\therefore \lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

Question 4. The graph of the function $f(x)$ is shown below. Use the graph to evaluate the following limits:



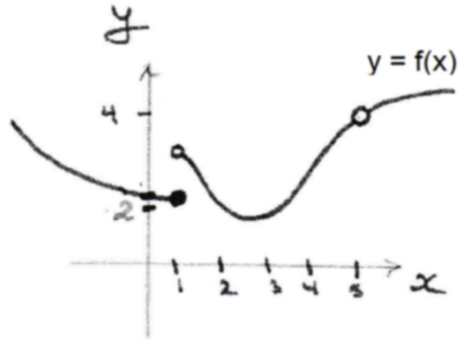
(a) $\lim_{x \to 1} f(x) =$ 1

(b) $\lim_{x \to 2^-} f(x) =$ $-\infty$

(c) $\lim_{x \to 2^+} f(x) =$ ∞

(d) $\lim_{x \to -1} f(x) =$ ∞

Question 5. The graph of the function $f(x)$ is shown below. Use the graph to evaluate the following limits:



(a) $\lim_{x\to 1^+} f(x) =$ 3

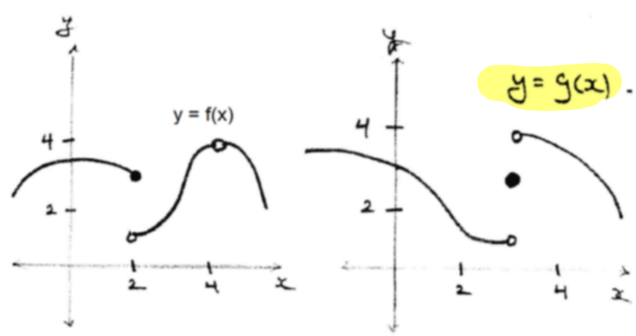
(b) $\lim_{x\to 1^-} f(x) =$ 2

(c) $\lim_{x\to 1} f(x) =$ DNE

(d) $\lim_{x\to 5^+} f(x) =$ 4

(e) $\lim_{x\to 5} f(x) =$ 4

Question 6. The graph of the function $f(x)$ is shown below. Use the graph to evaluate the following limits:



(a) $\lim_{x\to 4} f(x) =$ 4

(b) $\lim_{x\to 3} g(x) =$ DNE

(c) $\lim_{x\to 3^+} f(x)g(x) = \left(\lim_{x\to 3^+} f(x)\right)\left(\lim_{x\to 3^+} g(x)\right) = (2)(4) = 8$

(d) $\lim_{x\to 3^-} f(x)g(x) =$ → $(2)(1) = 2$

(e) $\lim_{x\to 3} f(x)g(x) =$

→ $(2)(\text{DNE}) = \text{DNE}$

Question 7. If

$$h(x) = \begin{cases} \sqrt{x-4}, & \text{if } x > 4, \\ 8 - 2x, & \text{if } x \le 4, \end{cases}$$

evaluate the followings:

(a) $f(4) =$ 0

(b) $\lim_{x \to 4^-} f(x) =$ 0

(c) $\lim_{x \to 4^+} f(x) =$ 0

(d) $\lim_{x \to 4} f(x) =$ 0

Question 8. If $f(x) = \dfrac{|x-1|}{x^2-1}$, evaluate the followings:

(a) $f(1) =$ $\frac{0}{0}$ = DNE

(b) $\lim_{x \to 1^-} f(x) =$ $-1/2$

(c) $\lim_{x \to 1^+} f(x) =$ $+1/2$

(d) $\lim_{x \to 1} f(x) =$ DNE, $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$

Calculus A

Chapter 2: Limits & Derivatives

Sections: 2.4 The Precise Definition of a Limit

Review

A+

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Prove the statement using the precise definition of a limit.

25. $\lim_{x \to 2} (14 - 5x) = 4$

$f(x) = 14 - 5x$
$L = 4$
$a = 2$

1) for any $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

$$|14 - 5x - 4| < \varepsilon$$

$$|10 - 5x| < \varepsilon \rightarrow |-5(x - 2)| < \varepsilon$$

$$\rightarrow 5|x - 2| < \varepsilon \rightarrow |x - 2| < \frac{\varepsilon}{5}$$

