ثمرة

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Calculus 1 – Limits

(Section 1.2 & 1.4 & 1.5)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

Outlines :-

1- Types of function.

2- Exponential & log properties.

3- Domain of functions.

4- One to One functions.

5- Range & Inverse of functions.

6- Trigonometric identities & Simplify.

7- Simplify the Trigonometric.

1) Type of functions

- $f(x) = x^2 + 3x + 2$

- $f(x) = \frac{x}{x+1}$

- $f(x) = |x|$

- $f(x) = \sqrt{x}$

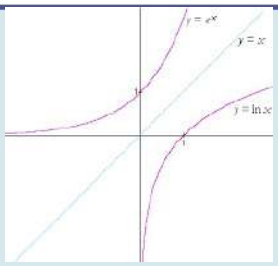
- $f(x) = e^x$

- $f(x) = \log$

- $f(x) = \sin, \cos, \tan$

رسمات الدوال

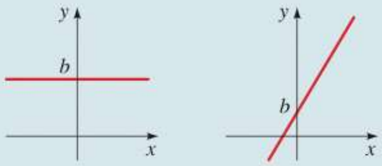
Log and expo graphs



SOME FUNCTIONS AND THEIR GRAPHS

Linear functions

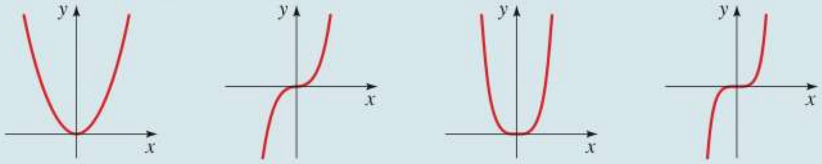
$f(x) = mx + b$



$f(x) = b$ $f(x) = mx + b$

Power functions

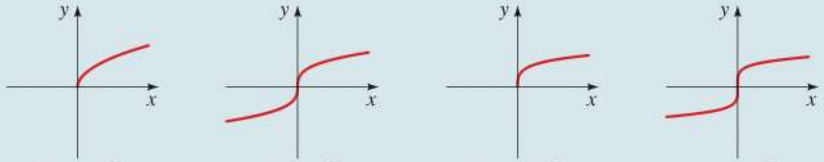
$f(x) = x^n$



$f(x) = x^2$ $f(x) = x^3$ $f(x) = x^4$ $f(x) = x^5$

Root functions

$f(x) = \sqrt[n]{x}$



$f(x) = \sqrt{x}$ $f(x) = \sqrt[3]{x}$ $f(x) = \sqrt[4]{x}$ $f(x) = \sqrt[5]{x}$

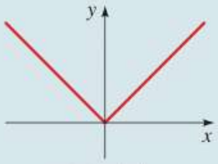
Reciprocal functions

$f(x) = \dfrac{1}{x^n}$

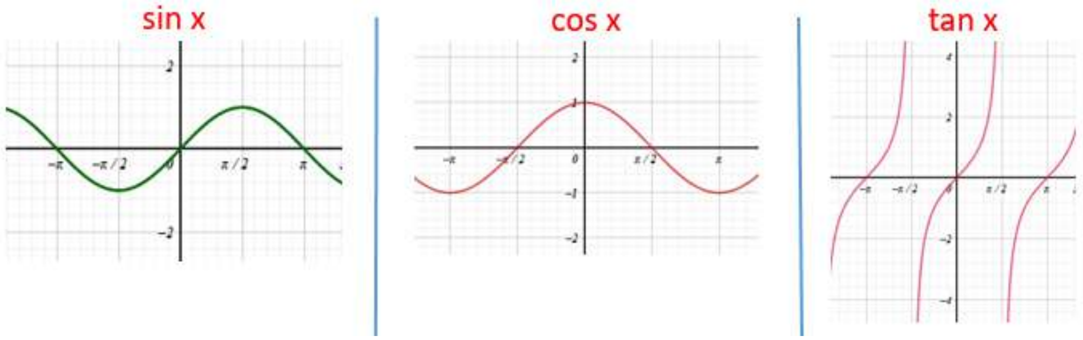
$f(x) = \dfrac{1}{x}$ $f(x) = \dfrac{1}{x^2}$ $f(x) = \dfrac{1}{x^3}$ $f(x) = \dfrac{1}{x^4}$

Absolute value function

$f(x) = |x|$



Trig Function Graphs



1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = \log_2 x$ logarithmic (b) $g(x) = \sqrt[4]{x}$ root

(c) $h(x) = \dfrac{2x^3}{1 - x^2}$ rational (d) $u(t) = 1 - 1.1t + 2.54t^2$ $\rightarrow$ polynomial Degree: 2

(e) $v(t) = 5^t$ exponential (f) $w(\theta) = \sin\theta\,\cos^2\theta$ $\rightarrow$ Trigonometric

2. (a) $y = \pi^x$ exponential (b) $y = x^\pi$ $\rightarrow$ power

(c) $y = x^2(2 - x^3)$ Degree: 5 $\rightarrow$ Polynomial (d) $y = \tan t - \cos t$ $\rightarrow$ Trigonometric

(e) $y = \dfrac{s}{1 + s}$ $\rightarrow$ rational (f) $y = \dfrac{\sqrt{x^3 - 1}}{1 + \sqrt[3]{x}}$ $\rightarrow$ algebraic

2 _ evaluating function.

1.4 EXERCISES

1–4 Use the Law of Exponents to rewrite and simplify the expression.

1. (a) $\frac{4^{-3}}{2^{-8}}$ (b) $\frac{1}{\sqrt[3]{x^4}}$

a) $\frac{2^8}{4^3} = \frac{2^8}{(2^2)^3} = \frac{2^8}{2^6} = 2^2 = 4$

b) $\frac{1}{(x^4)^{\frac{1}{3}}} = \frac{1}{x^{\frac{4}{3}}} = x^{\frac{-4}{3}}$

2. (a) $8^{4/3}$ (b) $x(3x^2)^3$

a) $(8^{\frac{1}{3}})^4 = 2^4 = 16$

b) $x(3^3x^{2.3}) = 27x^7$

3. (a) $b^8(2b)^4$

a) $b^8(2^4b^4) = 16b^{12}$

4. (a) $\dfrac{x^{2n}\cdot x^{3n-1}}{x^{n+2}}$ (b) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$

a) $\dfrac{x^{2n+3n-1}}{x^{n+2}} = \dfrac{x^{5n-1}}{x^{n+2}} = x^{5n-1-(n+2)}$

$= x^{5n-1-n-2} \qquad = x^{4n-3}$

b) $\dfrac{(ab^{\frac{1}{2}})^{\frac{1}{2}}}{a^{\frac{1}{3}}\ b^{\frac{1}{3}}} = \dfrac{a^{\frac{1}{2}}\ b^{\frac{1}{4}}}{a^{\frac{1}{3}}\ b^{\frac{1}{3}}}$

$= a^{(\frac{1}{2}-\frac{1}{3})}\ b^{(\frac{1}{4}-\frac{1}{3})} = a^{\frac{1}{6}}\ b^{\frac{-1}{12}}$

$= \dfrac{a^{\frac{1}{6}}}{b^{1/12}}$

23. If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

$\because f(x) = 5^x$

$\therefore f(x+h) = 5^{x+h}$

$5^{x+h} = 5^x \cdot 5^h$

$$\therefore \frac{f(x+h) - f(x)}{h} = \frac{5^{x+h} - 5^x}{h}$$

$$\therefore 5^x\left(\frac{5^h - 1}{h}\right)$$

Laws of Logarithms

EXAMPLE 1 ■ Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3}\log 8$

SOLUTION

$$\begin{aligned}
&\textbf{(a)}\ \log_4 2 + \log_4 32 = \log_4(2\cdot 32) && \text{Law 1}\\
&\qquad\qquad\qquad\quad = \log_4 64 = 3 && \text{Because } 64 = 4^3\\
&\textbf{(b)}\ \log_2 80 - \log_2 5 = \log_2\left(\tfrac{80}{5}\right) && \text{Law 2}\\
&\qquad\qquad\qquad\quad = \log_2 16 = 4 && \text{Because } 16 = 2^4\\
&\textbf{(c)}\ -\tfrac{1}{3}\log 8 = \log 8^{-1/3} && \text{Law 3}\\
&\qquad\qquad\quad = \log\left(\tfrac{1}{2}\right) && \text{Property of negative exponents}
\end{aligned}$$

Law

1. $\log_a(AB) = \log_a A + \log_a B$

2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$

3. $\log_a(A^C) = C\log_a A$

$$\frac{\log 6}{\log 2} \xcancel{=} \log\left(\frac{6}{2}\right) \qquad \text{and} \qquad (\log_2 x)^3 \xcancel{=} 3\log_2 x$$

$$\log_a(x + y) \xcancel{=} \log_a x + \log_a y$$

39–41 Express the given quantity as a single logarithm.

39. $\ln 10 + 2 \ln 5$

40. $\ln b + 2 \ln c - 3 \ln d$

41. $\frac{1}{3} \ln(x + 2)^3 + \frac{1}{2} [\ln x - \ln(x^2 + 3x + 2)^2]$

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39) $\ln 10 + \ln 5^2$

$= \ln 10 + \ln 25$

$\Rightarrow \ln (10 \cdot 25)$

$= \ln (250)$

---

40) $\ln b + \ln c^2 - \ln d^3$

$= \ln bc^2 - \ln d^3$

$= \ln \dfrac{bc^2}{d^3}$

39–41 Express the given quantity as a single logarithm.

✓ **39.** $\ln 10 + 2\ln 5$ ✓ **40.** $\ln b + 2\ln c - 3\ln d$

41. $\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2+3x+2)^2]$

---

41) $3\,\frac{1}{3}\ln(x+2) + \frac{1}{2}[\ln x - 2\ln(x^2+3x+2)]$

$= \ln(x+2) + \ln x^{\frac{1}{2}} - \ln(x^2+3x+2)$

$= \ln((x+2)(x)^{\frac{1}{2}}) - \ln(x^2+3x+2)$

$$= \ln \frac{(x+2)(x)^{\frac{1}{2}}}{x^2+3x+2}$$

$$= \ln \frac{(x+2)\,x^{\frac{1}{2}}}{(x+1)(x+2)}$$

$$= \ln \frac{x^{\frac{1}{2}}}{x+1}$$

35–38 Find the exact value of each expression.

35. (a) $\log_2 32$ (b) $\log_8 2$

---

a) $\log_2 32 = 5$

$2^x = 32$ لأنه

$x = 5$

---

b) $\log_8 2 = \frac{1}{3}$

$8^x = 2$ لأنه

$x = \frac{1}{3}$

EXAMPLE 8 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (9):

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \tfrac{1}{3}(5 - \ln 10) \end{aligned}$$

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

10 **Change of Base Formula** For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

51–54 Solve each equation for x.

51. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

52. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

53. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

---

51)a) $\ln e^{7-4x} = \ln 6$

$\Rightarrow 7 - 4x = \ln 6$

$\Rightarrow 7 - \ln 6 = 4x$

$\Rightarrow \frac{7 - \ln 6}{4} = x$

---

b) $e^{\ln(3x-10)} = e^2$

$\Rightarrow 3x - 10 = e^2$

$\Rightarrow 3x = e^2 + 10$

$\Rightarrow x = \frac{e^2 + 10}{3}$

51–54 Solve each equation for x.

51. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

52. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

53. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

---

52) $e^{\ln(x^2-1)} = e^3$

$\Rightarrow x^2 - 1 = e^3$

$\Rightarrow x^2 = e^3 + 1$

$\Rightarrow x = \pm\sqrt{e^3 + 1}$

---

b) $e^{2x} - 3e^x + 2 = 0$

Let $e^x = t$

$\therefore t^2 - 3t + 2 = 0$ $\Rightarrow (t-2)(t-1) = 0$

$\therefore t = 2$ and $t = 1$

$\therefore e^x = 2$

$\Rightarrow \ln e^x = \ln 2$

$\therefore x = \ln 2$

$e^x = 1$

$\ln e^x = \ln 1 \nearrow \ln 1 = 0$

$\therefore x = 0$

51–54 Solve each equation for x.

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---

53)

a) $\log_2 2^{x-5} = \log_2 3$

$\Rightarrow x - 5 = \log_2 3$

$\therefore x = \log_2(3) + 5$

b) $\ln[x(x-1)] = 1$

$e^{\ln[x(x-1)]} = e^1$

$x(x-1) = e^1 \Rightarrow x^2 - x = e$

$x^2 - x - e = 0$

استخدم قانون المميز

$x_{1,2} = \frac{1 \mp \sqrt{1 + 4e}}{2}$

$a = 1$
$b = -1$
$c = -e$

$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3) Domain

مثال	القاعدة	الحالة
$\frac{1}{x}$	المقام $\neq 0$	مقام
$\sqrt{x-1}$	ماداخل الجذر ≥ 0	جذر زوجي
$\frac{1}{\sqrt{x}}$	ماداخل الجذر > 0	جذر و مقام
$\ln(x)$	ماداخل اللوق > 0	Log
$\lvert x \rvert$	معرف دائماً	مطلق
$5^x , e^x$	معرفة دائماً	دالة أسية
$\sin^x , \cos^x$	معرفة دائماً	دالة مثلثية
$\sin^{-1}x , \cos^{-1}x$	$[-1, 1]$	دالة مثلثية عكسية

19–20 Find the domain of each function.

19. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

20. (a) $g(t) = \sqrt{10^t - 100}$ (b) $g(t) = \sin(e^t - 1)$

19) a) $\dfrac{1 - e^{x^2}}{1 - e^{1-x^2}} \Rightarrow 1 - e^{1-x^2} \neq 0$

$1 \neq e^{1-x^2} \Rightarrow \cancelto{0}{\ln 1} \neq \ln e^{1-x^2}$

$0 \neq 1 - x^2 \Rightarrow x^2 \neq 1$

$\therefore x \neq \pm 1$

∴ The domain is R / {-1, 1}

---

b) $\dfrac{1 + x}{e^{\cos x}} \Rightarrow e^{\cos x} \neq 0$

$\ln e^{\cos x} \neq \boxed{\ln 0}$ → No Solution

∴ The domain is R or $(-\infty, \infty)$

19–20 Find the domain of each function.

✓ **19.** (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ ✓(b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

20. (a) $g(t) = \sqrt{10^t - 100}$ (b) $g(t) = \sin(e^t - 1)$

20) a) $\sqrt{10^t - 100} \Rightarrow 10^t - 100 \geqslant 0$

$10^t \geqslant 100 \Rightarrow \log 10^t \geqslant \log 100$

$t \geqslant 2$ $\therefore$ The domain is $[2, \infty)$

---

b) $\sin(e^t - 1)$

The domain is $\mathbb{R}$

4) One to One Function

* كل قيمة x مختلفة تعطي قيمة y مختلفة

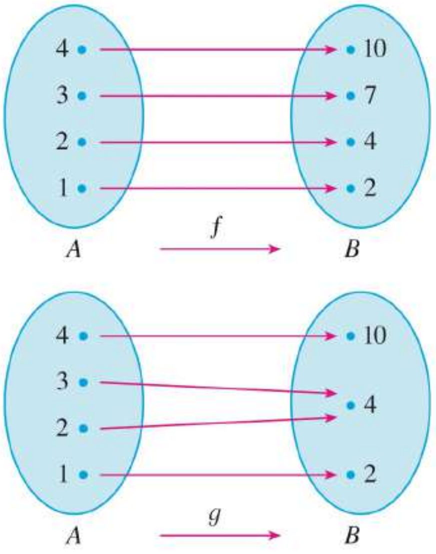
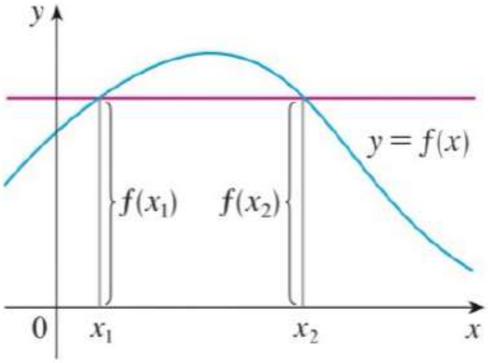


FIGURE 1
f is one-to-one; g is not.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$



* إذا قطع الخط الأفقي

أكثر من مرة ⇐ الدالة

not one_to_ one

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

طريقة الحل من خلال المعادلة

١) افرض $f(x_1) = f(x_2)$

* إذا الحل عطاني $x_1 = x_2$

∴ الدالة One-to-One

* إذا الحل عطاني $x_1 = \pm x_2$

∴ الدالة not One-to-One

طريقة الحل من خلال الرسم :-

١) نستخدم اختبار الأفقي

إذا أي خط أفقي يقطع الرسم ﻓ نقطة واحدة

فقط ⇐ الدالة One-to-One

إذا يقطع ﻓ أكثر من نقطة ⇐

not One-to-One.

إذا السؤال كان رسمة لازم تذكر السبب في حال الدالة one to one أو not one to one
... إنك تكتب جملة الخطوط الأفقية ماتقطع الرسمة اكثر من مرة في حالة one to one وفي
حالة not one to one تقول ان في خط أفقي قطع الرسمة اكثر من مرة

3–14 Determine whether it is one-to-one.

9. $f(x) = 2x - 3$ **10.** $f(x) = x^4 - 16$

9) $f(x_1) = f(x_2)$

$2x_1 - 3 = 2x_2 - 3$

$\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ One to one

$f(x_1) = f(x_2)$

10) $x_1^4 - 16 = x_2^4 - 16$

$x_1^4 = x_2^4 \Rightarrow \sqrt[4]{x_1^4} = \sqrt[4]{x_2^4}$

$x_1 = \pm x_2$ not One-to-One

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

1) let $f(x_1) = f(x_2)$

2) $x_1^3 = x_2^3 \Rightarrow \sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$

$\Rightarrow x_1 = x_2$ $\therefore$ One-to-one

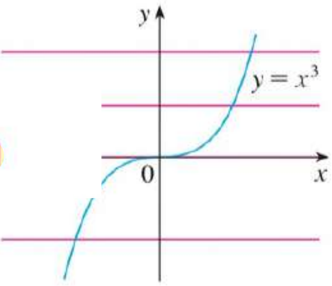


FIGURE 3
$f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

لازم تكتب الجملة في حالة رسمت أو السؤال في رسمة

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$x_1^2 = x_2^2 \Rightarrow \sqrt{x_1^2} = \sqrt{x_2^2}$

$= x_1 = \pm x_2$

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-on

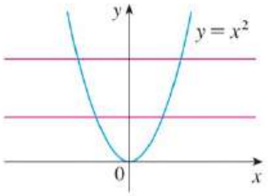


FIGURE 4
$g(x) = x^2$ is not one-to-one.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

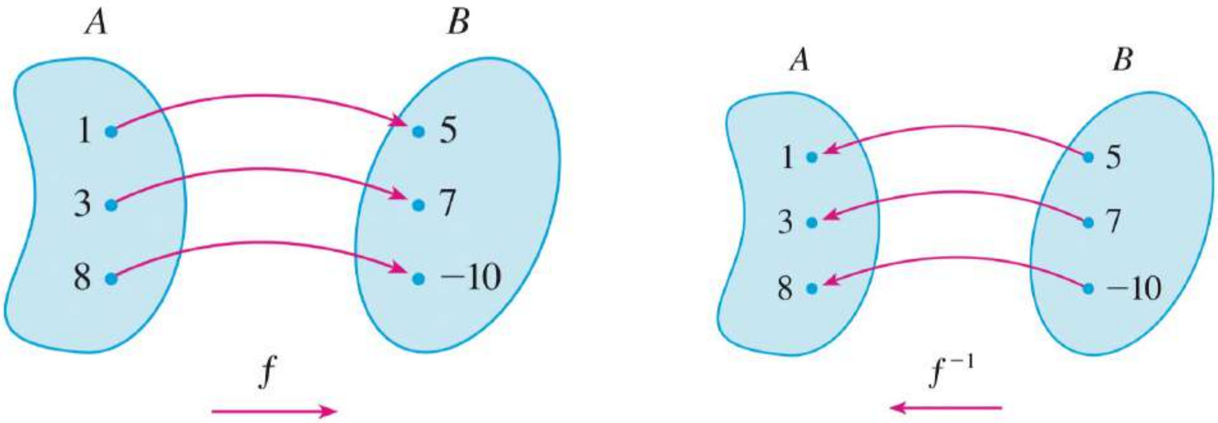


FIGURE 6
The inverse function reverses inputs and outputs.

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

3–14 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

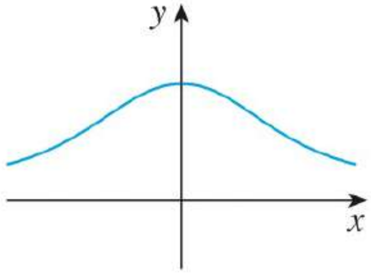
not one to one

4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

one to one

5.



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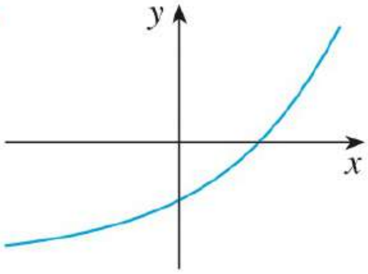

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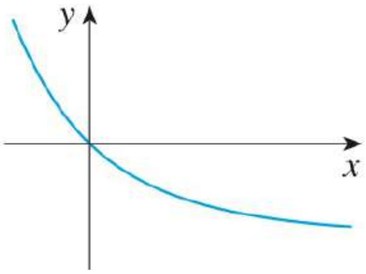

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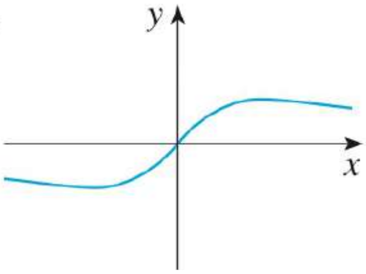

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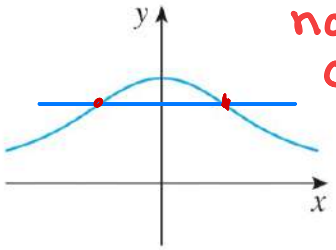
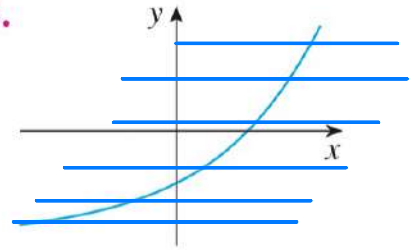


not one to one

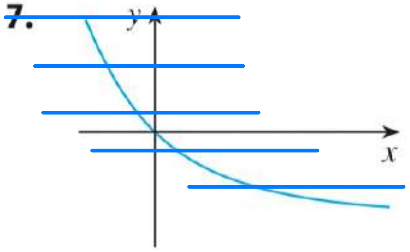
6.



one to one

one to one

7.



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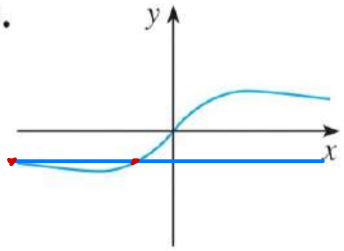


not one to one

From Figure we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

EXAMPLE 3 If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$
\begin{aligned}
f^{-1}(7) &= 3 &\quad \text{because} \quad & f(3) = 7 \\
f^{-1}(5) &= 1 &\quad \text{because} \quad & f(1) = 5 \\
f^{-1}(-10) &= 8 &\quad \text{because} \quad & f(8) = -10
\end{aligned}
$$

$$\boxed{f^{-1}(x) = y \iff f(y) = x}$$

cancellation equations:

$$
\boxed{
\begin{aligned}
f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in } A \\
f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in } B
\end{aligned}
}
$$

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

2. [5 + 5 = 10 pts.] The graph of f is given below.

(a) Explain why f is one-to-one.

(b) Find $f^{-1}(2)$.

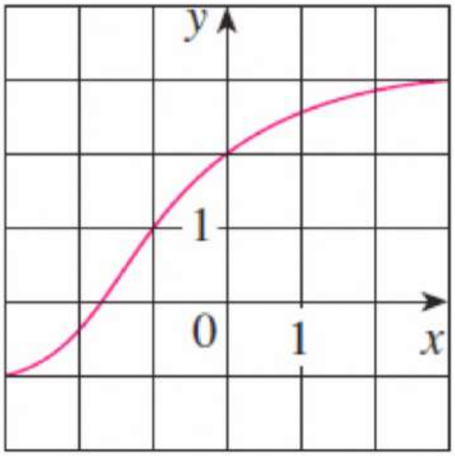


Figure 2: The graph of $y = f(x)$.

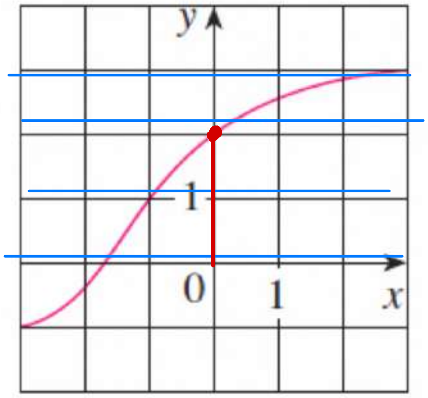


Figure 2: The graph of $y = f(x)$.

لازم تكتبها

(a) Any horizontal line intersects the graph of f at most once. Therefore, the function is one-to-one by the horizontal line test.

(b) $\boxed{f^{-1}(2) = 0 \text{ since } f(0) = 2}$.

21–26 Find a formula for the inverse of the function.

21. $f(x) = 1 + \sqrt{2 + 3x}$ **22.** $f(x) = \dfrac{4x - 1}{2x + 3}$

---

21) $y = 1 + \sqrt{2 + 3x}$ "نبي نخلي ال x بروحها"

$\Rightarrow y - 1 = \sqrt{2 + 3x}$

$\Rightarrow (y - 1)^2 = (\sqrt{2 + 3x})^2$

"لا تربّع من أول خطوة، لازم نخلي أول شيء الجذر بروح، بعدين نربّع"

$\Rightarrow (y - 1)^2 = 2 + 3x$

$\Rightarrow (y - 1)^2 - 2 = 3x$

$\Rightarrow \dfrac{(y - 1)^2 - 2}{3} = x$

$\therefore f^{-1}(x) = \dfrac{(x - 1)^2 - 2}{3}$

21–26 Find a formula for the inverse of the function.

✓**21.** $f(x) = 1 + \sqrt{2 + 3x}$ **22.** $f(x) = \dfrac{4x - 1}{2x + 3}$

---

22) $y = \dfrac{4x - 1}{2x + 3}$

$\Rightarrow y(2x+3) = (4x-1)$

$\Rightarrow 2xy + 3y = 4x - 1$

$\Rightarrow 3y + 1 = 4x - 2xy$

$\Rightarrow 3y + 1 = x(4 - 2y)$

$\Rightarrow \dfrac{3y+1}{4-2y} = x$

$\therefore f^{-1}(x) = \dfrac{3x+1}{4-2x}$

21–26 Find a formula for the inverse of the function.

23. $f(x) = e^{2x-1}$ **25.** $y = \ln(x + 3)$

---

23) $y = e^{2x-1} \Rightarrow \ln y = \ln e^{2x-1}$

$\Rightarrow \ln y = 2x - 1 \Rightarrow \ln y + 1 = 2x$

$$\Rightarrow \frac{\ln y + 1}{2} = x$$

$$\therefore f^{-1}(x) = \frac{\ln x + 1}{2}$$

---

25) $y = \ln(x+3) \Rightarrow e^{y} = e^{\ln(x+3)}$

$e^{y} = x + 3 \Rightarrow e^{y} - 3 = x$

$\therefore f^{-1}(x) = e^{x} - 3$

3. [10 pts.] Let $f(x) = \ln\left(\frac{x}{2x-1}\right)$. Find $f^{-1}(x)$.

Let $y = \ln\left(\frac{x}{2x-1}\right)$

$e^{y} = e^{\ln\left(\frac{x}{2x-1}\right)}$

$e^{y} = \frac{x}{2x-1}$

$e^{y}(2x-1) = x$

$2xe^{y} - e^{y} = x$

$2xe^{y} - x = e^{y}$

$x(2e^{y}-1) = e^{y}$

$x = \frac{e^{y}}{2e^{y}-1}$

$\therefore f^{-1}(x) = \frac{e^{x}}{2e^{x}-1}$

21–26 Find a formula for the inverse of the function.

26. $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

$$y(1 + e^{-x}) = 1 - e^{-x}$$

$$y + y e^{-x} = 1 - e^{-x}$$

$$y e^{-x} + e^{-x} = 1 - y$$

$$e^{-x}(y + 1) = 1 - y$$

$$e^{-x} = \frac{1 - y}{y + 1} \Rightarrow \ln e^{-x} = \ln\left(\frac{1 - y}{y + 1}\right)$$

$$-x = \ln\left(\frac{1 - y}{y + 1}\right) \Rightarrow x = -\ln\left(\frac{1 - y}{y + 1}\right)$$

$$\therefore x = \ln\left(\frac{1 - y}{y + 1}\right)^{-1}$$

$$\therefore x = \ln\left(\frac{y + 1}{1 - y}\right)$$

$$\therefore f^{-1}(x) = \ln\left(\frac{x + 1}{1 - x}\right)$$

5. [10 pts.] Let $f(x) = e^{x+1} + 2$. Find $f^{-1}(x)$.

$$\text{Let } y = e^{x+1} + 2$$

حاولوا تحلون السؤال هذا نفس الفكرة..نشوف فهمت الدرس ولا بعد 🤔؟

$$y - 2 = e^{x+1}$$

$$\ln(y-2) = \ln e^{x+1}$$

$$\ln(y-2) = x+1$$

$$\ln(y-2) - 1 = x$$

لاحظ وين ضفنا ال-1..برا الln

$$\therefore f^{-1}(x) = \ln(x-2) - 1$$

(1) Let $y = e^{x+1} + 2$.
(2) Solve for x in terms of y: $x = \ln(y-2) - 1$.
(3) Interchange x and y: $y = \ln(x-2) - 1$. Therefore, $\boxed{f^{-1}(x) = \ln(x-2) - 1}$.

✓

3. [10 pts.] Let $f(x) = \ln(2^x - 1)$. Find $f^{-1}(x)$.

Let $y = \ln(2^x - 1)$

$e^y = e^{\ln(2^x - 1)}$

$e^y = 2^x - 1$

$e^y + 1 = 2^x$

$\log_2(e^y + 1) = \log_2(2)^x$

$\log_2(e^y + 1) = x$

$\therefore f^{-1}(x) = \log_2(e^x + 1)$

6_range of the function :-

* هو جميع قيم x الناتجة من الدالة

لما نطبقها عل الـ Domain .

* لإيجاد Range :-

١_ أكتب $y = f(x)$

٢_ حاول تعبر من الـ x بدلالة y "$f^{-1}(x)$"

٣_ القيم المسموحة لـ y بعد الحل هي الـ Range .

مثال :-

$$f(x) = x^2$$

$$y = x^2$$

$$x = \sqrt{y}$$

$$f^{-1}(x) = \sqrt{x}$$

Range of $f(x) = (0, \infty) =$ Domain of $f^{-1}(x)$.

2. [5 + 5 + 5 = 15 pts.] Let $f(x) = \dfrac{x+1}{2x+1}$.

(a) Show that f is one-to-one.

(b) Find the inverse function of f.

(c) Find the domain and range of f.

---

a) $f(x_1) = f(x_2)$

$$\frac{x_1+1}{2x_1+1} = \frac{x_2+1}{2x_2+1}$$

$$\underbrace{(2x_1+1)(x_2+1)}_{\downarrow} = \underbrace{(x_1+1)(2x_2+1)}_{\downarrow}$$

$$2x_1x_2 + 2x_1 + x_2 + 1 = 2x_1x_2 + x_1 + 2x_2 + 1$$

$\therefore x_1 = x_2$ $\quad$ $\therefore$ f is one-to-one

---

b) $y = \dfrac{x+1}{2x+1} \Rightarrow 2xy + y = x + 1$

$\Rightarrow 2xy - x = 1 - y \Rightarrow x(2y-1) = 1 - y$

$\therefore x = \dfrac{1-y}{2y-1}$ $\quad$ $\therefore f^{-1}(x) = \dfrac{1-x}{2x-1}$

c) The domain is $\{R / \frac{-1}{2}\}$, The range $\{R, \frac{1}{2}\}$

Let $f(x) = \frac{e^x}{1+2e^x}$, Find $f^{-1}(x)$

Domain of $f(x)$:_

---

1) Let $y = \frac{e^x}{1+2e^x}$

2) Domain of $f(x) = \frac{e^x}{1+2e^x}$

$1 + 2e^x \neq 0$, $2e^x \neq -1$

impossible $\therefore D_f = R = (-\infty, \infty)$

$$y(1+2e^x) = e^x$$

$$y + 2ye^x = e^x$$

$$y = e^x - 2ye^x$$

$$y = e^x(1-2y)$$

$$\frac{y}{1-2y} = e^x$$

$$e^x = \frac{y}{1-2y}$$

$$\ln e^x = \ln\left(\frac{y}{1-2y}\right)$$

$$x = \ln\left(\frac{y}{1-2y}\right)$$

$$\therefore f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right)$$

57. (a) Find the domain of $f(x) = \ln(e^x - 3)$.
(b) Find f^{-1} and its domain.

a) $\ln(e^x - 3)$

$e^x - 3 > 0 \Rightarrow e^x > 3$

$\ln e^x > \ln 3 \Rightarrow x > \ln 3$

$\therefore$ The domain is $(\ln 3, \infty)$

---

b) $y = \ln(e^x - 3)$

$\Rightarrow e^y = e^{\ln(e^x - 3)} \Rightarrow e^y = e^x - 3$

$\Rightarrow e^y + 3 = e^x \Rightarrow \ln(e^y + 3) = \ln e^x$

$\Rightarrow \ln(e^y + 3) = x$

$\therefore f^{-1}(x) = \ln(e^x + 3)$

$e^x + 3 > 0 \Rightarrow e^x > -3$ The Domain is $\mathbb{R}$

3. [10 + 5 = 15 pts.] Let $f(x) = \ln\left(\frac{2}{x} + 3\right)$.

(a) Find the inverse function of f.

(b) Find the range of f.

---

a) $y = \ln\left(\frac{2}{x} + 3\right)$

$e^{y} = e^{\ln\left(\frac{2}{x} + 3\right)}$

$e^{y} = \frac{2}{x} + 3$

$e^{y} - 3 = \frac{2}{x}$

$x\,(e^{y} - 3) = 2$

$x = \frac{2}{e^{y} - 3}$ $\quad \therefore f^{-1}(x) = \frac{2}{e^{x} - 3}$

Now the range of the function f is the domain of the inverse function

b) $e^{x} - 3 \neq 0 \Rightarrow e^{x} \neq 3$

$\ln e^{x} \neq \ln 3 \Rightarrow x \neq \ln 3$ $\quad \{R / \ln 3\}$

4. [5+5 = 10 pts.] Let $f(x) = 3^x + 1$.

a) Find $f^{-1}(x)$.

b) Find the range of f.

---

a) Find $f^{-1}(x)$.

(1) Let $y = 3^x + 1$.

(2) Solve for x in terms of y: $x = \log_3(y-1)$.

(3) Interchange x and y: $y = \log_3(x-1)$. Therefore, $\boxed{f^{-1}(x) = \log_3(x-1)}$.

b) Find the range of f.

$\boxed{R_f = D_{f^{-1}} = (1, \infty)}$.

---

4. $(10 + 5 = 15$ pts) Let $f(x) = \ln(2x+3)$.

(a) Find $f^{-1}(x)$.

(b) Find the domain and the range of the function f.

(a) Find $f^{-1}(x)$.

$$y = f(x) = \ln(2x+3) \implies 2x + 3 = e^y \implies x = \tfrac{1}{2}\left(e^y - 3\right) \implies f^{-1}(x) = \tfrac{1}{2}\left(e^x - 3\right)$$

(b) Find the domain and the range of the function f.

The domain of f is $\left(-\frac{3}{2}, \infty\right)$. The range of f is $\mathbb{R}$ which is the domain of f^{-1}.

Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 17 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, *is* one-to-one (see Figure 18). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

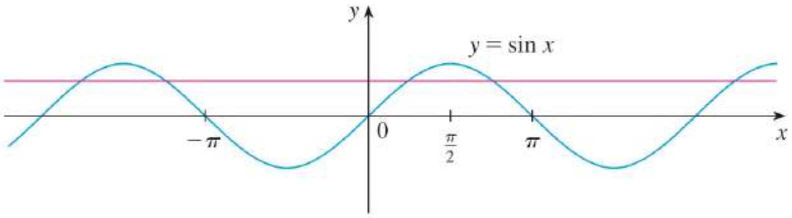


FIGURE 17

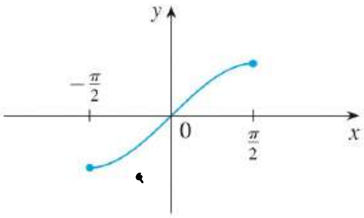


FIGURE 18
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

THE INVERSE SINE, INVERSE COSINE, AND INVERSE TANGENT FUNCTIONS

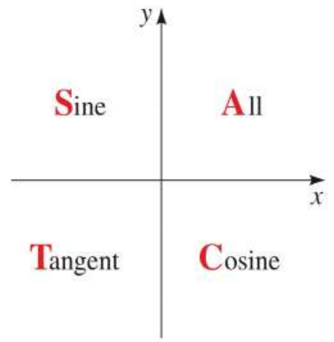
The sine, cosine, and tangent functions on the restricted domains $[-\pi/2, \pi/2]$, $[0, \pi]$, and $(-\pi/2, \pi/2)$, respectively, are one-to one and so have inverses. The inverse functions have domain and range as follows.

Function	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$(-\pi/2, \pi/2)$

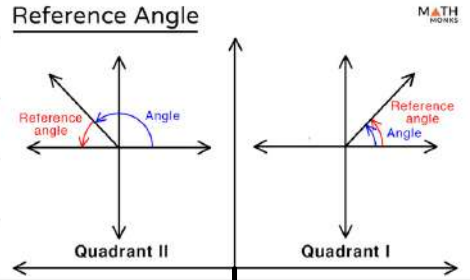
The functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are sometimes called **arcsine**, **arccosine**, and **arctangent**, respectively.

inverse Trigo معلومات مهمة

١) اعرف الاشارات في الأرباع



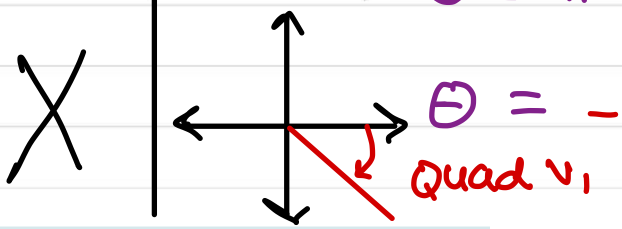
٢) اعرف الزاوية المرجعية $(\bar{\theta})$



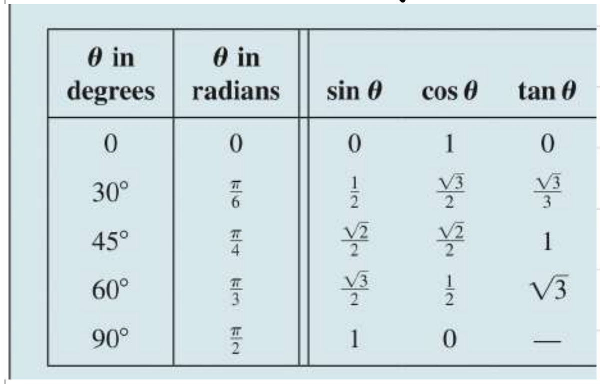
١) الربع الأول ⇐ $\theta = \bar{\theta}$

٢) الربع الثاني ⇐ $\theta = \pi - \bar{\theta}$

٣) الربع الثالث ⇐ $\theta = - \bar{\theta}$

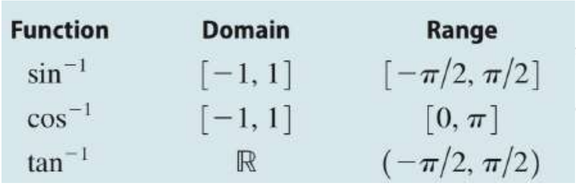


٣) اعرف القيم المحفوظة



θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	—

٤) Range of inverse



Function	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$(-\pi/2, \pi/2)$

خطوات إيجاد قيم inverse Tria

١) أوجد Range الدالة المثلثية المعكوسة

$\sin^{-1}x \Rightarrow$ الربع الأول + الرابع $\Rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1}x \Rightarrow$ الربع الأول + الثاني $\Rightarrow [0, \pi]$

$\tan^{-1}x \Rightarrow$ الربع الأول + الرابع $\Rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

٢) حدد الزاوية المرجعية

$\bar{\theta} = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ & in rad

٣) حدد الربع باستخدام إشارة القيمة (موجب / سالب)

٤) اكتب النتيجة θ in rad

١) الربع الأول $\bar{\theta} = \theta$

٢) الربع الثاني $\theta = \pi - \bar{\theta}$

٣) الربع الرابع $\theta = -\bar{\theta}$

EXAMPLE 1 ■ Evaluating Inverse Trigonometric Functions

Find the exact value.

(a) $\sin^{-1}\dfrac{\sqrt{3}}{2}$ **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$

---

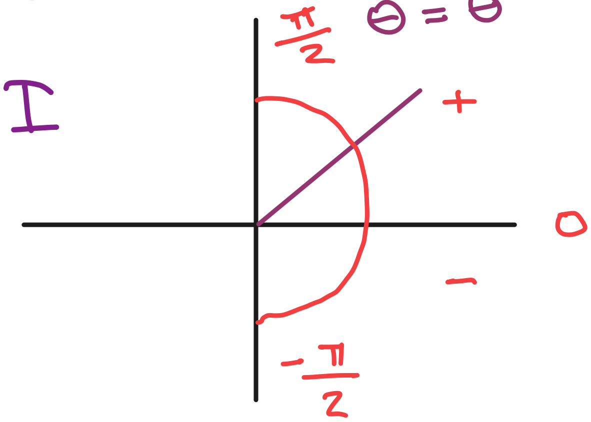
1) $\sin^{-1} x \in [-\pi/2, \pi/2]$

2) $\because \dfrac{\sqrt{3}}{2}$ Positive, Quad I

3) $\sin \bar{\theta} = \dfrac{\sqrt{3}}{2}$

4) $\theta = \bar{\theta} = 60^\circ = \dfrac{\pi}{3}$

$\therefore \sin^{-1} \theta = \dfrac{\pi}{3}$



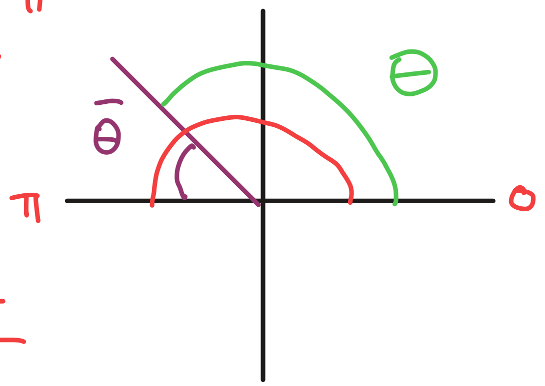
---

b) 1) $-\dfrac{1}{2}$ "2nd Quad"

2) $\cos^{-1}$ range :-

3) $\cos \bar{\theta} = -\dfrac{1}{2} \Rightarrow \bar{\theta} = \dfrac{\pi}{3}$

4) $\theta = \pi - \bar{\theta} = \pi - \dfrac{\pi}{3} = \dfrac{3\pi}{3} - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$



63–68 Find the exact value of each expression.