2) Let $0 < \delta < \frac{\varepsilon}{5}$

3) let $0 < |x - 2| < \delta \rightarrow 0 < |x - 2| < \frac{\varepsilon}{5}$

$$5|x - 2| < \varepsilon \rightarrow |5x - 10| < \varepsilon$$

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$$\rightarrow |10 - 5x| < \varepsilon \rightarrow |14 - 4 - 5x| < \varepsilon$$

$$\therefore \lim_{x \to 2} (14 - 5x) = 4$$

15–18 Prove the statement using the ε, δ definition of a limit

18. $\lim_{x \to -2} (3x + 5) = -1$

$f(x) = 3x+5$

$L = -1$

$a = -2$

1) for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

$$|3x+5+1| < \varepsilon$$

$$|3x+6| < \varepsilon \rightarrow |3(x+2)| < \varepsilon$$

$$\rightarrow 3|x+2| < \varepsilon \rightarrow |x+2| < \frac{\varepsilon}{3}$$

2) let $0 < |x+2| < \delta \rightarrow 0 < |x+2| < \varepsilon/3$

$$3|x+2| < \varepsilon \rightarrow |3(x+2)| < \varepsilon$$

$$\rightarrow |3x+6| < \varepsilon \rightarrow |3x+5+1| < \varepsilon$$

$$\therefore \lim_{x \to -2} (3x+5) = -1$$

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1. [10 pts.] Use the (ϵ, δ)–definition of the limit to show that $\lim_{x\to 1} f(x) = 3$, where $f(x) = 2x + 1$.

(a) Guessing a value for δ. Let ϵ be a given positive number. We want to find a number $\delta > 0$ such that $\boxed{\text{if } 0 < |x-1| < \delta \text{ then } |(2x+1)-3| < \epsilon}$. But $|(2x+1)-3| = |2x-2| = 2|x-1|$. Therefore, we want to find δ such that if $0 < |x-1| < \delta$ then $2|x-1| < \epsilon$. That is, if $0 < |x-1| < \delta$ then $|x-1| < \epsilon/2$. This suggests to take $\boxed{\delta \leqslant \epsilon/2}$.

(b) Showing that $\delta = \epsilon/2$ works. Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x-1| < \delta$ then

$$\boxed{|(2x+1)-3| = |2x-2| = 2|x-1| < 2\delta = \epsilon}.$$

Thus, if $0 < |x-1| < \delta$ then $|(2x+1)-3| < \epsilon$.

Therefore, by the definition of a limit $\lim_{x\to 1} f(x) = 3$

أو تقدر تحلها نفس طريقة حل مسألة الي فاتت

Calculus A

Chapter 2: Limits & Derivatives

Sections: 2.5 Continuity

Review

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8. [10+10 = 20 pts.] Use the graph of $y = f(x)$ shown below to answer the followings:

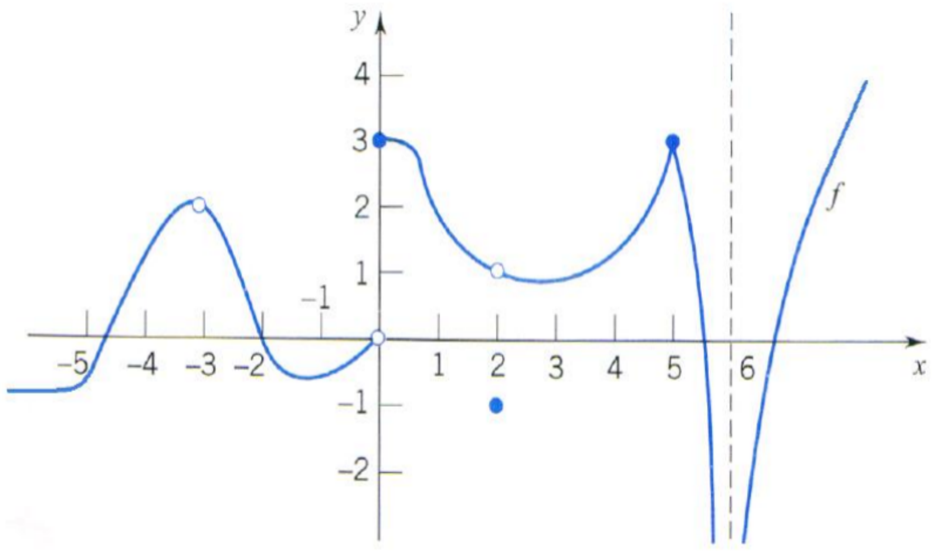

Figure 1: The graph of $y = f(x)$.