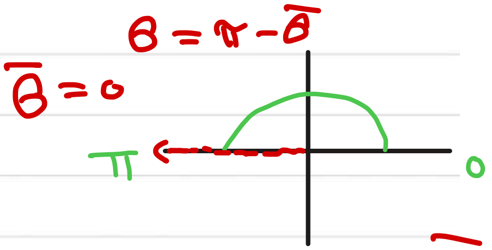
63. (a) $\cos^{-1}(-1)$ (b) $\sin^{-1}(0.5)$

64. (a) $\tan^{-1}\sqrt{3}$ (b) $\arctan(-1)$

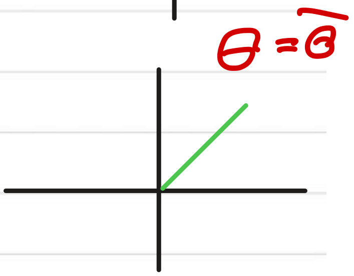
$* \tan^{-1}(1)$

---

63)



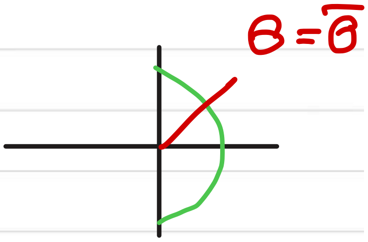
a) $\cos^{-1}(-1) = \pi$



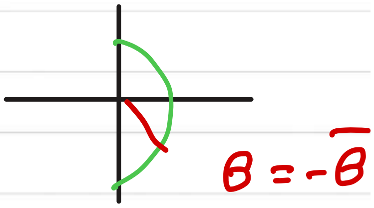
b) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

---

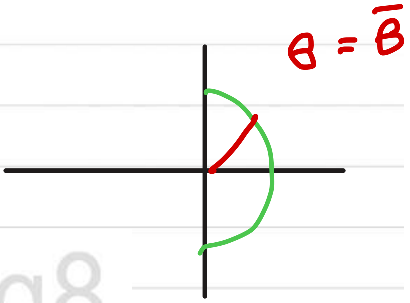
64)



a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$



b) $\tan^{-1}(-1) = -\frac{\pi}{4}$



$* \tan^{-1}(1) = \frac{\pi}{4}$

65. (a) $\csc^{-1}\sqrt{2}$ (b) $\arcsin 1$

66. (a) $\sin^{-1}(-1/\sqrt{2})$ (b) $\cos^{-1}(\sqrt{3}/2)$

65) a)

$\because \csc\theta = \sqrt{2}$, $\sin\theta = \frac{1}{\csc\theta}$

$\therefore \sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \frac{\pi}{4}$

$\therefore \csc^{-1}\sqrt{2} = \frac{\pi}{4}$

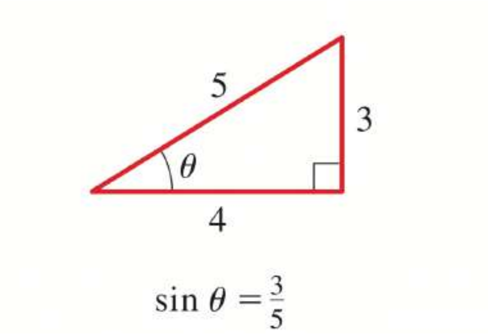
b) $\sin^{-1} 1 = \frac{\pi}{2}$

66. (a) $\sin^{-1}(-1/\sqrt{2})$ (b) $\cos^{-1}(\sqrt{3}/2)$

a) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

6 _ Trigonometric identities & Simplify



$\sin\theta = \frac{3}{5}$

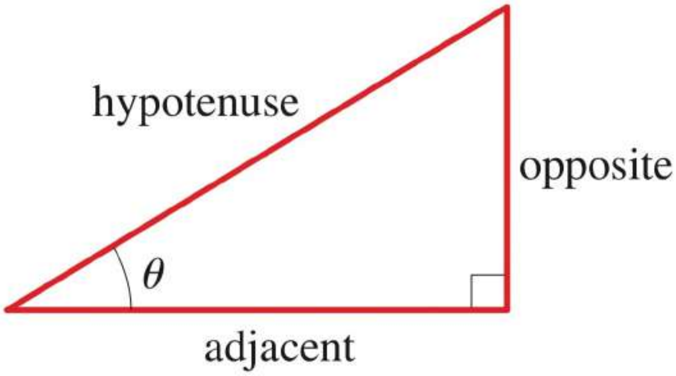
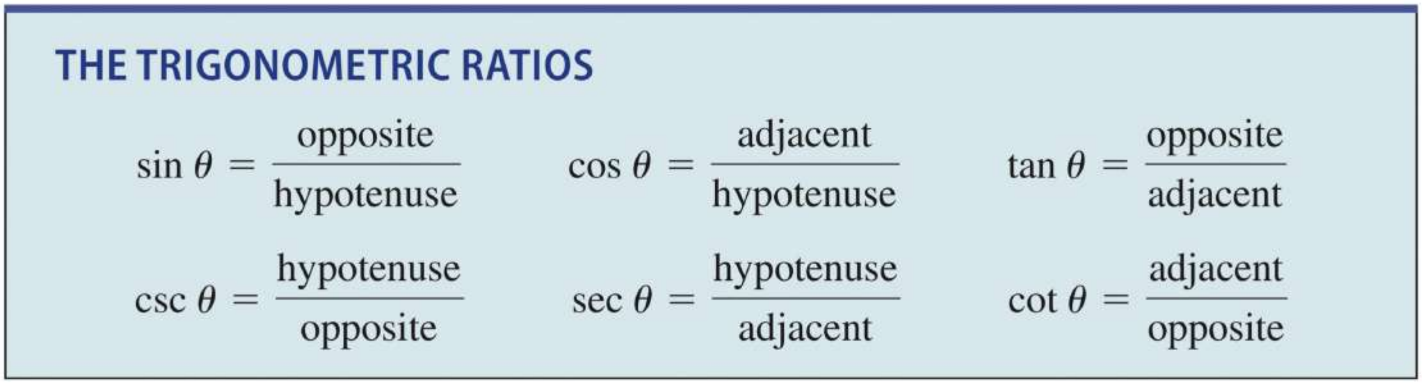


FIGURE 1

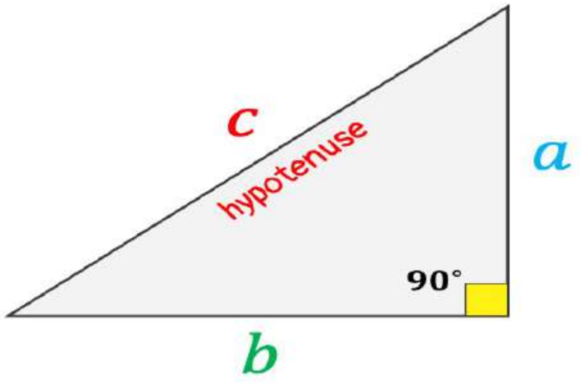


THE TRIGONOMETRIC RATIOS

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

PYTHAGOREAN THEOREM



$$\boxed{c^2 = a^2 + b^2}$$

★ $c = \sqrt{a^2 + b^2}$

★ $a = \sqrt{c^2 - b^2}$

★ $b = \sqrt{c^2 - a^2}$

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حل مسائل دوال المثلثية و الـ inverse لها :-

* prove: $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

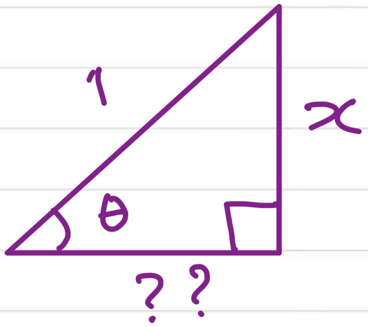
١) عيّن الزاوية

$\theta = \sin^{-1} x$

٢) حول النسبة المثلثية

$\sin\theta = \frac{x}{1}$

٣) ارسم مثلث قائم حسب النسبة



٤) أوجد الظلع المفقود بإستخدام فيثاغورث

$1^2 = ??^2 + x^2 \Rightarrow ?? = \sqrt{1-x^2}$

٥) اقرأ المطلوب:

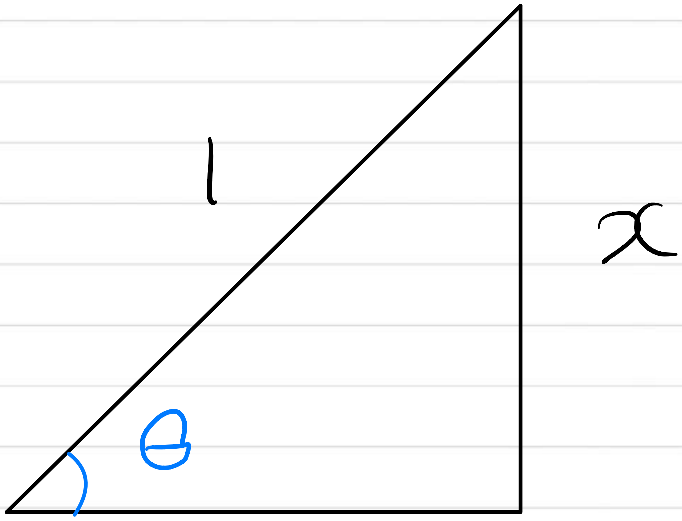
$\cos\theta = \frac{??}{1} = \sqrt{1-x^2}$

69. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

$$\text{Let } \theta = \sin^{-1} x$$

$$\therefore \sin\theta = x$$

$$\boxed{\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}}$$



$$z = \sqrt{1 - x^2}$$

بينا الضلع المفقود (z) عن طريق فيثاغورث

$$\because 1^2 = x^2 + z^2 \Rightarrow z = \sqrt{1 - x^2}$$

$$\therefore \cos(\sin^{-1} x) = \cos(\theta)$$

$$\cos(\theta) = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2} \quad ✓$$

4. [10 pts.] Find the exact value of each expression

(a) $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$.

(b) $\log_{10} 4 + 2\log_{10} 5$.

Let $y = \sin^{-1}\left(\frac{2}{3}\right)$, so $\sin y = \left(\frac{2}{3}\right)$ and $y \in [-\pi/2, \pi/2]$.

مثلث قائم

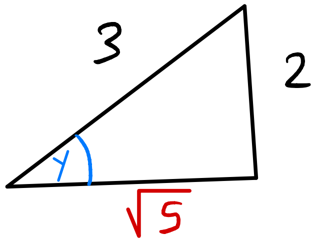
Then we can draw a right triangle with angle y as

نظرية فيثاغورث

shown below and deduce from the Pythagorean Theorem

ضلع الثالث

that the third side has length $\sqrt{9-4} = \sqrt{5}$.



This enables us to read from the triangle that $\boxed{\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \frac{2}{\sqrt{5}}}$.

(b) $\boxed{\log_{10} 4 + 2\log_{10} 5 = \log_{10} 4 + \log_{10} 5^2}$

$\boxed{= \log_{10} 4 + \log_{10} 25 = \log_{10} 100 = 2}$.

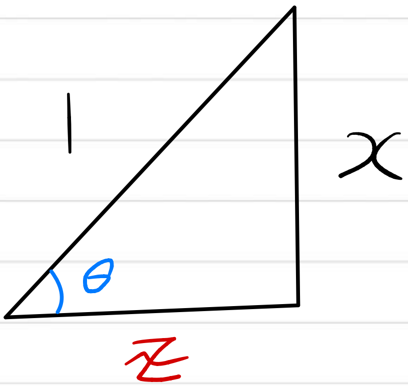
Log (25·4)

1. [10 pts.] Prove that: $\sec(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$ for any x in $(-1, 1)$.

Let $\theta = \sin^{-1} x$

$\therefore \sin\theta = x$

$z = \sqrt{1^2 - x^2}$ (Pythagorean Theorem)



$\because \sec(\sin^{-1} x) = \sec(\theta)$

$= \sec\theta = \dfrac{1}{\sqrt{1-x^2}}$

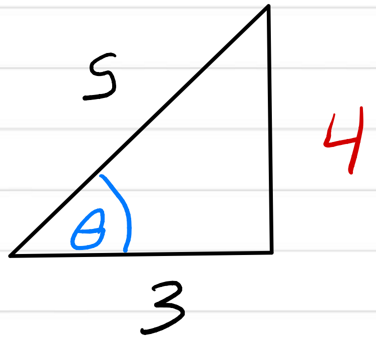
2. [10 pts.] Find the exact value of $\sin\left(\sec^{-1}\left(\frac{5}{3}\right)\right)$.

Let $\theta = \sec^{-1}\left(\frac{5}{3}\right)$

$\therefore \sec\theta = \frac{5}{3}$

$\because z = \sqrt{5^2 - 3^2}$

$= \sqrt{16} = 4$



$\therefore \sin\left(\sec^{-1}\left(\frac{5}{3}\right)\right) = \sin\theta$

$= \frac{4}{5}$

3. [5 pts.] Find the exact value of $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

We let $x = \cos^{-1}\left(\frac{4}{5}\right)$, then $\cos x = \frac{4}{5}$ and therefore we obtain $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin x = \frac{3}{5}$.

7. (5 + 5 = 10 pts)

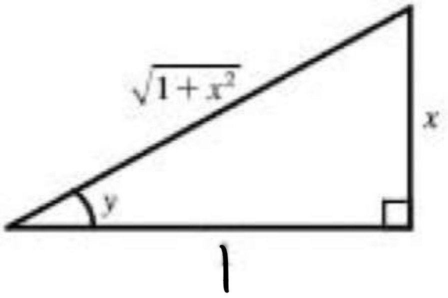
(a) Show that $\sin\left(\tan^{-1}x\right) = \dfrac{x}{\sqrt{1+x^2}}$.

(b) Find the exact value of $\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$.

(a) Show that $\sin\left(\tan^{-1}x\right) = \dfrac{x}{\sqrt{1+x^2}}$.

Let $y = \tan^{-1}x \implies x = \tan y;\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and we can read from the figure below (which illustrates the case where $y > 0$) that

$$\sin\left(\tan^{-1}x\right) = \sin y = \frac{x}{\sqrt{x^2+1}}.$$



(b) Find the exact value of $\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$. حل عن طريق المثلث أحسن

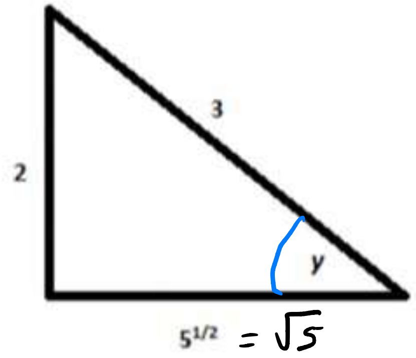
~~$\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \left(\frac{x}{\sqrt{1+x^2}}\right)$~~ $_{x=\frac{1}{\sqrt{3}}} = \dfrac{1}{2}$.

or $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} \implies \sin\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$.

حل آخر

(a) $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$.

Let $y = \sin^{-1}\left(\frac{2}{3}\right)$, so $\sin y = \left(\frac{2}{3}\right)$ and $y \in [-\pi/2, \pi/2]$. Then we can draw a right triangle with angle y as shown below and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-4} = \sqrt{5}$.



This enables us to read from the triangle that $\boxed{\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \frac{2}{\sqrt{5}}}$.

7_Simplify the Trigonometric.

70–72 Simplify the expression.

70. $\tan(\sin^{-1}x)$ **71.** $\sin(\tan^{-1}x)$

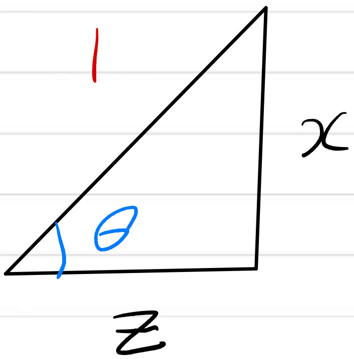
---

70) Let $\theta = \sin^{-1}x$

$\therefore \sin\theta = x$

using Pythagorean theorem

$z = \sqrt{1^2 - x^2}$



$\therefore \tan(\sin^{-1}x) = \tan\theta$

$\therefore \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

---

71) $\dfrac{x}{\sqrt{1+x^2}}$

ثمــرة

Thamrah

Kuwait University

Calculus 1 – Limits

(Section 2.2 & 2.3)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

Limits

1- Determine limit from graph

2- Direct Substitution.

3- Squeeze Theorem.

4- Continuity & Discontinuity.

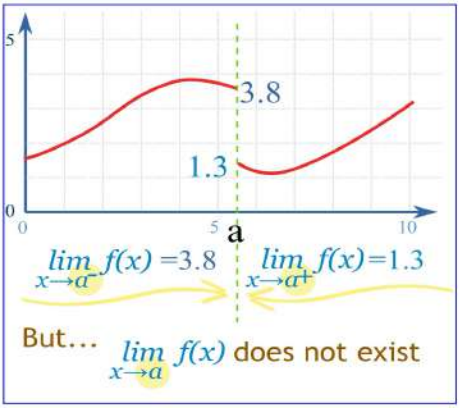
5- Find the values of a & b that make f continues everywhere.

6- IVT.

7- Asymptotes.

8- Definition of Derivative.

$$\lim_{x \to a} f(x)$$



- lim : النهاية
- $f(x)$: هي الدالة
- $x \to a$: x تقترب من a

الوصف	الرمز
تقترب من a من جهه اليمين	$x \to a^+$
تقترب من a من جهه اليسار	$x \to a^-$
تقترب من a من الجهتين	$x \to a$

1- Determine limit from graph

3 $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

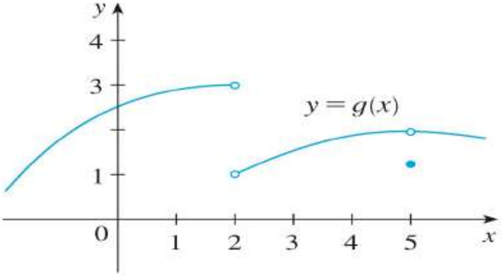


FIGURE 10

---

a) $\lim_{x\to 2^-} g(x) = 3$ b) $\lim_{x\to 2^+} g(x) = 1$

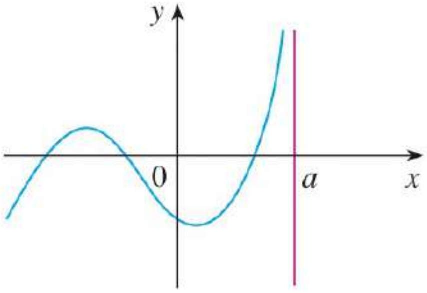
c) Since the left and right limits are different, we conclude

$\lim_{x\to 2} g(x)$ does not exist.

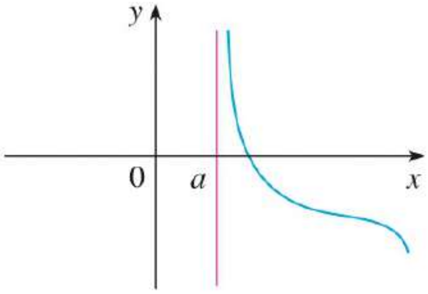
d) $\lim_{x\to 5^-} g(x) = 2$ and $\lim_{x\to 5^+} g(x) = 2$

we have $\lim_{x\to 5} g(x) = 2$

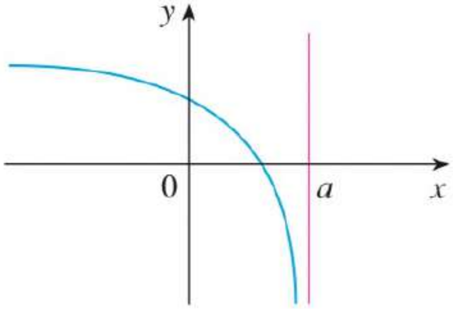
* Vertical Asymptotes



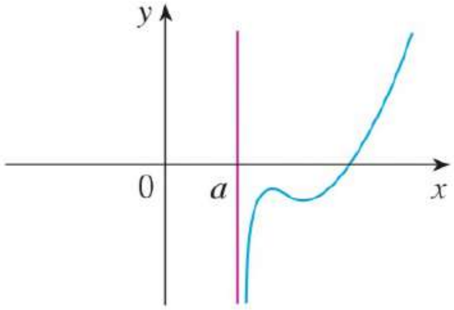
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

6 **Definition** The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

a is V.A

Infinite Limits

EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

To indicate the kind of behavior exhibited in Example 8, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

⊘ This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

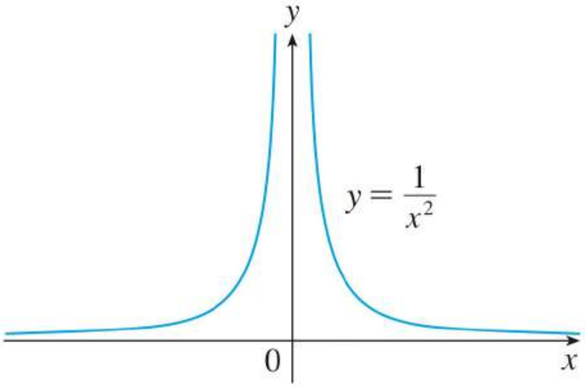


FIGURE 11

EXAMPLE 9 Find $\lim_{x\to 3^+} \dfrac{2x}{x-3}$ and $\lim_{x\to 3^-} \dfrac{2x}{x-3}$.

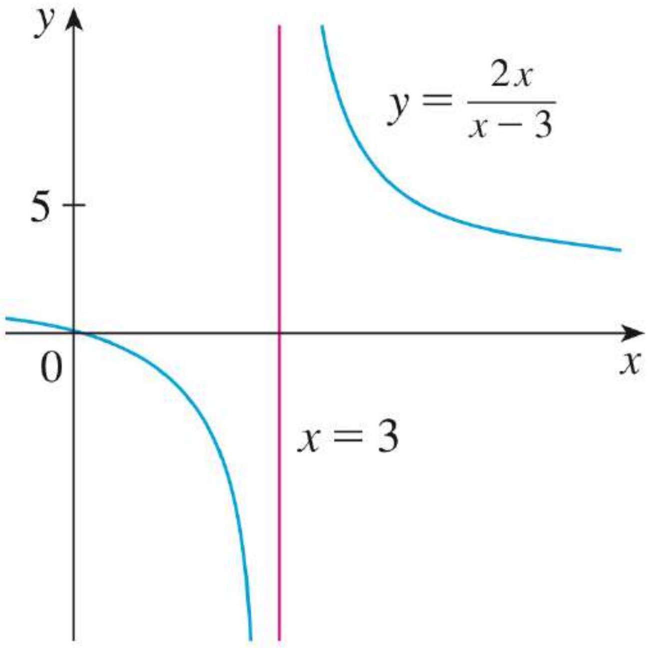


FIGURE 15

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$$

$x = 3$ is V.A

$$\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$$

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

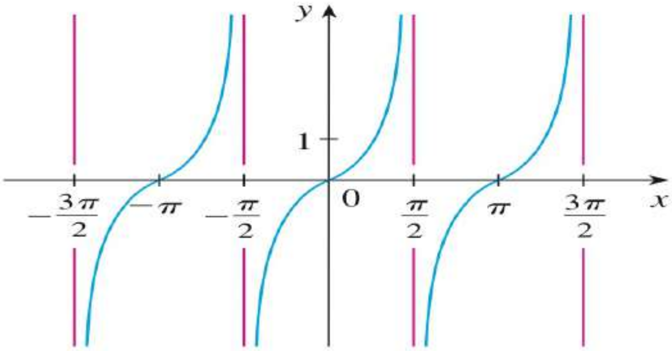
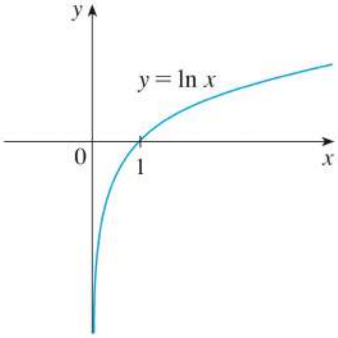


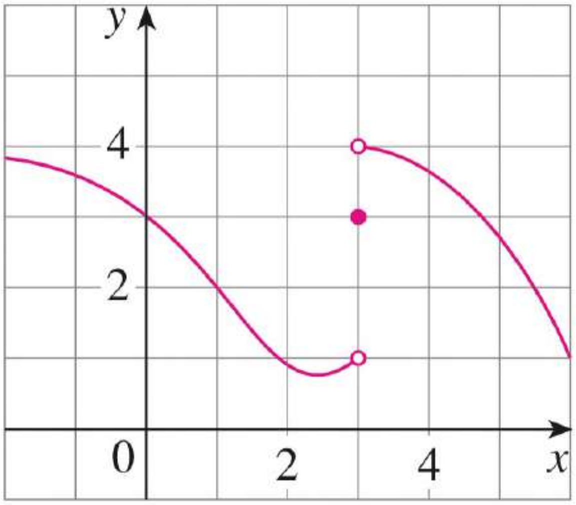
FIGURE 16
$y = \tan x$

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$



$$\lim_{x \to 2^+} \ln(x-2) = \ln(0^+) = -\infty$$

$x = 2$ is V.A



5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3^-} f(x)$ (c) $\lim_{x\to 3^+} f(x)$

(d) $\lim_{x\to 3} f(x)$ (e) $f(3)$ $= 3$

---

a) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$

$\therefore \lim_{x\to 1} f(x) = 2$

b) $\lim_{x\to 3^-} = 1$ c) $\lim_{x\to 3^+} f(x) = 4$

d) $\lim_{x\to 3^-} f(x) = 1$ $\lim_{x\to 3^+} f(x) = 4$

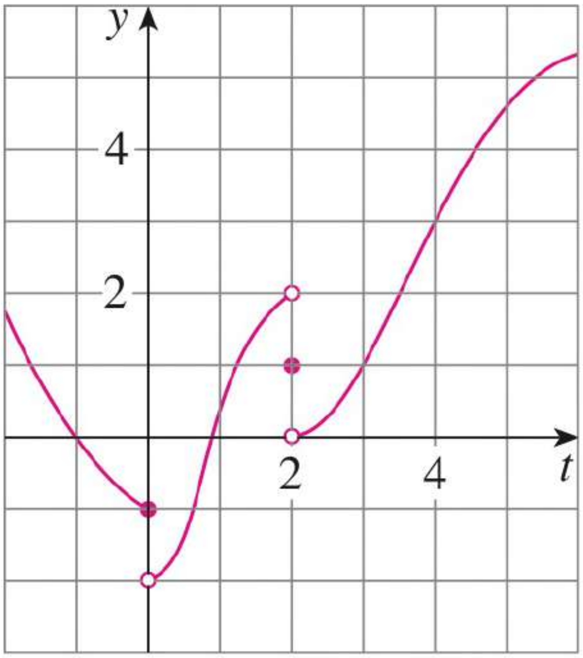
$\therefore \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$ $\therefore$ DNE

7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$



(a) $\lim_{t \to 0^-} g(t)$ -1 (b) $\lim_{t \to 0^+} g(t)$ -2 (c) $\lim_{t \to 0} g(t)$ DNE

(d) $\lim_{t \to 2^-} g(t)$ 2 (e) $\lim_{t \to 2^+} g(t)$ 0 (f) $\lim_{t \to 2} g(t)$ DNE

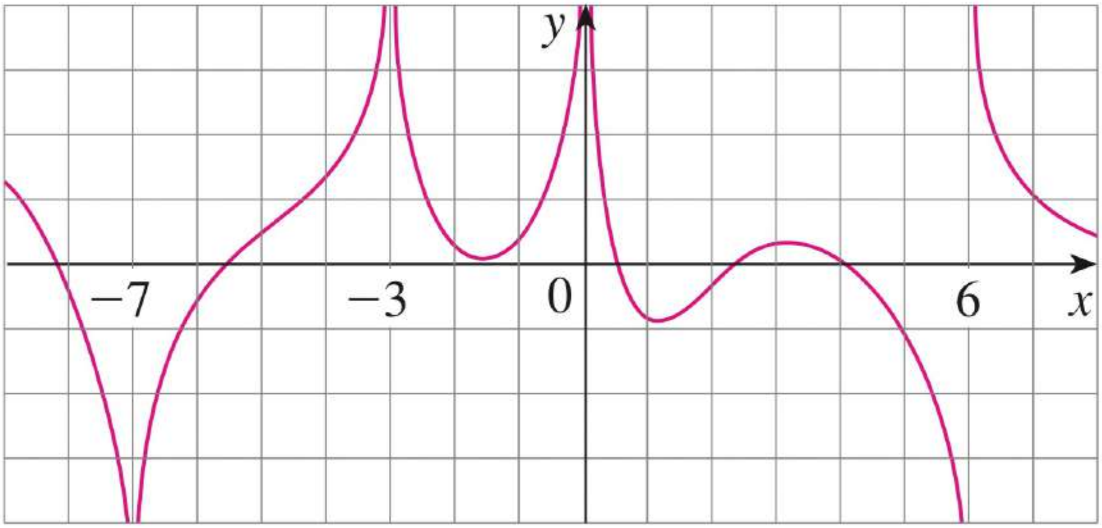
(g) $g(2)$ 1 (h) $\lim_{t \to 4} g(t)$ 3

9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

(f) The equations of the vertical asymptotes.



(a) $\lim_{x \to -7} f(x)$ $-\infty$ (b) $\lim_{x \to -3} f(x)$ ∞ (c) $\lim_{x \to 0} f(x)$ ∞

(d) $\lim_{x \to 6^-} f(x)$ $-\infty$ (e) $\lim_{x \to 6^+} f(x)$ ∞

1. [$2.5 \times 4 = 10$ pts.] Use the given graph of f to evaluate each of the following limits, if it exists.

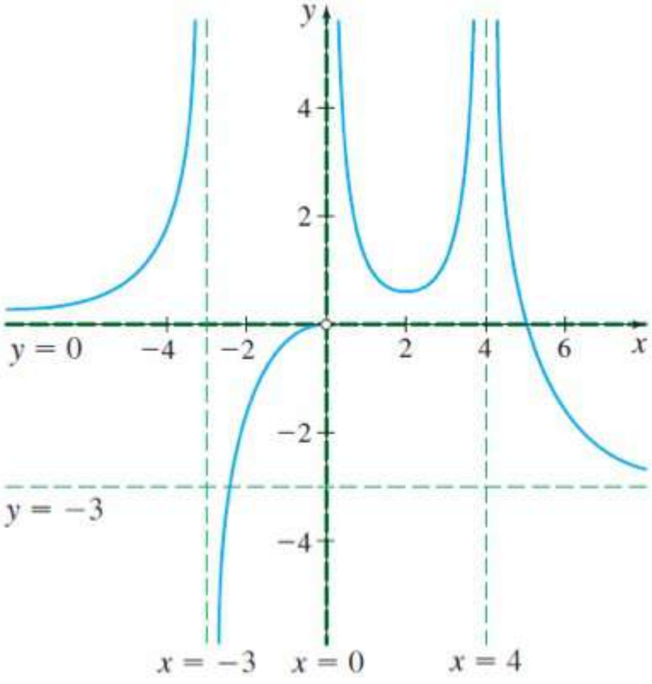


Figure 1: The graph of $y = f(x)$.

(a) $\lim\limits_{x \to -3} f(x) =$

(b) $\lim\limits_{x \to 0^-} f(x) =$

(c) $\lim\limits_{x \to 0^+} f(x) =$

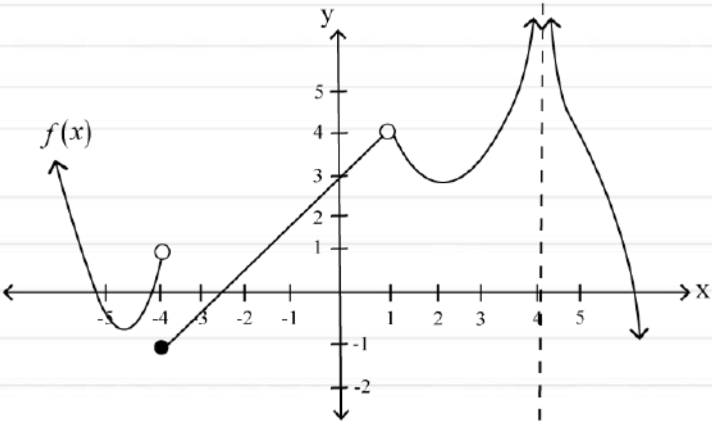
(d) $\lim\limits_{x \to -\infty} f(x) =$

(a) $\lim\limits_{x \to -3} f(x)$ $\boxed{\text{DNE}}$. (It is clear that $\lim\limits_{x \to -3^+} f(x) = -\infty$ and $\lim\limits_{x \to -3^-} f(x) = \infty$).

(b) $\lim\limits_{x \to 0^-} f(x) = \boxed{0}$.

(c) $\lim\limits_{x \to 0^+} f(x) = \boxed{\infty}$.

(d) $\lim\limits_{x \to -\infty} f(x) = \boxed{0}$.



(a) $\lim_{x\to -4^-} f(x) = 1$ (b) $\lim_{x\to -4^+} f(x) = -1$ (c) $\lim_{x\to -4} f(x) =$ does not exist

(d) $\lim_{x\to 1^-} f(x) = 4$ (e) $\lim_{x\to 1^+} f(x) = 4$ (f) $\lim_{x\to 1} f(x) = 4$

(g) $\lim_{x\to 4^-} f(x) = \infty$ (h) $\lim_{x\to 4^+} f(x) = \infty$ (i) $\lim_{x\to 4} f(x) = \infty$

Calc

ملخص :-

١- حدد $x = a$: شوف الخط العامودي عند هذه القيمة.

٢- افحص من اليسار و اليمين :-

* Continuous * Jump

* ASymptote * Hole

2- Direct Substitution.

Solution:-

$5 , \frac{0}{2} = 0 , \frac{1}{0} = \pm\infty , -2 , \frac{1}{3}$

---

1) $\lim_{x \to 3} (2x+1) = 2(3)+1 = 7$

2) $\lim_{x \to -2} (x^2-4) = (-2)^2-4 = 0$

3) $\lim_{x \to 3} \frac{x^2-2x-3}{x-3} = \frac{0}{0} \Rightarrow ??$

4) $\lim_{x \to \infty} x^{-2} \cos\left(\frac{1}{x}\right) = \infty \cdot \cos\infty$

$\Rightarrow ??$

قيمة غير معرفة

إذا عوضنا وطلع الناتج indeterminate forms :

Determinate-Indeterminate Forms Table	
Indeterminate Forms	Determinate Forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty / \pm\infty$	$-\infty - \infty = -\infty$
$\infty(-\infty)$	$0^{\infty} = 0$
$0(\infty)$	$0^{-\infty} = \infty$
0^0	$(\infty) \cdot (\infty) = \infty$
1^{∞}	
∞^0	
$\sin \infty$	S.T
$\cos \infty$	

a) $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$

$= \frac{3^2 - 2(3) - 3}{3 - 3} = \frac{9 - 6 - 3}{0} = \frac{0}{0}$

$\therefore \lim_{x \to 3} \frac{(x-3)(x+1)}{x - 3} = \lim_{x \to 3} \frac{x+1}{1}$

$= \lim_{x \to 3} x + 1 = 3 + 1 = 4$

---

b) $\lim_{x \to 5} \frac{x - 5}{x^2 - 6x + 5} = \frac{0}{0}$

$\therefore \lim_{x \to 5} \frac{x - 5}{(x-5)(x-1)} = \lim_{x \to 5} \frac{1}{x - 1}$

$= \frac{1}{5 - 1} = \frac{1}{4}$

c) $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$

$$\frac{4-4}{\sqrt{4}-2} = \frac{0}{0}$$

$$\therefore \lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} * \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$\lim_{x \to 4} \frac{(x-4)(\sqrt{x}+2)}{(x-4)} = \lim_{x \to 4} \sqrt{x}+2$$

$$= \sqrt{4}+2 = 2+2 = 4$$

---

d) $\lim_{x \to 4} \dfrac{x-55}{x^2-8x+16}$

$$= \frac{4-55}{4^2-8(4)+16}$$

$$= \frac{-51}{16-32+16} = \frac{-51}{32-32} = \frac{51}{0} = \pm\infty$$

e) $\lim_{h \to 0} \frac{(h-3)^2 - 9}{h}$

$$= \lim_{h \to 0} \frac{h^2 - 6h + 9 - 9}{h} = \lim_{h \to 0} \frac{h^2 - 6h}{h}$$

$$= \lim_{h \to 0} \frac{h(h-6)}{h} = \lim_{h \to 0} (h-6) = 0 - 6 = -6$$

f) $$\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \frac{0}{0}$$

$$\therefore \lim_{x\to -4} \frac{\frac{1*x}{4*x}+\frac{1*4}{x*4}}{4+x} = \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x}$$

$$= \lim_{x\to -4} \frac{x+4}{4x(4+x)} = \lim_{x\to -4} \frac{1}{4x}$$

$$= \frac{1}{4(-4)} = \frac{1}{-16}$$

---

g) $$\lim_{x\to -3^{+}} \frac{x+2}{x+3} = \frac{-3+2}{-3+3}$$

$$= \frac{-1}{0} = -\infty \text{ DNE}$$

h) $$\lim_{x \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$$

$$= \lim_{x \to -3} \frac{(t-3)(t+3)}{(2t+1)(t+3)} = \frac{t-3}{2t+1}$$

$$= \frac{-3-3}{2(-3)+1} = \frac{-6}{-6+1} = \frac{-6}{-5} = \frac{6}{5}$$

i) $$\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} * \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}$$

$$\lim_{h \to 0} \frac{9+h-9}{h\sqrt{9+h} + 3} = \lim_{h \to 0} \frac{h}{h\sqrt{9+h} + 3}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{\sqrt{9+0} + 3}$$

$$= \frac{1}{6}$$

J) $\lim\limits_{x \to 4} \dfrac{\sqrt{3} - \sqrt{x-1}}{x-4}$

$$\lim_{x \to 4} \frac{\sqrt{3} - \sqrt{x-1}}{x-4} * \frac{\sqrt{3} + \sqrt{x-1}}{\sqrt{3} + \sqrt{x-1}}$$

$$= \lim_{x \to 4} \frac{3 - (x-1)}{(x-4)(\sqrt{3} + \sqrt{x-1})} = \frac{3 - x + 1}{(x-4)(\sqrt{3} + \sqrt{x-1})}$$

$$= \lim_{x \to 4} \frac{4 - x}{(x-4)(\sqrt{3} + \sqrt{x-1})} = \frac{-(x-4)}{(x-4)(\sqrt{3} + \sqrt{x-1})}$$

$$= \lim_{x \to 4} \frac{-1}{(\sqrt{3} + \sqrt{x-1})} = \frac{-1}{\sqrt{3} + \sqrt{4-1}}$$

$$= \frac{-1}{\sqrt{3} + \sqrt{3}} = \frac{-1}{2\sqrt{3}}$$

4. [10 pts.] Evaluate the limit $\lim\limits_{x\to 3} \dfrac{x-3}{2-\sqrt{x+1}}$, if it exists.

$$\lim_{x\to 3} \frac{x-3}{2-\sqrt{x+1}} * \frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}$$

$$\lim_{x\to 3} \frac{(x-3)(2+\sqrt{x+1})}{4-(x+1)}$$

$$\lim_{x\to 3} \frac{(x-3)(2+\sqrt{x+1})}{3-x} = \frac{\cancel{(x-3)}(2+\sqrt{x+1})}{-\cancel{(x-3)}}$$

$$\lim_{x\to 3} \frac{(2+\sqrt{x+1})}{-1} = -(2+\sqrt{3+1})$$

$$= -(2+2) = -4$$

* piecewise function

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

SOLUTION Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

* ملاحظة :

١) عينك على a مطلق $\underset{x \to a}{\text{limit}}{}^{+}$

٢) صفر المطلق

$$K)\ \lim_{x \to -2^-} \frac{|x+2|}{x+2}$$

$$|x+2| \begin{cases} \rightarrow x+2 , & x \geqslant -2 \\ \rightarrow -(x+2) , & x < -2 \end{cases}$$

$$\lim_{x \to -2^-} \frac{-(x+2)}{x+2} = \frac{-1}{1} = -1$$

2) $\lim_{x \to 0} \frac{|x-4|}{x-4}$

تعويض مباشر

$\lim_{x \to 0} \frac{|0-4|}{0-4} = \frac{4}{-4} = -1$

---

or

$|x-4| = \begin{cases} x-4 & x \geq 4 \\ \boxed{-(x-4)} & x < 4 \end{cases}$

الصفر من جهة اليمين أو اليسار ينتمون للفترة x<4

$\lim_{x \to 0} \frac{-(x-4)}{x-4} = \lim_{x \to 0} -1 = -1$

10. $\lim\limits_{v \to 4^+} \dfrac{4-v}{|4-v|}$

$$\frac{4-4}{|4-4|} = \frac{0}{0}$$

$$|4-r| \begin{cases} \rightarrow 4-r \,, & 4 \geq r \\ \rightarrow -(4-r) \,, & 4 < r \end{cases}$$

$4 < r = r > 4$ لأن يبيها من جهة اليمين

نفس الشي .. لا تتخربط إذا تغيّرت أماكنهم .. بالنهاية أثنينهم يقولولي إن الv أكبر من 4

$$\therefore \lim_{x \to 4^+} \frac{4-r}{|4-r|} = \lim_{x \to 4^+} \frac{\cancel{4-r}}{-(\cancel{4-r})}$$

$$\lim_{x \to 4^+} \frac{1}{-1} = -1$$

2) $$\lim_{x\to -2} \frac{5|x+2|}{x+2}$$

$$|x+2| = \begin{cases} \to x+2 & , x \geq -2 \\ \to -(x+2) & , x < -2 \end{cases}$$

$$\lim_{x\to -2^-} \frac{5(-\cancel{(x+2)})}{\cancel{x+2}} = \frac{-5}{1} = -5$$

$$\lim_{x\to -2^+} \frac{5\cancel{(x+2)}}{\cancel{x+2}} = \frac{5}{1} = 5$$

$$\therefore \lim_{x\to -2} f(x) = DNE$$

---

3) $$\lim_{x\to 6^-} (|x-6|-3)$$

$$|x-6| \begin{cases} \to x-6 & , x \geq 6 \\ \to -(x-6) & , x < 6 \end{cases}$$

$$\lim_{x\to 6^-} f(x) = -(6-6)-3 = -0-3 = -3$$

Evaluate the following limits:

(a) $f(x) = \begin{cases} \frac{x^2-x-6}{|x+2|}, & \text{if } x \neq -2, \\ 5, & \text{if } x = -2. \end{cases}$ $\quad \lim\limits_{x \to -2} f(x)$

$$|x+2| = \begin{cases} \rightarrow x+2 \;, & x \geq -2 \\ \rightarrow -(x+2) \;, & x < -2 \end{cases}$$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{x^2-x-6}{-x-2} = \lim_{x \to -2^-} \frac{(x+2)(x-3)}{-x-2} = \lim_{x \to -2^-} -x+3 = 5.$$

Also, we have

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{x^2-x-6}{x+2} = \lim_{x \to -2^+} \frac{(x+2)(x-3)}{x+2} = \lim_{x \to -2^+} x-3 = -5.$$

$\because \lim_{x\to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$; $\therefore \lim\limits_{x \to -2} f(x) = \text{DNE}$

(b) $f(x) = \dfrac{1}{|e^x - e^{-x}|}$. , $\lim\limits_{x \to 0} f(x)$

$$\lim_{x \to 0^\pm} f(x) = \lim_{x \to 0^\pm} \frac{1}{|e^x - e^{-x}|} = \infty.$$

Evaluate each of the following limits, if it exists.

(a) $\lim\limits_{x \to 1} \dfrac{|x|(x-1)}{x^3 - x} =$

$$|x| \Rightarrow \begin{cases} \to \boxed{x \geq 0} \\ \to -x < 0 \end{cases}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x)$$

$$\lim_{x \to 1} \frac{x(x-1)}{x(x^2-1)} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \frac{1}{2}.$$

2) Squeeze theorem

نستخدم squeeze theorem مع الدوال المثلثية لما يطلع لي الناتج مثلاً :

$(\ldots\ 0 \cdot e^{\sin\infty},\ 0 \sin\infty,\ 0 \cos\infty,\ \infty \sin\infty$

Reminder :-

$$\boxed{-1 \leqslant \sin\theta \leqslant 1} \qquad \boxed{-1 \leqslant \cos\theta \leqslant 1}$$

EXAMPLE 1 Show that $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0.$ $= 0 \sin\infty$

---

1) $-1 \leqslant \sin\frac{1}{x} \leqslant 1$

2) $-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$

3) since $\lim\limits_{x\to 0} -x^2 = 0$

$\lim\limits_{x\to 0} x^2 = 0$, $\therefore$ by S.T

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

a) $\lim_{x\to\infty} x^{-6}\sin\left(\frac{2019}{x^2}\right) = \infty^{-6}\sin\infty$

We Know $-1 \leqslant \sin\left(\frac{2019}{x^2}\right) \leqslant 1$

$\Rightarrow -x^{-6} \leqslant x^{-6}\sin\left(\frac{2019}{x^2}\right) \leqslant x^{-6}$

$\lim_{x\to\infty} -x^{-6} \leqslant \lim_{x\to\infty} x^{-6}\sin\left(\frac{2019}{x^2}\right) \leqslant \lim_{x\to\infty} x^{-6}$

$\lim_{x\to\infty} -x^{-6} = \lim_{x\to\infty} x^{-6} = 0$

by S.T $\therefore \lim_{x\to\infty} x^{-6}\sin\left(\frac{2019}{x^2}\right) = 0$

b) $\lim\limits_{x\to 1}\left[x^4+(x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right]$

$= 1 + (1-2(1)+1)\sin\left(\frac{1}{1-1}\right)$

$= 1 + \boxed{0\ \sin\infty}$

By using Squeeze theorem

$$-1 \leqslant \sin\left(\frac{1}{x-1}\right) \leqslant 1$$

$$-(x^2-2x+1) \leqslant (x^2-2x+1)\sin\left(\frac{1}{x-1}\right) \leqslant (x^2-2x+1)$$

$\because \lim\limits_{x\to 1} -(x^2-2x+1) = -((1)^2-2(1)+1) = 0$

$\because \lim\limits_{x\to 1} (x^2-2x+1) = (1^2-2(1)+1 = 0$

$\therefore \lim\limits_{x\to 1} (x^2-2x+1)\sin\left(\frac{1}{x-1}\right) = 0$ by S.T

$\therefore \lim\limits_{x\to 1}\left[x^4+(x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right] = 1+0$

$= 1$ by squeeze theorem

c) $\lim\limits_{x \to 2} (x^2 - 4x + 4) \cos\left(\frac{2}{x-2}\right)$

1) $4 - 8 + 4 \cos \infty = 0 \cos \infty$

By Using Squeeze theorem

$$-1 \leqslant \cos\left(\frac{2}{x-2}\right) \leq 1$$

$$-(x^2 - 4x + 4) \leqslant (x^2 - 4x + 4) \cos\left(\frac{2}{x-2}\right) \leqslant (x^2 - 4x - 4)$$

$\because \lim\limits_{x \to 2} -(x^2 - 4x + 4) = 0$

$\because \lim\limits_{x \to 2} (x^2 - 4x + 4) = 0$

$\therefore \lim\limits_{x \to 2} (x^2 - 4x + 4) \cos\left(\frac{2}{2-x}\right) = 0$

By Squeeze theorem

d) $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin \frac{\pi}{x}}$

$$-1 \leqslant \sin \frac{\pi}{x} \leqslant 1$$

$$e^{-1} \leqslant e^{\sin \frac{\pi}{x}} \leqslant e$$

$$\sqrt{x}\,\frac{1}{e} \leqslant \sqrt{x}\, e^{\sin \frac{\pi}{x}} \leqslant \sqrt{x}\, e$$

$$\underset{\Downarrow \atop 0}{\lim_{x \to 0^+} \sqrt{x}\,\frac{1}{e}} \leqslant \lim_{x \to 0^+} \sqrt{x}\, e^{\sin \frac{\pi}{x}} \leqslant \underset{\Downarrow \atop 0}{\lim_{x \to 0^+} \sqrt{x}\, e}$$

$$\therefore \lim_{x \to 0^+} \sqrt{x}\, e^{\sin \frac{\pi}{x}} = 0$$

e) $\lim_{x\to\infty} \frac{\cos x}{e^{2x}}$

$$-1 \leqslant \cos x \leqslant 1$$

$$\frac{-1}{e^{2x}} \leqslant \frac{\cos x}{e^{2x}} \leqslant \frac{1}{e^{2x}}$$

$$\underbrace{\lim_{x\to\infty} \frac{-1}{e^{2x}}}_{\Downarrow \atop 0} \leqslant \lim_{x\to\infty} \frac{\cos x}{e^{2x}} \leqslant \underbrace{\lim_{x\to\infty} \frac{1}{e^{2x}}}_{\Downarrow \atop 0}$$

$$\therefore \lim_{x\to\infty} \frac{\cos x}{e^{2x}} = 0$$

1. Evaluate the limits if they exist.

f) $\lim_{x \to 0} x^2 \cos \frac{1}{x^2}$ $= 0 \cos \infty$

Since

$$-1 \leqslant \cos\left(\frac{1}{x^2}\right) \leqslant 1$$

$$-x^2 \leqslant x^2 \cos\left(\frac{1}{x^2}\right) \leqslant x^2$$

Since $\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$ by ST

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$$

b) Use the Squeeze Theorem to evaluate the Following limits:-

$$\lim_{x\to 0} \left(\frac{x}{2+x}\right)^2 \sin\left(\frac{2+x}{x}\right) = 0 \sin \infty$$

$$= \left(\frac{0}{2+0}\right)^2 \sin\left(\frac{2+0}{0}\right) = 0 \,.\, \sin\infty$$

$\therefore$ S.T

$$-1 \leq \sin\left(\frac{2+x}{x}\right) \leq 1$$

$$-\left(\frac{x}{2+x}\right)^2 \leq \left(\frac{x}{2+x}\right)^2 \sin\left(\frac{2+x}{x}\right) \leq \left(\frac{x}{2+x}\right)^2$$

$$\lim_{x\to 0} -\left(\frac{x}{2+x}\right)^2 = -\left(\frac{0}{2}\right)^2 = 0$$

$$\lim_{x\to 0} \left(\frac{x}{2+x}\right)^2 = -\left(\frac{0}{2}\right)^2 = 0$$

$\therefore$ By S.T $\lim_{x\to 0} \left(\frac{x}{2+x}\right)^2 \sin\left(\frac{2+x}{x}\right) = 0$

Evaluate each of the following limits, if it exists.