a) Find each of the following limits, if it exists:

i) $\lim_{x \to -3} f(x) =$

ii) $\lim_{x \to 0^-} f(x) =$

iii) $\lim_{x \to 0^+} f(x) =$

iv) $\lim_{x \to 2} f(x) =$

v) $\lim_{x \to 6} f(x) =$

Answers:

i) $\lim_{x \to -3} f(x) = \boxed{2}$

ii) $\lim_{x \to 0^-} f(x) = \boxed{0}$

iii) $\lim_{x \to 0^+} f(x) = \boxed{3}$

iv) $\lim_{x \to 2} f(x) = \boxed{1}$

v) $\lim_{x \to 6} f(x) = \boxed{-\infty}$

b) Find and classify the discontinuities of f as removable, infinite or jump.

f is discontinuous at $x = -3, 0, 2, 6$.

f has a removable discontinuity at at $x = -3$

f has a jump discontinuity at at $x = 0$

f has a removable discontinuity at at $x = 2$

f has an infinite discontinuity at at $x = 6$.

29. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leqslant x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x \to 0^+} f(x)$ (ii) $\lim_{x \to 0^-} f(x)$ (iii) $\lim_{x \to 0} f(x)$

(iv) $\lim_{x \to 3^-} f(x)$ (v) $\lim_{x \to 3^+} f(x)$ (vi) $\lim_{x \to 3} f(x)$

(b) Where is f discontinuous?

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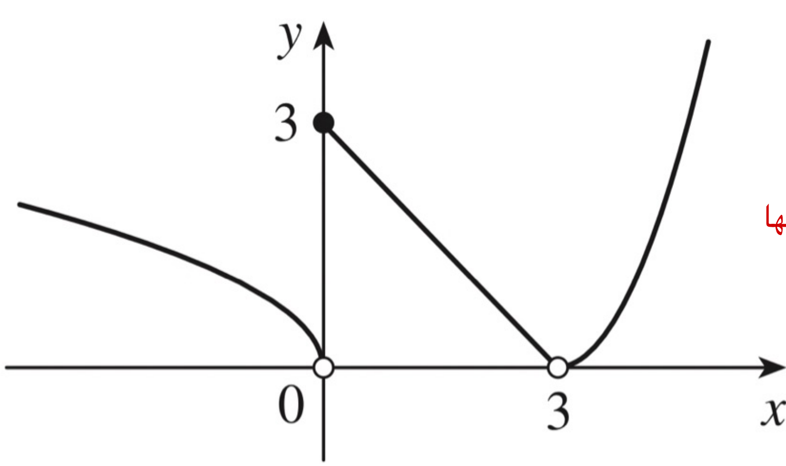
29. (a) (i) 3 (ii) 0 (iii) Does not exist

(iv) 0 (v) 0 (vi) 0

(b) At 0 and 3



مو مطلوب إنك ترسم بس أبيك تشوفها

@Precalculusq8

[20 pts.] Find nonzero values for the constants A and B, if any, that make the function f continuous at $x = 1$, where

$$f(x) = \begin{cases} \dfrac{A|x-1|}{x^2-5x+4} & \text{if } x < 1 \\ \dfrac{B}{4} & \text{if } x = 1 \\ \dfrac{x+\ln x}{x} & \text{if } x > 1. \end{cases}$$

for f to be continous at $x=1$, we must have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$$\lim_{x\to 1^-} \frac{-A(x-1)}{(x-4)(x-1)} = \lim_{x\to 1^-} \frac{-A}{(x-4)}$$

$$= \frac{-A}{-3} = \frac{A}{3}$$

$$\lim_{x\to 1^+} \frac{x+\ln x}{x} = \frac{1+0}{1} = 1$$

$$\therefore \frac{A}{3} = 1 \Rightarrow A = 3$$

$$\because \frac{B}{4} = 1 \Rightarrow B = 4$$

$$f(x)\begin{cases}\dfrac{8-2x^2}{2-x}, & x<2\\[2ex] \dfrac{B^{2x}-1}{B^x+1}, & x\geq 2\end{cases}$$

find B so that f is cont at $x=2$

$$\lim_{x\to 2^+}\frac{B^{2x}-1}{B^x+1}=\lim_{x\to 2^+}\frac{(B^x-1)\cancel{(B^x+1)}}{\cancel{B^x+1}}$$

$$\lim_{x\to 2^+} B^x-1 = B^2-1$$

تقدر تحلها بتعويض المباشر

$$\frac{B^4-1}{B^2+1}=\frac{(B^2-1)\cancel{(B^2+1)}}{\cancel{B^2+1}}$$

أول خطوة نحاول نعوض بتعويض المباشر وعطاني قيمة غير معرفة

$$\lim_{x\to 2^-}\frac{8-2x^2}{2-x}=\frac{2(4-x^2)}{2-x}=\frac{2\cancel{(2-x)}(2+x)}{\cancel{2-x}}$$

$$=\lim_{x\to 2^-} 2(2+x) = 8$$

$\because$ f is cont $\therefore \lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)$, $\therefore B^2-1=8$