(b) $\lim\limits_{x\to 2^+} \sqrt{x-2}\cos\left(\dfrac{1}{x-2}\right).$

We have

$$-1 \leqslant \cos\left(\frac{1}{x-2}\right) \leqslant 1 \Rightarrow -\sqrt{x-2} \leqslant \sqrt{x-2}\cos\left(\frac{1}{x-2}\right) \leqslant \sqrt{x-2}.$$

Since $\lim\limits_{x\to 2^+} \sqrt{x-2} = \lim\limits_{x\to 2^+} (-\sqrt{x-2}) = 0$, thus by Squeeze theorem, we have

$$\lim_{x\to 2^+} \sqrt{x-2}\cos\left(\frac{1}{x-2}\right) = 0.$$

* Squeeze theorem + inequity

If $2x - 1 \le f(x) \le x^2 - 2x + 3$ for $x \ge 0$, find $\lim_{x \to 2} f(x)$.

$$\lim_{x \to 2} 2x - 1 = 3$$

$$\lim_{x \to 2} x^2 - 2x + 3 = 3$$

$$\therefore \lim_{x \to 2} f(x) = 3 \text{ by S.T}$$

5. [10 pts.] If $e^x \leqslant f(x) \leqslant \dfrac{4x^2+1}{x^2}$, find $\lim\limits_{x\to 0} x^4 f(x)$, if it exists.

We multiply the given inequality by x^4 to obtain

$x^4 e^x \leqslant x^4 f(x) \leqslant \dfrac{4x^6+x^4}{x^2}$. Now we have

$\lim\limits_{x\to 0} x^4 e^x = 0$, $\lim\limits_{x\to 0} \dfrac{4x^6+x^4}{x^2} = \lim\limits_{x\to 0} 4x^4 + x^2 = 0$. As a result of Squeeze Theorem,

$\lim\limits_{x\to 0} x^4 f(x) = 0$.

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6. [10 pts.] Let $f(x)$ be a function which satisfies

$$5x - 6 \le f(x) \le x^2 + 3x - 5 \text{ for all } x \ge 0.$$

Find $\lim\limits_{x\to 1} f(x)$.

Since $\lim\limits_{x\to 1} 5x - 6 = -1$ and $\lim\limits_{x\to 1} x^2 + 3x - 5 = -1$,

then by the Squeeze Theorem $\boxed{\lim\limits_{x\to 1} f(x) = -1}$.

ثمرة

Thamrah

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Calculus 1 – Limits

(Section 2.4)

For Contact and Support:

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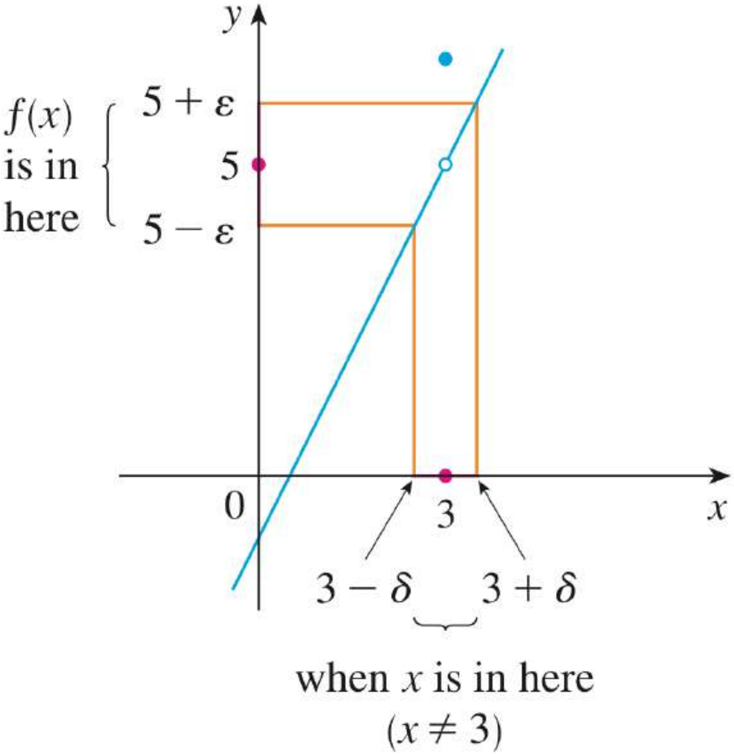
1 Use the ϵ-δ definition of a limit

2 **Precise Definition of a Limit** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$



حل عن طريق δ و ε :-

1) For any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

2) $|f(x) - L| < \varepsilon$ افرض المطلوب

3) $|x - c|$ حولها لشكل

4) $\delta \Leftarrow$ ε حدد قيمة δ بدلالة

5) $\therefore$ By definition $\lim\limits_{x \to c} f(x) = L$

15–18 Prove the statement using the ε, δ definition of a limit

15. $\lim_{x \to 3} \left(1 + \frac{1}{3}x\right) = 2$

$f(x) = 1 + \frac{1}{3}x$ $\qquad 0 < |x-3| < \delta$

$L = 2$

$a = 3$

1) For any $\varepsilon > 0$, $\exists\, \delta > 0$ Such that

$$|f(x) - L| < \varepsilon$$

$$|1 + \frac{1}{3}x - 2| < \varepsilon$$

$1 \div \frac{1}{3} = 1 * 3 = 3$

$|\frac{1}{3}x - 1| < \varepsilon \qquad |\frac{1}{3}(x-3)| < \varepsilon$

$\Rightarrow \frac{1}{3}|x-3| < \varepsilon \Rightarrow |x-3| < 3\varepsilon$

2) Let $0 < |x-3| < \delta \Rightarrow 0 < |x-3| < 3\varepsilon$

$\frac{1}{3}|x-3| < \varepsilon \Rightarrow |\frac{1}{3}x - 1| < \varepsilon$

$|\frac{1}{3}x - 2 + 1| < \varepsilon \Rightarrow |1 + \frac{1}{3}x - 2| < \varepsilon$

$\therefore$ By definition $\lim_{x \to 3} (1 + \frac{1}{3}x) = 2$

15–18 Prove the statement using the ε, δ definition of a limit

18. $\lim_{x \to -2} (3x + 5) = -1$

$f(x) = 3x + 5$
$L = -1$
$a = -2$

1) For any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - L| < \varepsilon$

$|3x + 5 + 1| < \varepsilon$

$|3x + 6| < \varepsilon \rightarrow |3(x+2)| < \varepsilon$

$\rightarrow 3|x+2| < \varepsilon \rightarrow |x+2| < \frac{\varepsilon}{3}$

2) Let $0 < |x+2| < \delta \rightarrow 0 < |x+2| < \frac{\varepsilon}{3}$

$3|x+2| < \varepsilon \rightarrow |3(x+2)| < \varepsilon$

$\rightarrow |3x+6| < \varepsilon \rightarrow |3x+5+1| < \varepsilon$

$\therefore \lim_{x \to -2} (3x+5) = -1$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x\to 1} \frac{2+4x}{3} = 2$

$f(x) = \frac{2+4x}{3}$ $0 < |x-1| < \delta$

$L = 2$

$a = 1$

1) For any $\varepsilon > 0$, $\exists$ $\delta > 0$ Such that

$$|f(x) - L| < \varepsilon$$

$$\left|\frac{2+4x}{3} - 2\right| < \varepsilon$$

$\left|\frac{4x-4}{3}\right| < \varepsilon$ $\left|\frac{4}{3}(x-1)\right| < \varepsilon$

$\Rightarrow \frac{4}{3}|x-1| < \varepsilon \Rightarrow |x-1| < \frac{3\varepsilon}{4}$

2) Let $0 < |x-1| < \delta$

$\frac{4}{3}|x-1| < \varepsilon \Rightarrow \left|\frac{4}{3}x - \frac{4}{3}\right| < \varepsilon$

$\left|\frac{4x-4}{3}\right| < \varepsilon \Rightarrow \left|\frac{4x+2-6}{3}\right| < \varepsilon$

$\left|\frac{4x+2}{3} - \frac{6}{3}\right| < \varepsilon$ $\left|\frac{4x+2}{3} - 2\right| < \varepsilon$

$\therefore$ By definition $\lim_{x\to 1} \frac{2+4x}{3} = 2$

19–32 Prove the statement using the ε, δ definition of a limit.

20. $\lim_{x \to 10} \left(3 - \frac{4}{5}x\right) = -5$

$f(x) = 3 - \frac{4}{5}x$, $L = -5$, $a = 10$

$0 < |x - 10| < \delta$

1) for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

$$\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$$

$\left|3 - \frac{4}{5}x + 5\right| < \varepsilon$ $\quad$ $\left|8 - \frac{4}{5}x\right| < \varepsilon$

$\Rightarrow \left|-\frac{4}{5}(x - 10)\right| < \varepsilon \Rightarrow \frac{4}{5}|x - 10| < \varepsilon \Rightarrow$
$|x - 10| < \frac{5}{4}\varepsilon$

2) Let $0 < |x - 10| < \delta \Rightarrow 0 < |x - 10| < \frac{5}{4}\varepsilon$
$\frac{4}{5}|x - 10| < \varepsilon \Rightarrow \left|\frac{4}{5}x - 8\right| < \varepsilon$

$\left|-\left(8 - \frac{4}{5}x\right)\right| < \varepsilon \Rightarrow \left|8 - \frac{4}{5}x\right| < \varepsilon \Rightarrow \left|3 - \frac{4}{5}x + 5\right| < \varepsilon$

$\therefore$ By definition $\lim_{x \to 10} \left(3 - \frac{4}{5}x\right) = -5$

19–32 Prove the statement using the ε, δ definition of a limit.

23. $\lim_{x \to a} x = a$

$f(x) = x$

$L = a$

$a = a$

$0 < |x - a| < \varepsilon$

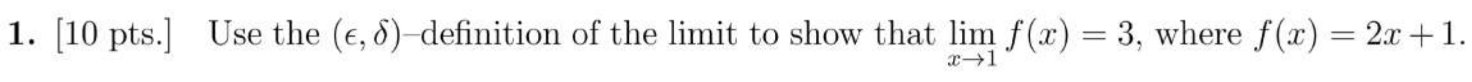
1) for any $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

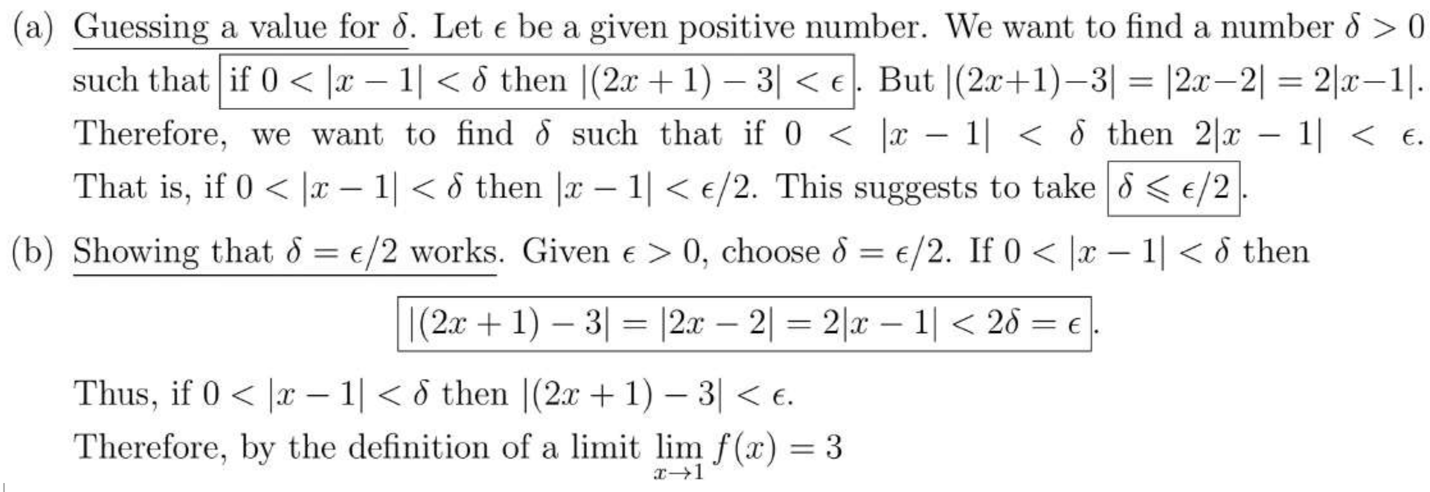
$$|x - a| < \varepsilon$$

2) Let $0 < |x - a| < \delta \Rightarrow 0 < |x - a| < \varepsilon$

$$\therefore \lim_{x \to a} x = a$$



1. [10 pts.] Use the (ϵ, δ)–definition of the limit to show that $\lim_{x\to 1} f(x) = 3$, where $f(x) = 2x + 1$.



(a) Guessing a value for δ. Let ϵ be a given positive number. We want to find a number $\delta > 0$ such that $\boxed{\text{if } 0 < |x-1| < \delta \text{ then } |(2x+1)-3| < \epsilon}$. But $|(2x+1)-3| = |2x-2| = 2|x-1|$. Therefore, we want to find δ such that if $0 < |x-1| < \delta$ then $2|x-1| < \epsilon$. That is, if $0 < |x-1| < \delta$ then $|x-1| < \epsilon/2$. This suggests to take $\boxed{\delta \leqslant \epsilon/2}$.

(b) Showing that $\delta = \epsilon/2$ works. Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x-1| < \delta$ then

$$\boxed{|(2x+1)-3| = |2x-2| = 2|x-1| < 2\delta = \epsilon}.$$

Thus, if $0 < |x-1| < \delta$ then $|(2x+1)-3| < \epsilon$.

Therefore, by the definition of a limit $\lim_{x\to 1} f(x) = 3$

أو تقدر تحلها بنفس الطريقة الي حلينا فيها مسائل الي فاتت

$$\lim_{x\to 1}(2x+1)=3$$

Kuwait University

Calculus 1 – Continuity

(Section 2.5)

For Contact and Support:

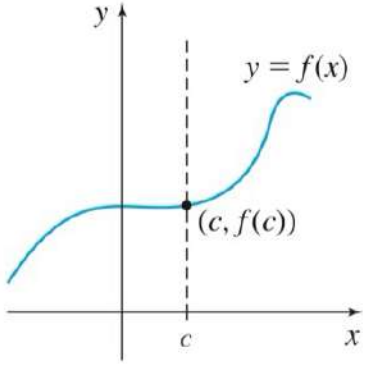
YouTube: Precalculusq8 Twitter: Precalculusq8

3) Continuity and discontinuity

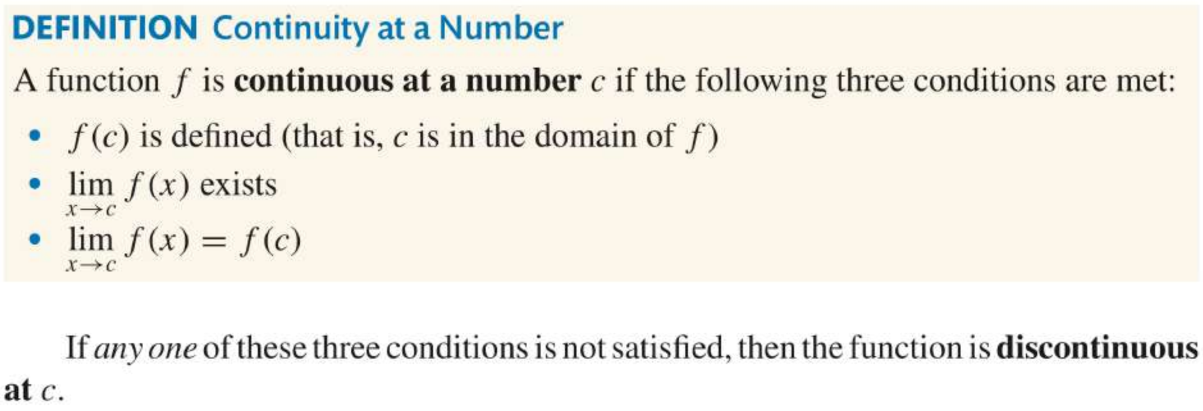
a) $f(a)$ is defined.

b) $\lim_{x \to a} f(x)$ exist.

c) $\lim_{x \to a} f(x) = f(a)$.



(a) $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$



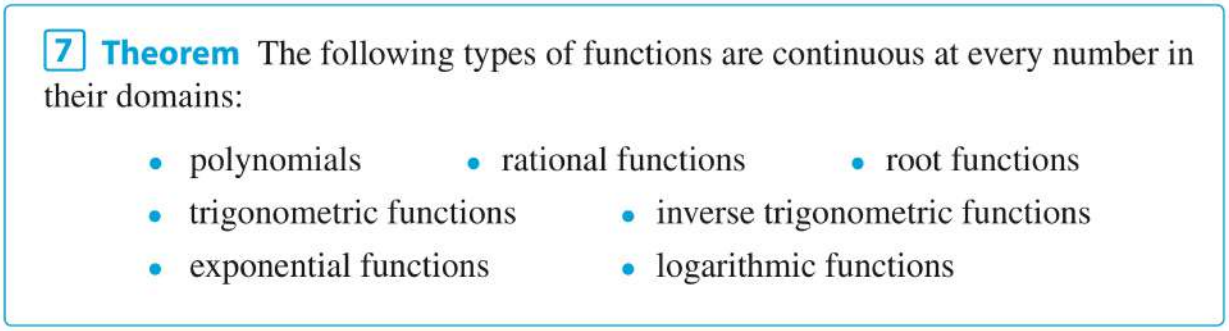
DEFINITION Continuity at a Number

A function f is **continuous at a number** c if the following three conditions are met:

- $f(c)$ is defined (that is, c is in the domain of f)
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

If *any one* of these three conditions is not satisfied, then the function is **discontinuous at** c.

* Where the function Continuous :-



7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

* أي دالة فوق فهي مستمرة في مجالها

* الحل = حدد الـ Domain ، وبعدين نقول إن الدالة مستمرة (cont..) في هذا المجال .

تذكر :-

١) المقام ≠ ٠ .

٢) داخل جذر الزوجي ≥ ٠ .

٣) داخل اللوغاريتم ln( ) ، log( ) .

٤) الدوال المجمعة :

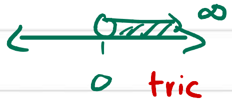
إذا في أكثر من شرط (كسر ، جذر ، لوغاريتم) ناخذ التقاطع بين الشروط .

EXAMPLE 6 Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

---

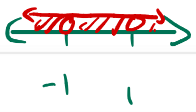
f(x) is Combination of : logarithmic , inverse trigonome and rational function "All are Continuous on their Domain".

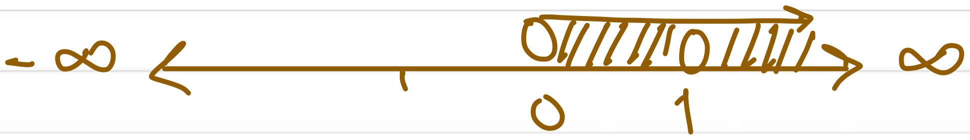
$y = \ln x$, D: $(0, \infty)$



$y = \tan^{-1} x$ D: R

$y = x^2 - 1$ D: $R / \{-1, 1\}$





$\therefore$ f is Cont on $(0,1) \cup (1,\infty)$

On what interval are the following function continuous

$$f(x) = \sqrt{x} + \frac{2x}{x^2+1}$$

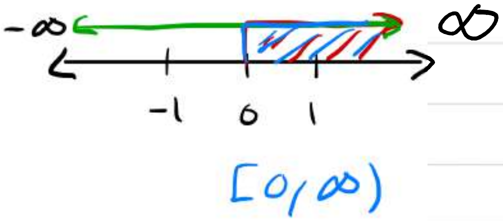
f(x) is combination of : root and Rational functions

" all are continuous on their Domain" [Theorem]

Domain of $\sqrt{x} = [0, \infty)$

Domain of $\frac{2x}{x^2+1} = \mathbb{R}$

∴ Domain of $f = [0, \infty)$



On what interval are the following function continuous

$$\text{Evaluate } \lim_{x \to \pi} \frac{\sin x}{2 + \cos x}.$$

f(x) is Trigonometric function

" continuous on it's Domain" [Theorem]

Domain of $\sin x$: $\mathbb{R}$

Domain of $2 + \cos x$: $2 + \cos x \neq 0$

$\cos x \neq -2$ Never

∴ $\mathbb{R}$

$$\therefore \lim_{x \to \pi} \frac{\sin x}{2+\cos x} = \frac{\sin \pi}{2+\cos \pi} = \frac{0}{2+(-1)} = \frac{0}{1} = 0$$

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 9 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

---

a) $h(x) = \sin(x^2)$

Trig.

→ $\sin x$, continuous on $\mathbb{R}$

→ x^2 continuous on $\mathbb{R}$

→ poly

b) $\ln(1+\cos x)$

→ $\ln x$, continuous on $(0,\infty)$

→ $1+\cos x$, continuous on $\mathbb{R}$

$\therefore 1+\cos x > 0 \Rightarrow \cos x > -1$

we know $-1 \leq \cos x \leq 1$

$\Rightarrow \cos x \neq -1$

$\{x \mid x \neq n\pi$, n is odd number

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

28. $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$

$f(t)$ is combination of : exponential & Trigonometric

Domain of $e^{\sin t}$: $\mathbb{R}$

Domain of $2+\cos \pi t$: $\mathbb{R}$

$\because 2+\cos\pi t \neq 0$, $\Rightarrow \cos\pi t \neq -2$

impossible

« all are continuous on their Domain" [Theorem]

I. The function $f(x) = \dfrac{3^x}{(x-2)\sqrt{1-x^2}}$ is continuous on the interval

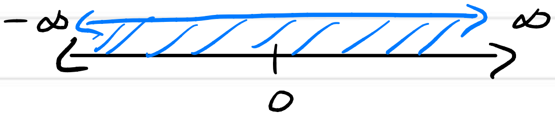
a) $(-\infty, \infty)$. c) $(-1, 2)$.

✓ b) $(-1, 1)$. d) $(1, 2)$.

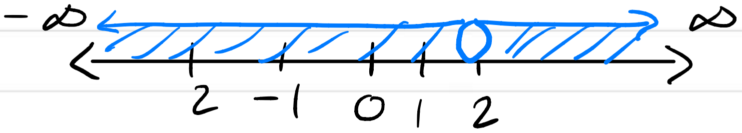
c) $(-1, 2)$. e) None of the above.

The function is continuous on its domain

1) The domain of 3^x is $\mathbb{R}$



2) The domain of $x-2$ is $\mathbb{R}/\{2\}$

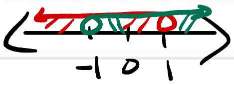


3) The domain of $\sqrt{1-x^2}$:-

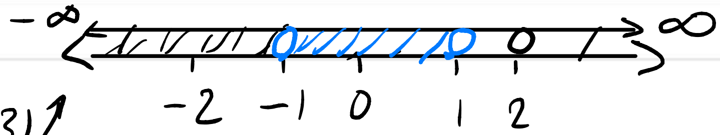
$1-x^2 > 0 \Rightarrow 1 > x^2 \Rightarrow \sqrt{1} > \sqrt{x^2}$

$\boxed{1 > |x| \text{ or } |x| < 1}$ same

$-1 \leq x \leq 1$

∴ 

∴ $D_{\sqrt{1-x^2}}$: $(-1, 1)$

∴ 

cont. on $(-1, 1)$

1),2),3) ↗

١) شروط الاستمرارية عند $x = a$

1. $f(a)$ مُعرفة

2. $\lim_{x \to a^-} f(x)$ و $\lim_{x \to a^+} f(x)$ exist.

3. $\lim_{x \to a} f(x) = f(a)$

* الخلاصة : قيمة الدالة = نهاية اليمين = نهاية اليسار عند النقطة.

* طريقة حل سؤال الاستمرارية.

١) حدد نقطة الشك.

٢) احسب $\lim_{x \to a^-} f(x)$ و $\lim_{x \to a^+} f(x)$

٣) احسب $f(a)$

٤) إن وجد مجهول / مجاهيل : ساوي النهايات ببعض أو مع الصورة لتحصل على معادلة / معادلتين.

Show that f is Continuous on $(-\infty, \infty)$.

$$\lim_{x \to 1^-} f(x) = f(1)$$

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \boxed{\leq} 1 \\ \ln x & \text{if } x > 1 \end{cases}$$

$$f(1) = 1 - 1^2 = 0$$

$$\lim_{x \to 1^-} f(x) = 1 - 1^2 = 0$$

$$\lim_{x \to 1^+} f(x) = \ln 1 = 0$$

$$\therefore f(1) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 0$$

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$$

$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$f\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^+} f(x) = \frac{\sqrt{2}}{2}$$

For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if} \quad x < 2 \\ x^3 - cx & \text{if} \quad x \geq 2 \end{cases} \longrightarrow f(2) = \lim_{x \to 2^+} f(x)$$

---

f(x) will be continuous at $x = 2$ if and only if :-

$$f(2) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

$$\lim_{x \to 2^+} (x^3 - cx) = \lim_{x \to 2^-} (cx^2 + 2x)$$

$$8 - 2c = 4c + 4$$

$$4 = 6c$$

$$c = 4/6 = \frac{2}{3}$$

$$f(x) = \begin{cases} ax^2+3, & \text{if } x \leqslant -1 \\ 3x+4, & \text{if } x > -1 \end{cases}$$

$f(x)$ Continuous , $a = ??$

$\because$ Continuous , $\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(1)$

1) $\lim_{x \to -1^+} 3x+4 = -3+4 = 1$

2) $\lim_{x \to -1^-} ax^2+3 = a+3$

3) $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$a+3 = 1 \Rightarrow a = 1-3 = -2$

$a = -2$

2. (3 points) For what value of the constant a is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^2 + 2a & \text{if} \quad x < 2 \\ xa^2 & \text{if} \quad x \geq 2 \end{cases}$$

$\because$ Continuous , $\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$

1) $\lim_{x \to 2^-} x^2 + 2a = 4 + 2a$

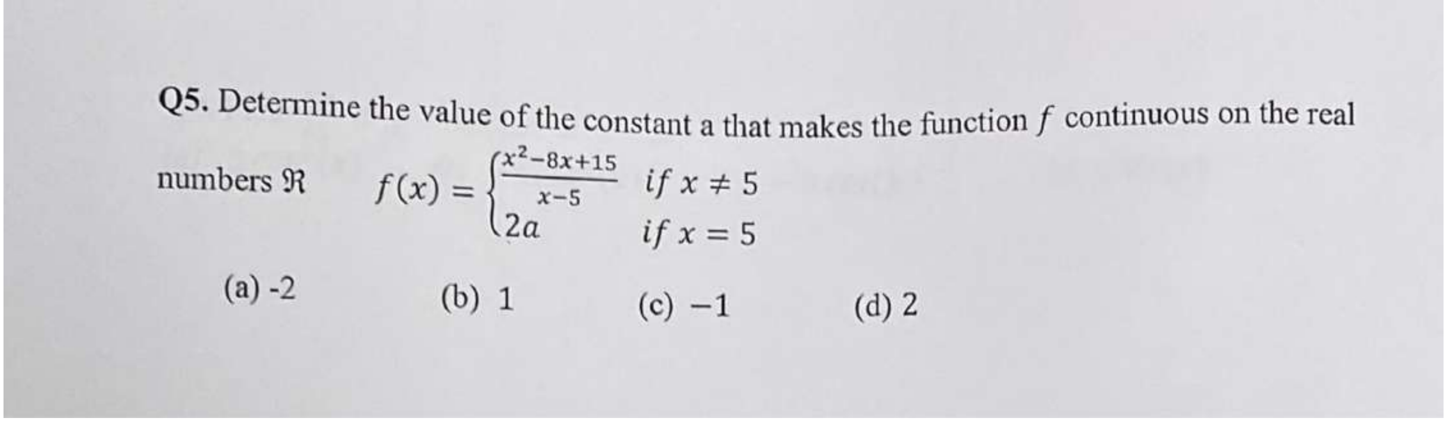
2) $\lim_{x \to 2^+} xa^2 = 2a^2$

3) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

$4 + 2a = 2a^2 \Rightarrow 2a^2 - 2a - 4 = 0$ $\div 2$

$a^2 - a - 2 = 0 \Rightarrow (a-2)(a+1)$

$\Rightarrow a = 2$, $a = -1$, Check !



Q5. Determine the value of the constant a that makes the function f continuous on the real numbers $\Re$

$$f(x) = \begin{cases} \frac{x^2-8x+15}{x-5} & if\ x \neq 5 \\ 2a & if\ x = 5 \end{cases}$$

(a) -2　　(b) 1　　(c) -1　　(d) 2

$\because f(x)$ is Cont...

$$\therefore f(5) = \lim_{x \to 5} f(x)$$

$$2a = \lim_{x \to 5} \frac{x^2 - 8x + 15}{x - 5}$$

$$2a = \lim_{x \to 5} \frac{(x-5)(x-3)}{x-5}$$

$$2a = \lim_{x \to 5} (x-3) = 2$$

$$\therefore 2a = 2 \qquad \Rightarrow a = 1$$

*طريقة حل معادلتين بالمجهولين :-

الطريقة الأولى :-

١) رتب المعادلتين .

٢) وحد معامل أحد المجهولين اضرب المعادلة / معادلتين .

٣) اجمع / اطرح المعادلتين لإلغاء مجهول واحد .

٤) احسب المجهول المتبقي .

٥) عوض به في إحدى المعادلتين لإيجاد المجهول الآخر .

٦) تحقق سريعاً بالتعويض في الشرط الأصلي .

الطريقة الثانية ، التعويض (Subsitution) :-

١) من إحدى المعادلتين عزل مجهولاً (مثلاً = b) .

٢) عوض في المعادلة التالية .

٣) احسب المجهول الأول .

٤) عوض به لإيجاد المجهول الثاني .

٥) تحقق بالتعويض في الشروط الأصلية .

2. For what value of a and b is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^3 - 6a & \text{if} \quad x < -3 \\ 3 & \text{if} \quad x = -3 \\ bx - a & \text{if} \quad x > -3 \end{cases}$$

Since $f(x)$ is Continuous on $(-\infty, \infty)$

$\Rightarrow f(x)$ is Continuous at $x = -3$

$$\Rightarrow \lim_{x\to -3^+} f(x) = \lim_{x\to -3^-} f(x) = f(3)$$

$$\lim_{x\to -3^+} f(x) = f(-3) \Rightarrow \lim_{x\to -3^+} (bx - a) = 3$$

$$\Rightarrow -3b - a = 3$$

$$\Rightarrow \boxed{a + 3b = -3} \rightarrow ①$$

Als $\lim_{x\to -3^-} f(x) = f(-3)$

$$\Rightarrow \lim_{x\to -3^-} (x^3 - 6a) = 3$$

$$-27 - 6a = 3$$

$$-6a = 3 + 27$$

$$-6a = 30 \Rightarrow \boxed{a = -5}$$

① $\Rightarrow$ $-5 + 3b = -3$

$$3b = 5 - 3 \Rightarrow 3b = 2$$

$$\boxed{b = 2/3}$$

48. $$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

At $x=2$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x+2)(x-2)}{x-2}$

$= \lim_{x\to 2^-} (x+2) = 2+2 = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$

we must have $4a - 2b + 3 = 4$, or $4a - 2b = 1$ ①

At $x=3$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x - a + b) = 6 - a + b$

we must have $9a - 3b + 3 = 6 - a + b$, or $10a - 4b = 3$ ②

$$\begin{array}{r} -8a + 4b = -2 \\ 10a - 4b = 3 \\ \hline 2a \quad = 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in (1) gives us $-2b = -1$ so $b = \frac{1}{2}$ as well, Thus for f to be continous on $(-\infty, \infty)$

$a = b = \frac{1}{2}$

2. For what value of a and b is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^3 - 6a + 3 & \text{if} \quad x < -5 \\ 3 & \text{if} \quad x = -5 \\ bx - a + 2 & \text{if} \quad x > -5 \end{cases}$$

f(x) is Contiuous at $x = -5$

$$\Rightarrow \lim_{x \to -5^-} f(x) = \lim_{x \to -5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to -5^-} (x^3 - 6a + 3) = \lim_{x \to -5^+} (bx - a + 2) = 3$$

$$\Rightarrow \lim_{x \to -5^-} (x^3 - 6a + 3) = 3$$

$$(-5)^3 - 6a + 3 = 3$$

$$\Rightarrow 6a = -125$$

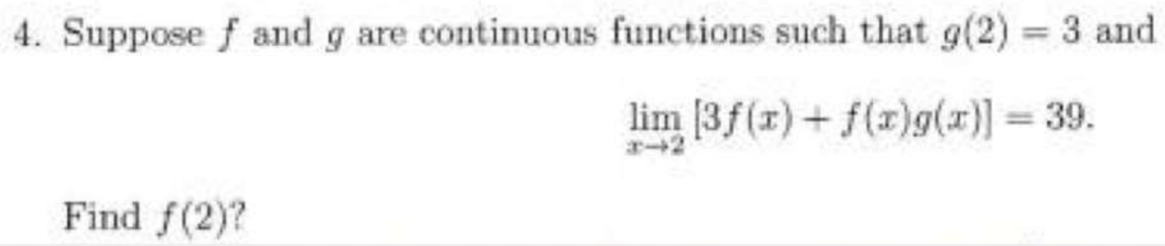
$$\boxed{a = -\frac{125}{6}}$$

Also $\lim_{x \to -5^+} (bx - a + 2) = 3$

$$\Rightarrow -5b + \frac{125}{6} + 2 = 3$$

$$\boxed{\Rightarrow b = \frac{119}{30}}$$

* finding function value using limits & continuity



4. Suppose f and g are continuous functions such that $g(2) = 3$ and

$$\lim_{x \to 2} [3f(x) + f(x)g(x)] = 39.$$

Find $f(2)$?

$$\lim_{x \to 2} [3f(x) + f(x)g(x)] = 39$$

$$[3f(2) + f(2)g(2)] = 39$$

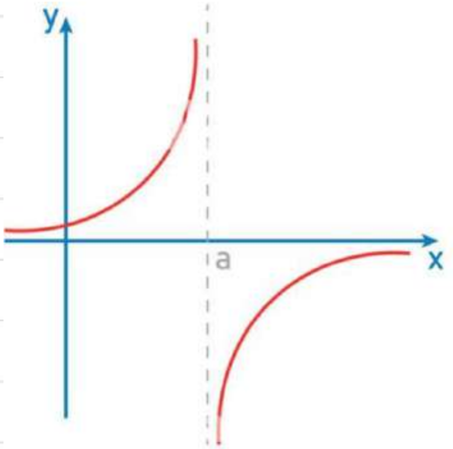
$$3f(2) + f(2)(3) = 39$$

$$3f(2) + 3f(2) = 39$$

$$6f(2) = 39 \Rightarrow f(2) = \frac{39}{6}$$

$$= \frac{13}{2}$$

* Types of discontinuity



Infinite Discontinuity

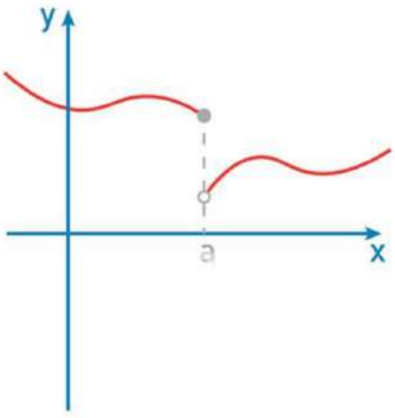
$$\lim_{x\to a^+} f(x) = \pm\infty$$

or

$$\lim_{x\to a^-} f(x) = \pm\infty$$

النهاية إلى خط مقارب عامودي لي انفنتي

* infinite discontinuity

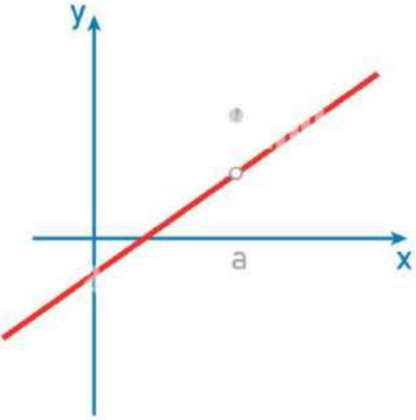


Jump Discontinuity

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$$

نهاية اليمين ≠ نهاية اليسار

* Jump discontinuity



Removable Discontinuity

$\lim_{x\to a} f(x)$ exist

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} f(x)$$

$$\lim_{x\to a} f(x) \neq f(a)$$

النهاية موجودة لكن غير معرفة

* Removable discontinuity

b) Find and classify the discontinuities of f as removable, infinite or jump.

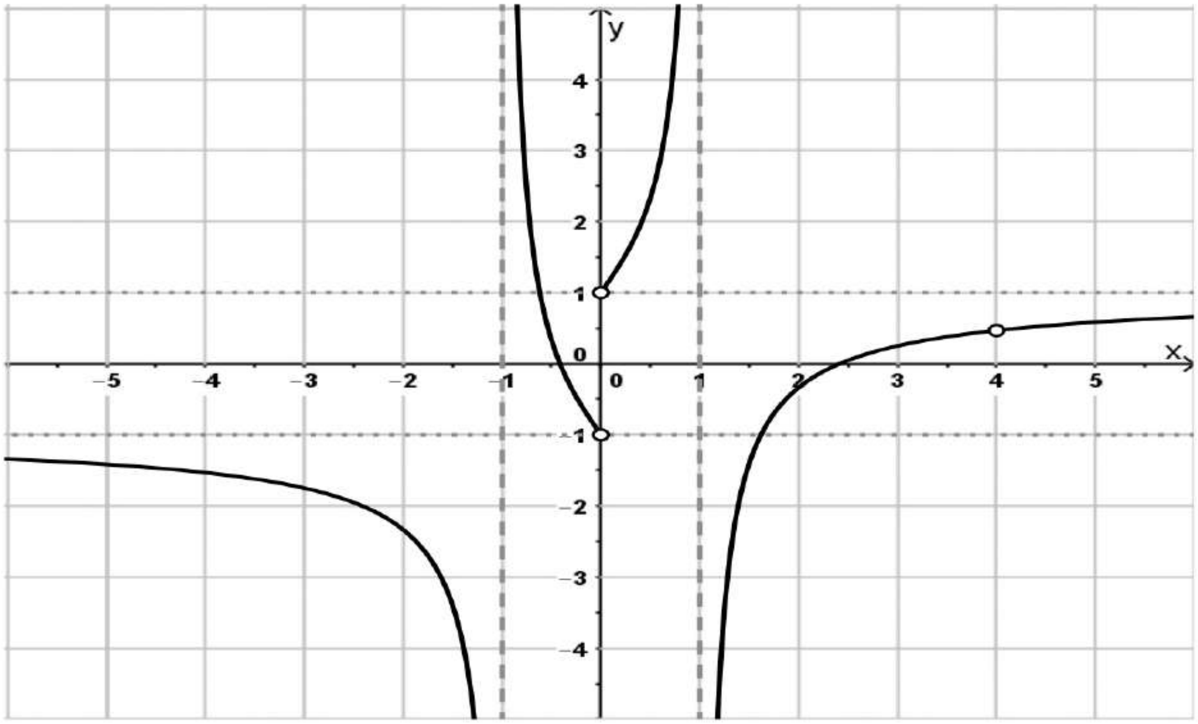
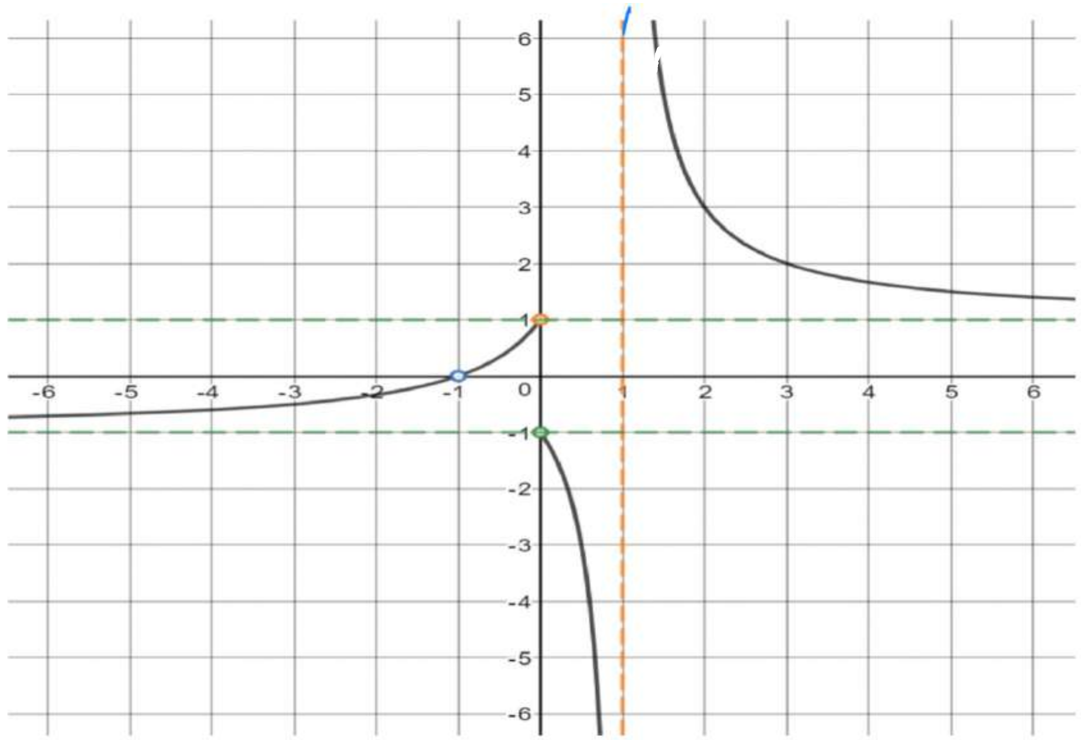


Figure 1: The graph of $y = f(x)$.

---

b) f is discontinuous at $\boxed{x = -1 \text{ (infinite)}, 0 \text{ (jump)}, 1 \text{ (infinite)}, 4 \text{ (removable)}}$.

1. [$5 \times 4 = 20$ pts.] Use the given graph of f to select the correct answer.



(I) $\lim_{x \to 0} f(x) =$

(a) 1 (b) 0 (c) ∞ (d) -1 [✓ (e) None of the mentioned]

(II) $\lim_{x \to 1^+} f(x) =$

[✓ (a) $+\infty$] (b) $-\infty$ (c) 1 (d) 0 (e) None of the mentioned

---

(IV) The function f has

a) a jump discontinuity at $x = 1$

b) a removable discontinuity at $x = 0$

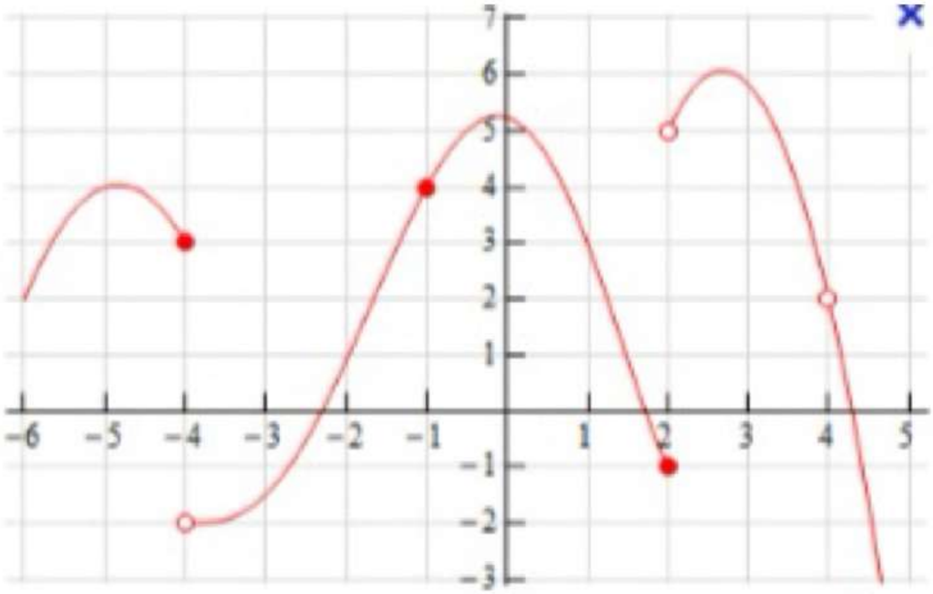
c) a jump discontinuity at $x = -1$

[✓ d) a removable discontinuity at $x = -1$]

e) None of the above

The graph of function f(x) is given below. Find the following limits. If limit does not exist, write DNE.

b) Find and classify the discontinuities of f as removable, infinite or jump.