$B^2=9$

$B=\pm 3$

$$f(x) = \begin{cases} \dfrac{\sqrt{6x-5} - \sqrt{3x-2}}{x^2+4x-5} & \text{if } x>1 \\ \dfrac{A}{4} & \text{if } x=1 \\ \dfrac{5|x-1|}{x^2-3x+2} + B & \text{if } x<1 \end{cases}$$

find A and B so that f is cont. at $x=1$

$A = 1$, $B = \dfrac{-19}{4}$

1. [10 pts.] Let $f(x) = \begin{cases} x, & \text{if } x < a, \\ x^2 - 2, & \text{if } x \geqslant a. \end{cases}$

Find all values of a, if any, for which f is continuous everywhere.

$\boxed{\text{The function } f \text{ is continuous for } x < a, \text{ and for } x > a}$. Now f is continuous at $x = a$ if $\boxed{\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)}$. That is; if $a = a^2 - 2$. Therefore, $\boxed{a = 2 \text{ or } a = -1}$.

---

6. [10 pts.] Use the Intermediate Value Theorem to show that the equation $\cos x - \ln(x+1) = 0$ has a real root.

$\boxed{\text{Let } f(x) = \cos x - \ln(x+1)}$.

The function f is continuous on its domain $(-1, \infty)$, hence its continuous on $[0, \pi]$. Since $f(0) = 1 > 0$ and $f(\pi) = -1 - \ln(\pi + 1) < 0$, therefore by the IVT there is a number $c \in (0, \pi)$ s.t. $f(c) = 0$. Hence, the given equation has a real root.

3. [10 pts.] Use the Intermediate Value Theorem to show that the equation $x^3 + x^2 - 2x - 1 = 0$ has a real root.

---

let $f(x) = x^3 + x^2 - 2x - 1 = 0$

Domain of f is $\mathbb{R}$ and Continuous

"it is polynomial"

$f(1) = (1)^3 + (1)^2 - 2(1) - 1 = -1 < 0$

$f(-1) = (-1)^3 + (-1)^2 - 2(-1) - 1 = 7 > 0$

f is cont on $(-1, 1)$

By I.V.T, There exist a "c" in $(-1, 1)$ such that $f(c) = 0$

$\therefore$ c is a root

Let $f(x) = x^3 + x^2 - 2x - 1$.

حل النموذجي

The function f is continuous everywhere (it is a polynomial). Since $f(1) = -1 < 0$ and $f(2) = 7 > 0$, therefore, by the IVT $\exists\, c \in (1, 2)$ s.t. $f(c) = 0$. Hence, the given equation has a real root.

Q6. [10 pts.] Let $f(x) = 2 + x\sin(x)$ and $g(x) = x^2$. Use the Intermediate Value Theorem to show that the graphs of f and g intersect at least once.

Let $h(x) = 2 + x\sin(x) - x^2$.

We have:

(i) $\boxed{h \text{ is continuous on } [0, \pi]}$ (since $\sin(x), x$ and x^2 are continuous everywhere).

(ii) $\boxed{h(0) = 2 + 0 - 0 = 2 > 0}$ and $\boxed{h(\pi) = 2 + 0 - \pi^2 < 0}$.

Hence by the IVT there is at least one $c \in (0, \pi)$ such that $\boxed{h(c) = f(c) - g(c) = 0}$. Thus, the graphs of f and g intersect at least once.

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7. [10 pts.] Use the Intermediate Value Theorem to show that the equation $e^x + \cos x = 4$ has a real root.

Let $\boxed{f(x) = e^x + \cos x - 4}$. The function f is continuous everywhere as e^x, $\cos(x)$ and 4 are continuous everywhere. Moreover, $\boxed{f(0) = 1 + 1 - 4 = -2 < 0}$ and $\boxed{f(2\pi) = e^{2\pi} + 1 - 4 > 0}$. Thereafter, the IVT implies that there exists at least one c in $(0, 2\pi)$ such that $f(c) = 0$. Thus, the given equation has a real root.

I. Suppose f and g are continuous functions. If $\lim_{x \to 3} (2f(x) + f(x)g(x)) = 20$ and $f(3) = 5$, then: $g(3) =$

$$2f(3) + f(3)g(3) = 20$$

$$\because f(3) = 5$$

$$2(5) + (5)g(3) = 20$$

$$\Rightarrow 10 + 5g(3) = 20$$

$$\Rightarrow 5g(3) = 10$$

$$g(3) = 2$$

5. [3+7=10 pts.] Let $f(x) = \dfrac{|x|}{x^2 - x}$.

(a) Find all points of discontinuity of f, if any.

$\boxed{f \text{ is discontinuous at } x = 0, 1}$.

(b) Classify each discontinuity as removable, jump, or infinite (justify your answers).

$$\lim_{x\to 0^\pm} f(x) = \lim_{x\to 0^\pm} \frac{|x|}{x^2 - x} = \lim_{x\to 0^\pm} \frac{\pm x}{x(x-1)} = \lim_{x\to 0^\pm} \frac{\pm 1}{(x-1)} = \mp 1$$

$\boxed{\text{Thus, } f \text{ has a jump discontinuity at } x = 0}$.

$$\lim_{x\to 1^\pm} f(x) = \lim_{x\to 1^\pm} \frac{|x|}{x^2 - x} = \pm\infty$$

$\boxed{\text{Thus, } f \text{ has an infinite discontinuity at } x = 1}$.

4. [10 pts.] Find a value for k, if any, such that $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{k}{x^2-1}\right)$ exists and is finite.

المطلوب من السؤال أنّ يبي يعرف. جم قيمة الk الي رح تخلي الدالة معرفة (يعني تحلها عادي بطرق حل اللمت وتشوف شنو ممكن تكون قيمة الk الي تخلي دالة معرفة

$$\frac{1}{1-1}-\frac{k}{1^2-1}=\frac{1}{0}-\frac{k}{0}=\infty-\infty$$

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{k}{(x-1)(x+1)}\right)$$

$$\lim_{x\to 1}\left(\frac{1(x+1)}{(x-1)(x+1)}-\frac{k}{(x-1)(x+1)}\right)$$

$$\lim_{x\to 1}\left(\frac{x+1-k}{(x-1)(x+1)}\right)$$

طبعا هني ما أقدر أشطب دايركت الx+1 مع الي تحت .. ليش ؟

let $k=2$

الحين قيمة الk المناسبة المفروض موجب ٢ .. لأن في هذي الحالة أقدر أشطب البسط مع المقام x-1

$$\lim_{x\to 1}\left(\frac{x+1-(2)}{(x-1)(x+1)}\right)=\lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)$$

$$\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}\quad,\ \therefore\ k=2$$

Calculus A

Chapter 2: Limits & Derivatives

Sections: 2.6 Limits at Infinity; Horizontal Asymptotes

Review

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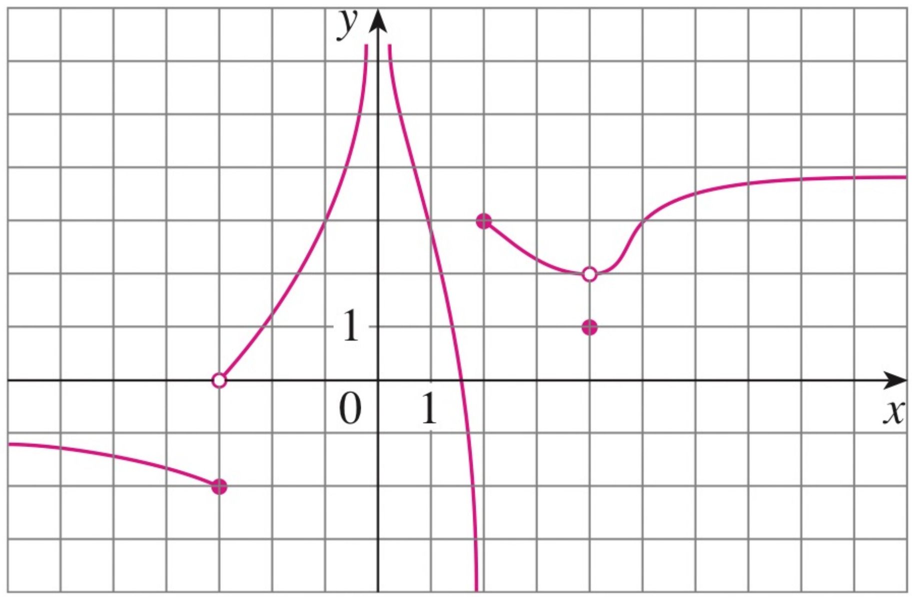
1. The graph of f is given.