1) $f(4) + \lim_{x\to 4} f(x)$ = undefind + 2 = DNE

2) $\lim_{x\to -4^-} f(x) + \lim_{x\to 4} f(x)$ = 3 + 2 = 5

3) $\lim_{x\to 2^-} f(x)$ = -1

4) $\lim_{x\to 4^+} f(x) + 2 \lim_{x\to 2^+} f(x)$ = 2 + 2 (5) = 12

5) $f(-1) + 2 \lim_{x\to -4^+} f(x)$ = 4 + 2 (-2) = 0

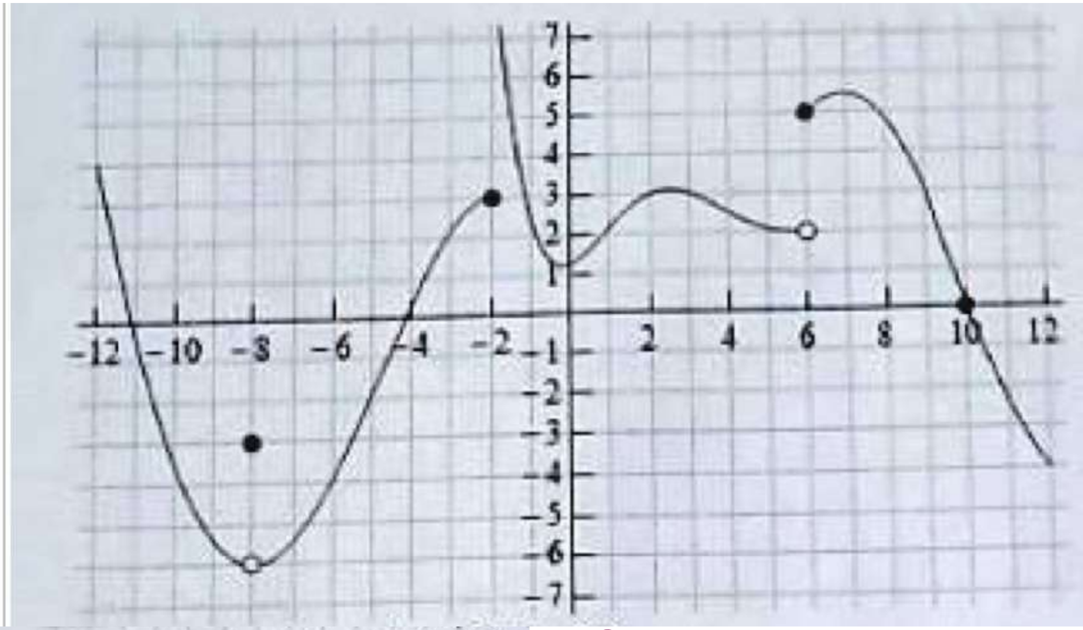
b) f is discontinuous at $x = -4, 2, 4$

at $x = -4$, $\lim_{x \to -4^-} f(x) \neq \lim_{x \to -4^+} f(x)$ $\therefore$ J.D

at $x = 2$, $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$ $\therefore$ J.D

at $x = 4$, $f(4) \neq \lim_{x \to 4} f(x)$ $\therefore$ R.D

6. Answer the following questions using the graph of $f(x)$ shown below.



$\lim_{x \to -2^-} f(x) = 3$

$\lim_{x \to -2^+} f(x) = \infty$

a. Find the value of $f(-8)$.

$f(-8) = -3$

b. Find the following limits. If a limit does not exist explain why.

1) $\lim_{x \to -8} f(x) = -6$

$\lim_{x \to -8^-} f(x) = \lim_{x \to -8^+} f(x) = -6$

2) $\lim_{x \to 6^-} f(x)$ $\lim_{x \to 6^-} f(x) = 2$

3) $\lim_{x \to 6^+} f(x) = 5$

4) $\lim_{x \to 6} f(x) = \text{DNE}$, $\lim_{x \to 6^-} f(x) \neq \lim_{x \to 6^+} f(x)$

c) Find and classify the discontinuities of f as removable, infinite or jump.

f is discontinuous at $x = -8, -2, 6$

at $x = -8$, $f(-8) \neq \lim_{x \to -8} f(x)$ $\therefore$ R.D

at $x = -2$, $\lim_{x \to -2^+} f(x) = \infty$ $\therefore$ I.D

at $x = 6$, $\lim_{x \to 6^-} f(x) \neq \lim_{x \to 6^+} f(x)$ $\therefore$ J.D

Discontinuity أنواع ال *

1) Removable :-

$$f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

2) Jump

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

3) Infinite (vertical Asy)

$$f(x) = \lim_{x \to 3} \frac{1}{x - 3}$$

خطوات حل الـDiscontinuity

١) افحص المقام إذا كان = ٥ .

٢) إذا دالة مقطعية (piece wise)

افحص limit من اليمين و اليسار

٣) حدد إذا Cont أو discont و النوع

Jump or Removable or infinite

Classify the discontinuities

$$f(x) = \frac{x^2-2x}{|x|(x^2-x-2)}$$

$$|x|(x^2-x-2)=0$$

$$|x|(x-2)(x+1)=0$$

$$\therefore x=0 \quad , \quad x=2 \quad , \quad x=-1$$

$$\because |x| = \begin{cases} \rightarrow x \geq 0 \\ \rightarrow -x < 0 \end{cases}$$

$x=0$	$x=2$	$x=-1$
$f(0)=\frac{0}{0}$	$f(2)=\frac{0}{0}$	$f(-1)=\frac{3}{0}$
$\lim\limits_{x\to 0^+} \frac{x(x-2)}{x(x-2)(x+1)}$ $\lim\limits_{x\to 0^+} \frac{1}{x+1}=1$ $\lim\limits_{x\to 0^-} \frac{1}{-(x+1)}=-1$ at $x=0$ f has Jump discontinuities	$\lim\limits_{x\to 2} \frac{x(x-2)}{x(x-2)(x+1)}$ $\lim\limits_{x\to 2} \frac{1}{x+1} = \frac{1}{3}$ but f(2) undefined at $x=2$ $\therefore$ f has Removable discontinuities	$\lim\limits_{x\to -1} \frac{x(x-2)}{-x(x-2)(x+1)}$ $=\lim\limits_{x\to -1} \frac{-1}{x+1} = -\infty$ at $x=-1$ it has infinite discontinuities

2. (20 pts) Let $f(x) = \begin{cases} \dfrac{x^2+x-6}{x+3}, & \text{if } x < 0 \\[2ex] \dfrac{x+1}{(x-1)^2}, & \text{if } x \geq 0 \end{cases}$

Find all the points of discontinuity of f. Classify each discontinuity as removable, jump, or infinite.

---

f is discontinuous at $x = -3, 1$ and also at $x = 0$ (see below).

i) $\lim\limits_{x\to-3} \dfrac{x^2+x-6}{x+3} = \lim\limits_{x\to-3} \dfrac{(x+3)(x-2)}{x+3} = \lim\limits_{x\to-3} (x-2) = -5.$

So, f has a removable discontinuity at $x = -3$ since $\lim\limits_{x\to-3} f(x) = -5$ but $f(-3)$ is undefined.

ii) $\lim\limits_{x\to1^\pm} f(x) = \lim\limits_{x\to1^\pm} \dfrac{x+1}{(x-1)^2} = \infty$. So, f has an infinite discontinuity at $x = 1$.

iii) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{x^2+x-6}{x+3} = -2$, and $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} \dfrac{x+1}{(x-1)^2} = 1.$

So, f has a jump discontinuity at $x = 0$ since $\lim\limits_{x\to0^-} f(x)$ and $\lim\limits_{x\to0^+} f(x)$ both exist but are not equal.

---

6. $[10 + 5 = 15 \text{ pts.}]$ Let

$$f(x) = \begin{cases} x^2 - \ln(x+e), & \text{if } -1 < x \leqslant 0, \\ e^x - \sin x, & \text{if } x > 0. \end{cases}$$

(a) Find and classify the discontinuities of f as removable, infinite, or jump.

(b) Does $\lim\limits_{x\to0}(f(x))^2$ exist? explain why or why not?

We have $f(0) = -1$, $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (e^x - \sin x) = 1$ and $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} (x^2 - \ln(x+e)) = -1$. Therefore, the function f is discontinuous at $x = 0$ since the limit does not exit. The type of discontinuity is jump since $\lim\limits_{x\to0^+} f(x) \neq \lim\limits_{x\to0^-} f(x)$. The limit in part (b) exists as shown below: we have $\lim\limits_{x\to0^+} (f(x))^2 = \lim\limits_{x\to0^+} f(x) \times \lim\limits_{x\to0^+} f(x) = 1$ and $\lim\limits_{x\to0^-} (f(x))^2 = \lim\limits_{x\to0^-} f(x) \times \lim\limits_{x\to0^-} f(x) = 1$. Therefore, $\lim\limits_{x\to0} (f(x))^2 = 1$.

17–22 Explain why the function is discontinuous at the given number a.

18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$ $\qquad a = -2$

$$\lim_{x \to -2^-} f(x) = \frac{1}{-2^- + 2} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to 2^+} f(x) = \frac{1}{-2^+ + 2} = \frac{1}{0^+} = \infty$$

$$\because \lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$$

$$\therefore \lim_{x \to 2} f(x) = \text{DNE}$$

$\therefore$ f is discontinuous at $a = -2$

Type: infinity "I.D"

17–22 Explain why the function is discontinuous at the given number a.

19. $f(x) = \begin{cases} x+3 & \text{if } x \leqslant -1 \\ 2^x & \text{if } x > -1 \end{cases}$ $\quad a = -1$

$$\lim_{x \to -1^-} f(x) = -1 + 3 = 2$$

$$\lim_{x \to -1^+} f(x) = 2^{-1} = \frac{1}{2}$$

$$\because \lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$$

$$\therefore \lim_{x \to -1} f(x) = DNE$$

$\therefore$ f is discontinuous at $a = -1$

Type:- Jump J.D

17–22 Explain why the function is discontinuous at the given number a.

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $\qquad a = 0$

$$\lim_{x \to 0^-} f(x) = \cos 0 = 1$$

$$\lim_{x \to 0^+} f(x) = 1 - 0^2 = 1$$

$$f(0) = 0$$

$$\because \lim_{x \to 0} f(x) \neq f(0)$$

$\therefore$ f is discontinuous at $a = 0$

Type :- removable R.D

5. [10 pts.] Determine whether f is continuous at $x = 1$, where:

$$f(x) = \begin{cases} 2^x, & \text{if } x \geq 1 \\ \dfrac{1-x^2}{|1-x|}, & \text{if } x < 1. \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1-x^2}{|1-x|}$$

$$|1-x| \begin{cases} \rightarrow -(1-x) & x > 1 \\ \rightarrow 1-x & x < 1 \end{cases}$$

$$\lim_{x \to 1^-} \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{1-x}$$

$$= \lim_{x \to 1^-} 1+x = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2^x = 2 = f(1)$$

$\therefore$ f is Continous at $x=1$

4. [2 + 8 = 10 pts.] Define $f(x) = \tan^{-1}\left(\frac{1}{x-2}\right)$ for $x \neq 2$.

(a) Explain why f is discontinuous at $x = 2$.

(b) Classify the type of this discontinuity.

$$\because \lim_{x \to 2^-} \tan^{-1}\left(\frac{1}{x-2}\right) = \tan^{-1} - \infty = -\frac{\pi}{2}$$

$$\because \lim_{x \to 2^+} \tan^{-1}\left(\frac{1}{x-2}\right) = \tan^{-1} \infty = \frac{\pi}{2}$$

$$\because \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \quad \therefore \lim_{x \to 2} f(x) = DNE$$

$\therefore$ f is discontinuous at $x=2$

(b) Since $\boxed{\lim_{x \to 2^\pm} f(x) = \pm\frac{\pi}{2}}$, the function $\boxed{f \text{ has a jump discontinuity at } x = 2}$.

3. [10 + 10 = 20 pts.] Let $f(x) = \dfrac{x^2 - 4x}{|x|(x+2)}$.

(a) Find and classify the discontinuities of f as removable, infinite or jump.

---

f is discontinuous at $\boxed{x = -2, 0}$.

$\boxed{\text{For } x = -2}$: since $\boxed{\lim_{x \to -2^\pm} \dfrac{x^2 - 4x}{|x|(x+2)} = \pm\infty}$, $\boxed{f \text{ has an \underline{infinite} discontinuity at } x = -2}$.

$\boxed{\text{For } x = 0}$: since $\boxed{\lim_{x \to 0^\pm} \dfrac{x^2 - 4x}{|x|(x+2)} = \lim_{x \to 0^\pm} \dfrac{x(x-4)}{\pm x(x+2)} = \mp 2}$, $\boxed{f \text{ has a \underline{jump} discontinuity at } x = 0}$.

Q2. [5+5=10 pts.] Let $f(x) = \begin{cases} x^2 - 3 & \text{if } x \neq 3 \\ 3 & \text{if } x = 3. \end{cases}$

(a) Determine whether f is continuous at $x = 3$.

(a) f is discontinuous at $x = 3$ since $\boxed{\lim_{x \to 3} f(x) = \lim_{x \to 3} (x^2 - 3) = 6 \neq f(3) = 3}$.

Type: Removable "R.D"

Type: Removable "R.D

Q1. [5+5=10 pts.] Let $f(x) = 3 + \ln(2x+5)$.

(a) Find the interval(s) where f is continuous.

(a) f is continuous on its domain which is $\boxed{D_f = \left(\frac{-5}{2}, \infty\right)}$.

---

5. [10 pts.] Find a value for the constant A, if any, that makes the function f continuous at $x = 1$, where

$$f(x) = \begin{cases} \dfrac{\sqrt{x+1} - \sqrt{2x}}{x-1} & \text{if } x \geqslant 0,\ x \neq 1 \\ \\ A & \text{if } x = 1. \end{cases}$$

It is clear that $\boxed{f(1) = A}$ and

$$\begin{aligned} \lim_{x \to 1} f(x) &= \lim_{x \to 1} \frac{\sqrt{x+1} - \sqrt{2x}}{x-1} \\ &= \lim_{x \to 1} \frac{\sqrt{x+1} - \sqrt{2x}}{x-1} \times \frac{\sqrt{x+1} + \sqrt{2x}}{\sqrt{x+1} + \sqrt{2x}} \\ &= \lim_{x \to 1} \frac{1-x}{(x-1)(\sqrt{x+1} + \sqrt{2x})} \\ &= \lim_{x \to 1} \frac{-1}{\sqrt{x+1} + \sqrt{2x}} = \frac{-1}{2\sqrt{2}} \end{aligned}$$

$\boxed{\text{The function is continuous at } x = 1 \text{ if } f(1) = \lim_{x \to 1} f(x) = \frac{-1}{2\sqrt{2}}, \text{ i.e., } A = \frac{-1}{2\sqrt{2}}}$.

EXAMPLE 5 Determining Whether a Function Is Continuous on a Closed Interval

Is the function $f(x) = \sqrt{4 - x^2}$ continuous on the closed interval $[-2, 2]$?

Solution The domain of f is $\{x| -2 \leq x \leq 2\}$. So, f is defined for every number in the closed interval $[-2, 2]$.

For any number c in the open interval $(-2, 2)$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{4 - x^2} = \sqrt{\lim_{x \to c}(4 - x^2)} = \sqrt{4 - c^2} = f(c)$$

So, f is continuous on the open interval $(-2, 2)$.

To determine whether f is continuous on $[-2, 2]$, we investigate the limit from the right at -2 and the limit from the left at 2. Then,

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \sqrt{4 - x^2} = 0 = f(-2)$$

So, f is continuous from the right at -2. Similarly,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{4 - x^2} = 0 = f(2)$$

So, f is continuous from the left at 2. We conclude that f is continuous on the closed interval $[-2, 2]$. ■

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

15. $f(x) = x + \sqrt{x - 4}, \quad [4, \infty)$

The domain of f is $x \geq 4 \Rightarrow [4, \infty)$

so f is defined in every number $[4, \infty)$

for any number of "c" in the open interval $(4, \infty)$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(x + \sqrt{x-4}\right) = c + \sqrt{c-4} = f(c)$$

f is continuous on $(4, \infty)$

$$\lim_{x \to 4^+} \left(x + \sqrt{x-4}\right) = 4 + \sqrt{4-4} = 4 = f(4)$$

f(x) is continuous from the right at 4

$\therefore$ f(x) is continuous on $[4, \infty)$

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

16. $g(x) = \dfrac{x-1}{3x+6}$, $(-\infty, -2)$

The domain of g is $\mathbb{R}/\{-2\}$

so g is defined in every number $(-\infty,-2)$

for any number of "c" in the open interval $(-\infty,-2)$

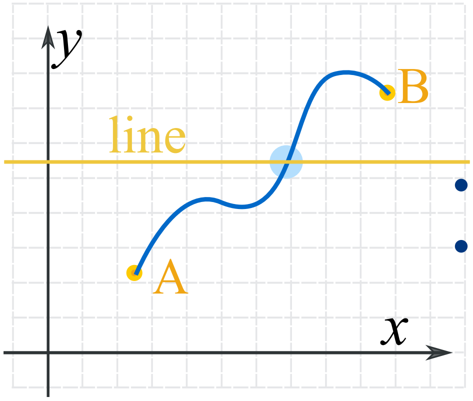
$$\lim_{x\to c} g(x) = \lim_{x\to c}\left(\frac{x-1}{3x+6}\right) = \frac{c-1}{3c+6} = g(c)$$

$\therefore$ $g(x)$ is continuous on $(-\infty,-2)$

③

Intermediate Value Theorem

The idea behind the Intermediate Value Theorem is this:



When we have **two points** connected by a continuous curve:

- one point below the line
- the other point above the line

... then there will be **at least one place** where the curve crosses the line!

Well **of course** we must cross the line to get from A to B!

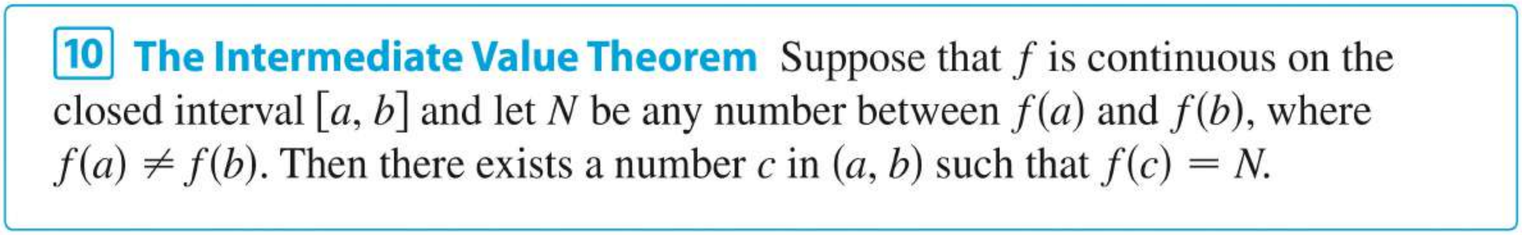
Now that you know the **idea**, let's look more closely at the details.

Continuous

The curve must be **continuous** ... no gaps or jumps in it.

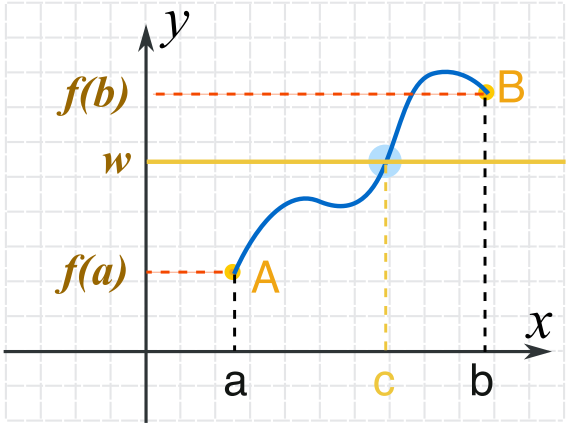
Continuous is a special term with an exact definition in calculus, but here we will use this simplified definition:

we can draw it without lifting our pen from the paper



10 **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Here is the Intermediate Value Theorem stated more formally:



When:

- The curve is the function y = f(x),
- which is continuous on the interval [a, b],
- and w is a number between f(a) and f(b),

Then ...

> ... there must be at least one value **c** within [a, b] such that **f(c) = w**

In other words the function y = f(x) at some point must be w = f(c)

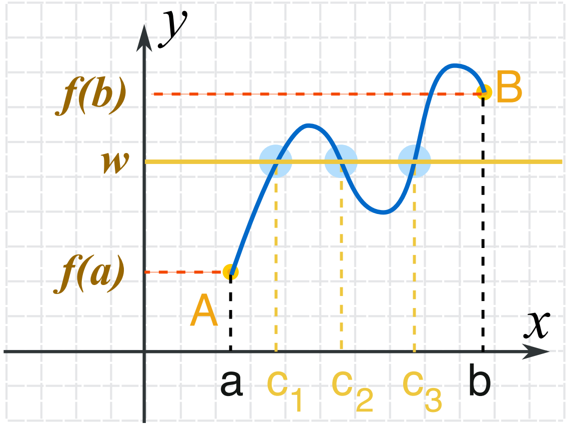
Notice that:

- **w** is between f(a) and f(b), *which leads to ...*
- **c** must be between a and b

At Least One

It also says "at least one value c", which means we **could** have more.

Here, for example, are 3 points where f(x)=w:



How Is This Useful?

١) أتأكد المعادلة صفرية

٢) أتأكد أن continuous

٣) إذا عطاني فترة (ح أعوض فيها

٤) حَموله الاستنتاج

Whenever we can show that:

- there is a point above some line
- and a point below that line, and
- that the curve is continuous,

we can then safely say "yes, there is a value somewhere **in between** that is on the line".

Example: is there a solution to $x^5 - 2x^3 - 2 = 0$ between x=0 and x=2?

At x=0:

$$0^5 - 2 \times 0^3 - 2 = \mathbf{-2} < 0$$

At x=2:

$$2^5 - 2 \times 2^3 - 2 = \mathbf{14} > 0$$

Now we know:

- at x=~~0, the curve is below zero~~
- at x=~~2, the curve is above zero~~

And, ~~being a polynomial, the curve will be continuous,~~

so **~~somewhere in between~~** ~~the curve must cross through y=0~~

~~Yes, there is a solution to $x^5 - 2x^3 - 2 = 0$ in the interval [0, 2]~~

f is cont. on (0,2) Then by I.V.T, There exists a c in (0,2) such that $f(c) = 0$ ∴ c is a root

نظرية مهم مهم مهم

10 **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

EXAMPLE 10 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$

Domain of f is $\mathbb{R}$ & continous "poly."

$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2 = -1 < 0$

$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2 = 12 > 0$

f is cont. on $(1,2)$ Then by I.V.T, there exist c in $(1,2)$ such that $f(c) = 0$

$\therefore$ c is a root

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

53. $x^4 + x - 3 = 0, \quad (1, 2)$

Let $f(x) = x^4 + x - 3 = 0$

لازم تتأكد أن المعادلة
صفرية .. إذا ما كانت
صفرية ... خلها صفرية 🤷‍♂️

Domain of f is $\mathbb{R}$ and $\underline{\text{continuous}} \rightarrow$ poly.