(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x \to 2^+} f(x)$ (ii) $\lim_{x \to -3^+} f(x)$ (iii) $\lim_{x \to -3} f(x)$

(iv) $\lim_{x \to 4} f(x)$ (v) $\lim_{x \to 0} f(x)$ (vi) $\lim_{x \to 2^-} f(x)$

(vii) $\lim_{x \to \infty} f(x)$ (viii) $\lim_{x \to -\infty} f(x)$

(b) State the equations of the horizontal asymptotes.

(c) State the equations of the vertical asymptotes.

(d) At what numbers is f discontinuous? ~~Explain~~. Types

---

1. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2
(v) ∞ (vi) $-\infty$ (vii) 4 (viii) -1
(b) $y = 4$, $y = -1$ (c) $x = 0$, $x = 2$ (d) $-3, 0, 2, 4$

infinite discontinuity

removable discontinuity

Jump discontinuity

$\lim_{x \to 2^-} f(x) = -\infty$

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Find the limit.

17. $\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+4x+1}-x}{1}\cdot\frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}$$

$$\lim_{x\to\infty} \frac{x^2+4x+1-x^2}{\sqrt{x^2+4x+1}+x}$$

$$\lim_{x\to\infty} \frac{4x+1}{\sqrt{x^2+4x+1}+x}$$

$$\lim_{x\to\infty} \frac{x\left(\frac{4x}{x}+\frac{1}{x}\right)}{|x|\sqrt{\frac{x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}}+\frac{x}{x}}$$

$$|x|\begin{cases} \to x, & x\geq 0 \\ -x & x<0 \end{cases}$$

$$\lim_{x\to\infty} \frac{x\left(4+\frac{1}{x}\right)}{x\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1} = \frac{4+0}{\sqrt{1+0+0}+1}$$

$$= \frac{4}{2} = 2$$

28. $\lim_{x \to -\infty} \left(\sqrt{4x^2 + 3x} + 2x\right)$

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2+3x}+2x}{1} * \frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}$$

$$\lim_{x \to -\infty} \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x} = \lim_{x \to -\infty} \frac{3x}{\sqrt{4x^2+3x}-2x}$$

عامل مشترك x فلما آخذ عامل مشترك أقسمه على العناصر

$$\lim_{x \to -\infty} \frac{x(3)}{\sqrt{x^2\left(\frac{4x^2}{x^2}+\frac{3x}{x^2}\right)} - \frac{2x}{x}}$$

$$|x| \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\lim_{x \to -\infty} \frac{x(3)}{|x|\sqrt{4+\frac{3}{x}}-2} = \frac{x(3)}{-x\sqrt{4+\frac{3}{x}}-2}$$

$$\lim_{x \to -\infty} \frac{3}{-\sqrt{4+0}-2} = \frac{3}{-2-2} = \frac{3}{-4} = -\frac{3}{4}$$

Let $f(x) = \dfrac{3x^2|x-2|}{x^3-8}$

find the vertical and Horizontal asymptotes

$$f(x) = \frac{3x^2|x-2|}{(x-2)(x^2+2x+4)}$$

$\therefore\ x = 2$

$$|x-2| \begin{cases} \to x-2, & x \geq 2 \\ \to -(x-2), & x < 2 \end{cases}$$

$$\lim_{x \to 2^{\pm}} \frac{\pm 3x^2(x-2)}{(x-2)(x^2+2x+4)}$$

$$\lim_{x \to 2^{\pm}} \frac{\pm 3x^2}{(x^2+2x+4)}$$

$$= \frac{\pm 3(2)^2}{2^2+2(2)+4} = \frac{\pm 12}{4+4+4} = \frac{\pm 12}{12} = \pm 1$$

No V.A

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for H.A

$$\lim_{x \to \pm\infty} \frac{\pm 3x^2(x-2)}{(x-2)(x^2+2x+4)}$$

$$\lim_{x \to \pm\infty} \frac{\pm 3x^2}{(x^2+2x+4)} = \pm 3$$

$y = -3$ & $y = 3$ are H.A

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find the V.A and H.A if any

$$f(x) = \frac{5x^2+x}{|x|(x-3)}$$

---

for V.A

$x = 0$, $x = 3$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\lim_{x \to 0^{\pm}} \frac{x(5x+1)}{\pm x(x-3)} = \lim_{x \to 0} \frac{5x+1}{\pm(x-3)}$$

$$= \frac{0+1}{\pm(-3)} = \frac{1}{\mp 3} = \pm \frac{1}{3}$$ Not V.A

$$\lim_{x \to 3} \frac{x(5x+1)}{x(x-3)} = \lim_{x \to 3} \frac{5x+1}{x-3}$$

$$= \frac{16}{0} = \infty$$

$x = 3$ is V.A

for H.A

$$\lim_{x \to \pm\infty} \frac{x(5x+1)}{\pm x(x-3)} = \lim_{x \to \pm\infty} \frac{5x+1}{\pm(x-3)}$$

$$= \pm 5$$

$y = 5$ & $y = -5$ are H.A

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let $f(x) = \dfrac{x^3+x^2-2x}{|x|(x^2-1)}$

a) classify the discontinuity

b) find V.A & H.A if any

---

a) f is discontinuous at $x = 0, -1, 1$

for $x = 0$, $\lim\limits_{x \to 0^{\pm}} \dfrac{x^3+x^2-2x}{|x|(x^2-1)}$

$$\lim_{x \to 0^{\pm}} \frac{x(x^2+x-2)}{\pm x(x^2-1)} = \frac{x(x+2)(x-1)}{\pm x(x-1)(x+1)}$$

$$\lim_{x \to 0^{\pm}} \frac{(x+2)}{\pm(x+1)} = \frac{2}{\pm 1} = \pm 2$$

f has jump discontinuity at $x = 0$

لان لما أعوض الصفر من جهة اليمين رح يطلعلي موجب 2 و لما أعوض صفر من جهة اليسار يطلعلي سالب 2

$\Rightarrow$

at $x = 1$

for $x = 1$, $\lim_{x \to 1^{\pm}} \frac{x^3 + x^2 - 2x}{|x|(x^2-1)}$

$$\lim_{x \to 1^{\pm}} \frac{x(x^2+x-2)}{\pm x(x^2-1)} = \frac{x(x+2)(x-1)}{\pm x(x-1)(x+1)}$$

$$\lim_{x \to 1^{\pm}} \frac{(x+2)}{\pm(x+1)} = \frac{3}{\pm 2} = \pm \frac{3}{2}$$

f has jump discontinuity at $x = 1$

---

for $x = -1$, $\lim_{x \to -1^{\pm}} \frac{x^3 + x^2 - 2x}{|x|(x^2-1)}$

$$\lim_{x \to -1^{\pm}} \frac{x(x^2+x-2)}{\pm x(x^2-1)} = \frac{x(x+2)(x-1)}{\pm x(x-1)(x+1)}$$

$$\lim_{x \to -1^{\pm}} \frac{(x+2)}{\pm(x+1)} = \frac{1}{\pm 0} = \pm \infty$$

f has an infinite discontinuity at $x = -1$

for V.A

from part ⓐ we know

اختصرت الحل لانه حالينها
بالجزء الأول من الحل

$$\lim_{x \to -1^{\pm}} \frac{x^3+x^2-2x}{|x|(x^2-1)}$$

$$\lim_{x \to -1^{\pm}} \frac{(x+2)}{\pm(x+1)} = \frac{1}{\pm 0} = \pm\infty$$

$x = -1$ is V.A

---

for H.A

$$\lim_{x \to \pm\infty} \frac{x^3+x^2-2x}{|x|(x^2-1)} = \lim_{x \to \pm\infty} \frac{x^3+x^2-2x}{\pm x(x^2-1)}$$

$$\lim_{x \to \pm\infty} \pm \frac{x^3+x^2-2x}{x^3-x} = \lim_{x \to \pm\infty} \pm \frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}-\frac{2x}{x^3}}{\frac{x^3}{x^3}-\frac{x}{x^3}}$$

$$\lim_{x \to \pm\infty} \pm \frac{1+\frac{1}{x}-\frac{2}{x^2}}{1-\frac{1}{x^2}} = \pm\frac{1+0-0}{1-0} = \pm 1$$

$y = +1$, $y = -1$ are H.A

8. [10 + 10 = 20 pts.] Let $f(x) = \begin{cases} \sqrt{x^2+5}+x, & \text{if } x < 2 \\ \ln(x-2), & \text{if } x > 2. \end{cases}$

(a) Find all points of discontinuity of f, if any. Then classify each discontinuity as removable, jump, or infinite (justify your answers).

The function f is defined and continuous everywhere except at $x = 2$. Since

$$\lim_{x\to 2^+} \ln(x-2) = -\infty.$$

thus, f is discontinuous at $x = 2$ (f has an infinite discontinuity at $x = 2$) (full mark). (Note that $\lim_{x\to 2^-} \sqrt{x^2+5}+x = 5$.)

(b) Find the vertical and horizontal asymptotes of the graph of f, if any.

Vertical asymptotes: Since $\lim_{x\to 2^+} \ln(x-2) = -\infty$, thus the line $x = 2$ is a vertical asymptote.

Horizontal asymptotes: Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln(x-2) = \infty,$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{x^2+5}+x = \lim_{x\to-\infty} \left((\sqrt{x^2+5}+x) \left(\frac{\sqrt{x^2+5}-x}{\sqrt{x^2+5}-x} \right) \right) = \lim_{x\to-\infty} \frac{5}{\sqrt{x^2+5}-x} = 0$$

thus, the line $y = 0$ is a horizontal asymptote.

Calculus A

Chapter 2: Limits & Derivatives

Sections: 2.7 Derivatives and Rate of Change

Sections: 2.8 The Derivatives as a Function

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6. [10 pts.] Use the definition of the derivative to find the derivative of the function $f(x) = x^2 + 3kx + 1$, where k is a constant.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 3k(x+h) + 1 - (x^2 + 3kx + 1)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 + 3kx + 3kh + 1 - x^2 - 3kx - 1}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2 + 3kh}{h} = \lim_{h\to 0} \frac{h(2x + h + 3k)}{h} = \lim_{h\to 0} 2x + h + 3k = 2x + 3k.
\end{aligned}$$

6. (10 pts) Let $f(x) = \sqrt{2x+1}$. Use the definition of the derivative to find $f'(4)$

$$f'(4) = \lim_{x\to 4} \frac{f(x) - f(4)}{x-4} = \lim_{x\to 4} \frac{\sqrt{2x+1} - 3}{x-4} = \lim_{x\to 4} \frac{2x+1-9}{(x-4)\left(\sqrt{2x+1}+3\right)}$$

$$= \lim_{x\to 4} \frac{2(x-4)}{(x-4)\left(\sqrt{2x+1}+3\right)} = \lim_{x\to 4} \frac{2}{\sqrt{2x+1}+3} = \frac{1}{3}.$$

2. [10 pts.] Let $f(x) = \dfrac{2}{x-1}$. Use the definition of the derivative to find $f'(3)$.

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

$$f(x) = \frac{2}{x-1}$$

$$f(3) = \frac{2}{3-1} = \frac{2}{2} = 1$$

$$f(x) - f(3) = \frac{2}{x-1} - 1$$

توحيد مقامات

$$= \frac{2}{x-1} - \frac{x-1}{x-1} = \frac{2-(x-1)}{x-1}$$

$$= \frac{2-x+1}{x-1} = \frac{3-x}{x-1}$$

تذكر إن الكسر
عبارة عن قسمة

$$\frac{3-x}{x-1} \div x-3$$

$$\frac{3-x}{x-1} * \frac{1}{x-3}$$

$$= \frac{3-x}{(x-1)(x-3)}$$

$$f'(3) = \lim_{x \to 3} \frac{\frac{3-x}{x-1}}{x-3} = \lim_{x \to 3} \frac{3-x}{(x-1)(x-3)}$$

خذينا السالب عامل مشترك بالبسط
عشان نشطبها مع الي تشبها بالمقام

$$\lim_{x \to 3} \frac{-(x-3)}{(x-1)(x-3)} = \lim_{x \to 3} \frac{-1}{x-1} = \frac{-1}{3-1} = \frac{-1}{2}$$

5. [10 pts.] Let $f(x) = \dfrac{2}{x+1}$. Use the definition of the derivative to find $f'(0)$.

$$\boxed{f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{2}{x+1}-2}{x} = \lim_{x\to 0} \frac{2-2x-2}{x(x+1)} = \lim_{x\to 0} \frac{-2\cancel{x}}{\cancel{x}(x+1)} = -2}.$$

7. [10 pts.] Use the **definition of the derivative** to find $f'(3)$ of $f(x) = \sqrt{x-2}+3$.

We have

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{3+h-2}+3-\sqrt{3-2}-3}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3+h-2}-1}{h} = \lim_{h\to 0} \frac{(3+h-2)-1}{h\left(\sqrt{3+h-2}+1\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{3+h-2}+1} = \frac{1}{2} \end{aligned}$$