$f(1) = (1)^4 + 1 - 3 = -1 < 0$

$f(2) = (2)^4 + 2 - 3 = 15 > 0$

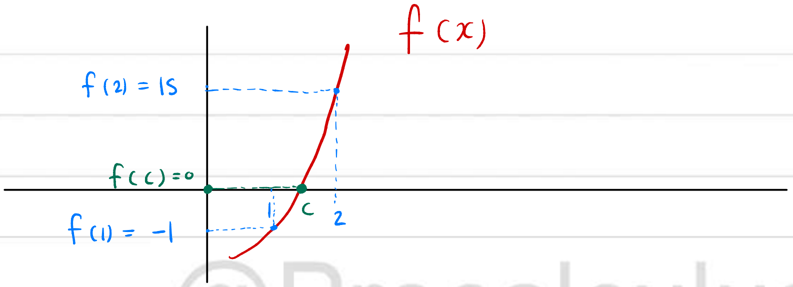
f is cont. on $(1, 2)$ Then

by I.V.T, There exists a c in $(1,2)$

Such that $f(c) = 0$ $\therefore$ c is a root

للتوضيح شنو معنات النظرية عشان تفهمها !!!!!

يعني بما إن أحنا طلعنا ناتجين .. وواحد فيهم أكبر من الصفر والثاني أصغر من الصفر .. والدالة
متصلة ومستمرة .. يعني أكيد في قيمة لو أعوضها بالدالة رح تعطيني صفر .. بس ما أعرفها عشان
جذي سميتها c ..



يعني قيمة لو أعوضها : Root
بالدالة تعطيني الناتج صفر

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

54. $\ln x = x - \sqrt{x}, \quad (2, 3)$

Let $f(x) = \ln x - x + \sqrt{x}$

لازم تتأكد أن المعادلة
صفرية .. إذا ما كانت
صفرية ... خلها صفرية 🤷‍♂️

Domain of f is $(0, \infty)$ and continuous

"poly, log, root functions"

$f(2) = \ln 2 - 2 + \sqrt{2} < 0$

$f(3) = \ln 2 - 3 + \sqrt{3} > 0$

صعب إني أحدد
الناتج موجب أو سالب
بدون آلة فلا تحاتي من
الارقام استفيد من فكرة
المثال لا أكثر

f is cont. on $(2, 3)$ Then

by I.V.T, There exists a c in $(2, 3)$

Such that $f(c) = 0$ $\therefore$ c is a root

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

55. $e^x = 3 - 2x, \quad (0, 1)$

Let $f(x) = e^x + 2x - 3 = 0$

لازم تتأكد أن المعادلة
صفرية .. إذا ما كانت
صفرية ... خلها
صفرية 🤷‍♂️

The domain of f is $\mathbb{R}$ and continuous

"exponential, poly functions"

$f(0) = e^0 + 2(0) - 3 = -2 < 0$

$f(1) = e^1 + 2(1) - 3 = e - 1 > 0$

f is cont. on $(0, 1)$ Then

by I.V.T, There exists a c in $(0, 1)$

Such that $f(c) = 0$ $\therefore$ c is a root

7. [10 pts.] Use the Intermediate Value Theorem to show that the equation

$$e^x + \cos x = 4 \quad \text{has a real root.}$$

Let $f(x) = e^x + \cos x - 4$

لازم تتأكد أن
المعادلة صفرية ..
إذا ما كانت
صفرية ... خلها
صفرية 🤷‍♂️

إذا السؤال ما عطاني أرقام أعوض فيهم أبيب أرقام من عندي سهلة التعويض وتكون من ضمن الـdomain للدالة ولازم الناتج الرقمين الي ابيبهم واحد يعطيني الناتج موجب والثاني يعطيني سالب

Domain of f is $\mathbb{R}$ & continuous

"exponential, Trigonometric functions"

$f(0) = e^0 + \cos 0 - 4 = -2 < 0$

$f(2\pi) = (e^{2\pi} + \cos 2\pi - 4) > 0$

f is cont. on $(0, 2\pi)$ Then by I.V.T, There exists a c in $(0, 2\pi)$ such that $f(c) = 0$ $\therefore$ c is a root

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

56. $\sin x = x^2 - x, \quad (1, 2)$

Let $f(x) = \sin x - x^2 + x = 0$

لازم تتأكد أن المعادلة صفرية .. إذا ما كانت صفرية ... خلها صفرية 🤷🏻‍♂️

Domain of f is $\mathbb{R}$ and continuous

"Trigonometric & poly. function"

$f(1) = \sin 1 - (1)^2 + 1 = \sin 1 > 0$

$f(2) = \sin 2 - (2)^2 + 2 = \sin(2) - 2 < 0$

f is cont. on $(1, 2)$ Then

by I.V.T, There exists a c in $(1, 2)$ such that $f(c) = 0$

$\therefore$ c is root

1. [10 pts.] Let $f(x) = \begin{cases} x, & \text{if } x < a, \\ x^2 - 2, & \text{if } x \geqslant a. \end{cases}$

Find all values of a, if any, for which f is continuous everywhere.

$\boxed{\text{The function } f \text{ is continuous for } x < a, \text{ and for } x > a}$. Now f is continuous at $x = a$ if $\boxed{\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)}$. That is; if $a = a^2 - 2$. Therefore, $\boxed{a = 2 \text{ or } a = -1}$.

6. [10 pts.] Use the Intermediate Value Theorem to show that the equation $\cos x - \ln(x+1) = 0$ has a real root.

$\boxed{\text{Let } f(x) = \cos x - \ln(x+1)}$.

The function f is continuous on its domain $(-1, \infty)$, hence its continuous on $[0, \pi]$. Since $f(0) = 1 > 0$ and $f(\pi) = -1 - \ln(\pi + 1) < 0$, therefore by the IVT there is a number $c \in (0, \pi)$ s.t. $f(c) = 0$. Hence, the given equation has a real root.

6. [10 pts.] Suppose that the function f is continuous on $[0,1]$ and satisfies $0 < f(x) < 1$ for all x in $[0,1]$. Use the Intermediate Value Theorem to show that the equation $f(x) - x = 0$ has a solution in $(0,1)$.

[10 pts.] Define $\boxed{g(x) = f(x) - x}$. It is clear that the function $\boxed{g \text{ is continuous on } [0,1]}$ since it is the difference of two continuous functions on $[0,1]$. Also since $0 < f(x) < 1$, we have $\boxed{g(0) = f(0) - 0 = f(0) > 0}$ and $\boxed{g(1) = f(1) - 1 < 0}$. Therefore, by the IVT there exists a c in $(0,1)$ such that $\boxed{g(c) = f(c) - c = 0}$ or equivalently $f(c) = c$. Thus c is a solution for $f(x) - x = 0$.

8. [5 + 10 = 15 pts.] a) State the Intermediate Value Theorem.

b) Show that there is at least one real root of the equation $\cos(\sqrt{x}) = e^x - 2$.

a) Suppose f is a continuous function on $[a,b]$ with N between $f(a)$ and $f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = N$.

b) The function $f(x) = \cos(\sqrt{x}) - e^x + 2$ has domain $[0,\infty]$. Now we have $f(0) = 1 - 1 + 2 > 0$ and $f(2) = \cos(\sqrt{2}) - e^2 + 2 < 0$. The function f is continuous as it is a linear combination of continuous functions. Hence by the IVT, there exits a number c in $(0,2)$ such that $f(c) = 0$. This is equivalent to say that $\cos(\sqrt{c}) - e^c + 2 = 0$ or $\cos(\sqrt{c}) = e^c - 2$. Therefore, c is a real root of the given equation.

* IVT + intersection

Show that the graphs

$f(x) = -3x^3 - 2x + 1$

$g(x) = 2x^3 - x^2 + 4$

intersect

إذا طلب منك intersect يعني يبي يعرف وين
تتقاطع الدالتين.
نفس فكرة حل IVT بس الفرق بالبداية تطرح الدالتين

---

let $h(x) = f(x) - g(x)$

$= 3x^2 - 2x + 1 - (2x^3 - x^2 + 4)$

$h(x) = x^3 + x^2 - 2x - 3$

$h(0) = 0 + 0 - 2(0) - 3 = -3 < 0$

$h(2) = 2^3 + 2^2 - 2(3) - 3 = 5 > 0$

h is cont on $[0, 2]$ (poly)

and since $h(0) < 0 < h(2)$ then

by I.V.T, There exists c in $(0, 2)$

such that $h(c) = f(c) - g(c) = 0$, Thus the graphs f, g intersect at least once

Q6. [10 pts.] Let $f(x) = 2 + x\sin(x)$ and $g(x) = x^2$. Use the Intermediate Value Theorem to show that the graphs of f and g intersect at least once.

Let $h(x) = 2 + x\sin(x) - x^2$.

We have:

(i) $\boxed{h \text{ is continuous on } [0, \pi]}$ (since $\sin(x), x$ and x^2 are continuous everywhere).

(ii) $\boxed{h(0) = 2 + 0 - 0 = 2 > 0}$ and $\boxed{h(\pi) = 2 + 0 - \pi^2 < 0}$.

Hence by the IVT there is at least one $c \in (0, \pi)$ such that $\boxed{h(c) = f(c) - g(c) = 0}$. Thus, the graphs of f and g intersect at least once.

ثمرة

Thamrah

Kuwait University

Calculus 1 – Asymptotes

(Section 2.6)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

* Asymptotes

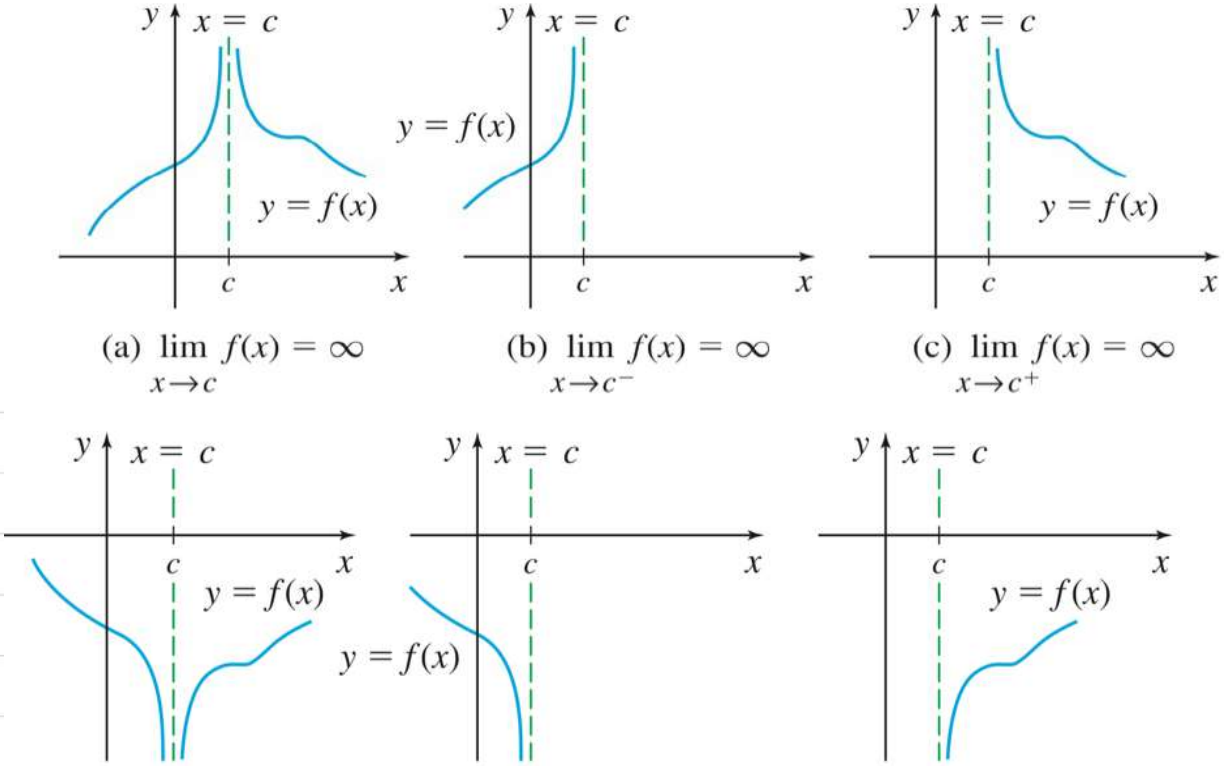
* How determine Asymptotes from graph.

* evaluating $\lim_{x \to \pm\infty} f(x)$.

* Find the vertical & horizontal asymptotes "if any".

* How determine asymptotes from graph.

Vertical Asymptotes



(a) $\lim_{x\to c} f(x) = \infty$ (b) $\lim_{x\to c^-} f(x) = \infty$ (c) $\lim_{x\to c^+} f(x) = \infty$

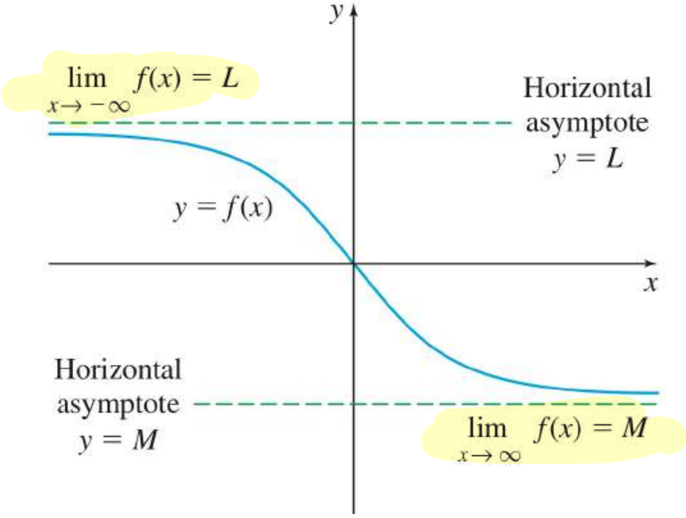
(d) $\lim_{x\to c} f(x) = -\infty$ (e) $\lim_{x\to c^-} f(x) = -\infty$ (f) $\lim_{x\to c^+} f(x) = -\infty$

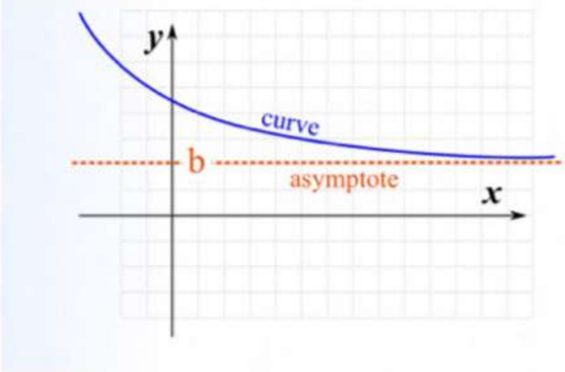
---

* بالرسم: نشوف خط عامودي يقترب منه المنحنى

⟵ الدالة تروح ∞ أو $-\infty$.

Horizontal Asymptotes





If $\lim_{x \to -\infty} f(x) = L$

then $y = L$ is H.A

$\lim_{x \to \infty} f(x) = M$

then $y = M$ is H.A

In Problems 9–16, use the accompanying graph of $y = f(x)$.

9. Find $\lim_{x\to\infty} f(x)$. $= 2$

10. Find $\lim_{x\to-\infty} f(x)$. $= 0$

11. Find $\lim_{x\to-1^-} f(x)$. $= \infty$

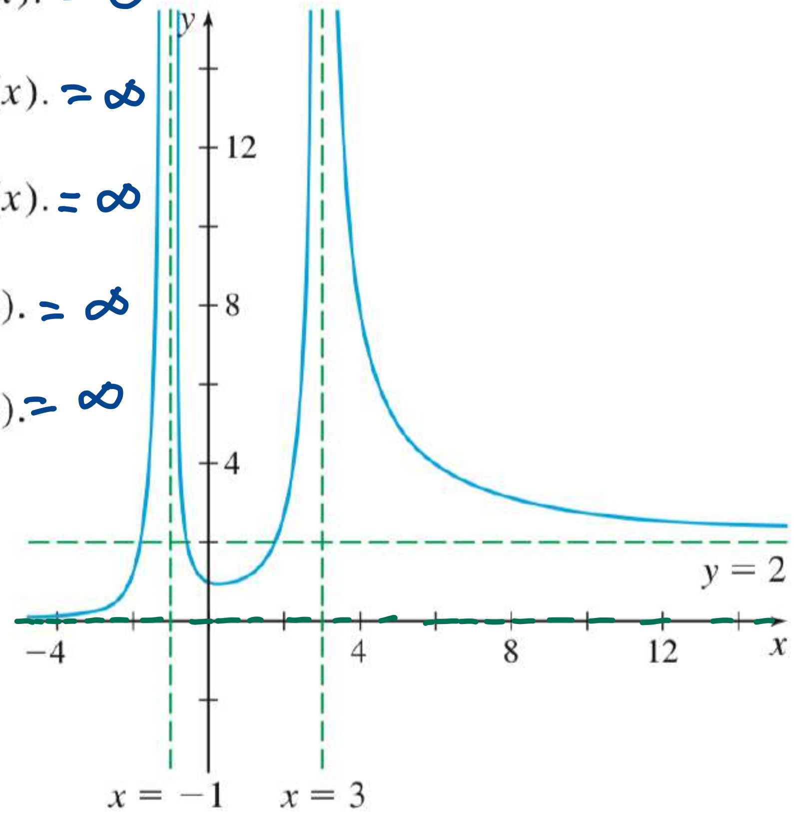
12. Find $\lim_{x\to-1^+} f(x)$. $= \infty$

13. Find $\lim_{x\to3^-} f(x)$. $= \infty$

14. Find $\lim_{x\to3^+} f(x)$. $= \infty$

15. Identify all vertical asymptotes.

16. Identify all horizontal asymptotes.



15. $x = -1$, $x = 3$

16. $Y = 2$, $Y = 0$

In Problems 17–26, use the accompanying graph of $y = f(x)$.

17. Find $\lim_{x\to\infty} f(x)$. $= -3$ **18.** Find $\lim_{x\to-\infty} f(x)$ $= 0$

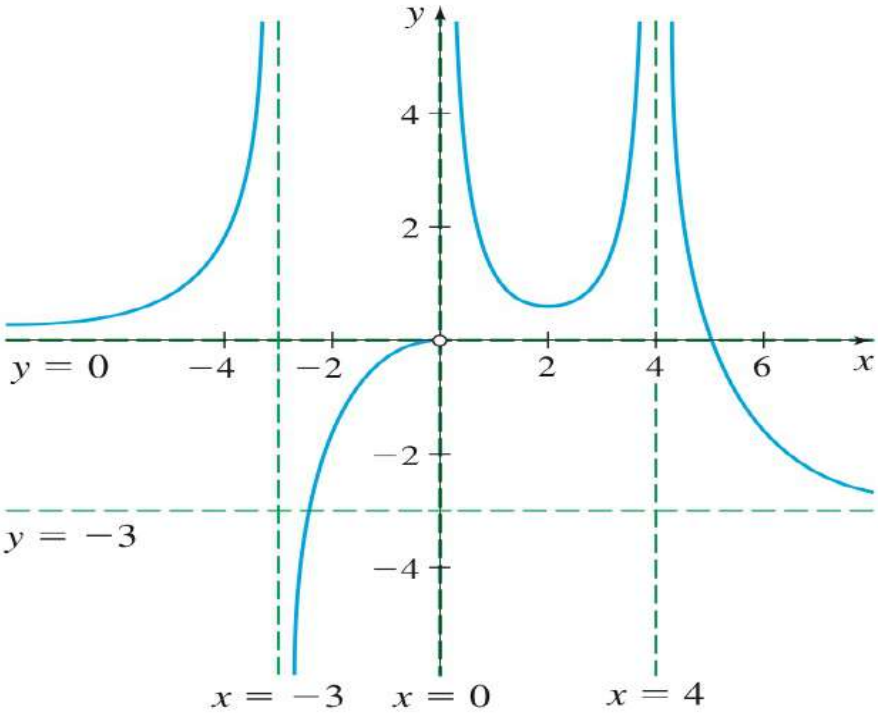
19. Find $\lim_{x\to-3^-} f(x)$. $= \infty$ **20.** Find $\lim_{x\to-3^+} f(x)$. $= -\infty$

21. Find $\lim_{x\to0^-} f(x)$. $= 0$ **22.** Find $\lim_{x\to0^+} f(x)$. $= \infty$

23. Find $\lim_{x\to4^-} f(x)$. $= \infty$ **24.** Find $\lim_{x\to4^+} f(x)$. $= \infty$

25. Identify all vertical asymptotes.

26. Identify all horizontal asymptotes.

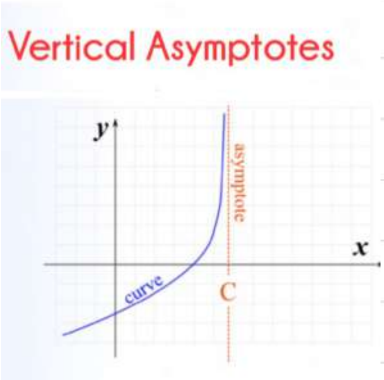


25. $x = -3$, $x = 0$, $x = 4$

26. $y = -3$, $y = 0$

*خطوات التعامل مع Asymptotes

١_ Vertical Asymptote



*حل معادلة المقام = 0

* افحص limit من اليمين و اليسار

إذا راحت لـ ±∞ معناته الدالة عند

هالنقطة vertical Asymptote .

* إذا عطاني رقم إذاً NO V.A

٢_ Horizantal Asymptote

* أفحص $\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$

* إذا عطاني رقم ، هذا الرقم أهوا H.A .

* إذا عطاني ∞ أو -∞ إذاً NO H.A .

* ملخص *

find the vertical & horizontal asymptote "if any".

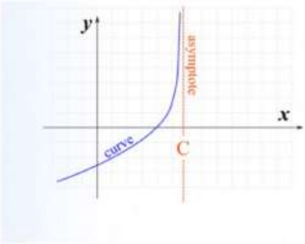
- vertical Asymptote :

(١ أصفار المقام ، $x = c$

(٢ $\lim_{x \to c} f(x) = \infty$

(٣ $\therefore c$ is V.A

Vertical Asymptotes

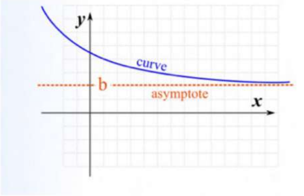


- Horizontal Asymptote :

$\lim_{x \to \infty^+} f(x) = c$ or $\lim_{x \to \infty^-} f(x) = -c$ (1

$\therefore c$ & $-c$ are H.A

Horizontal Asymptotes



limit تذكر أن !

1- $e^{0} = 1$

2- $e^{\infty} = \infty$

3- $e^{-\infty} = 0$

4- $\tan^{-1}(-\infty) = \frac{-\pi}{2}$

5- $\ln 0^{+} = -\infty$

6- $\tan^{-1}(\infty) = \frac{\pi}{2}$

7- $\ln e = 1$

8- $\ln 1 = 0$

9- $\ln \infty = \infty$

10- $\sin 0 = 0$

11- $\cos 0 = 1$

12- $\frac{1}{0} = \mp\infty$

13- $\frac{1}{\pm\infty} = 0$

14- $\frac{0}{\pm\infty} = 0$

15- $\frac{\pm 3}{\infty} = 0$

16- $\frac{0}{\pm 4} = 0$

17- $0^{\infty} = 0$

18- $3^{\infty} = \infty$

19- $(\frac{1}{4})^{\infty} = 0$

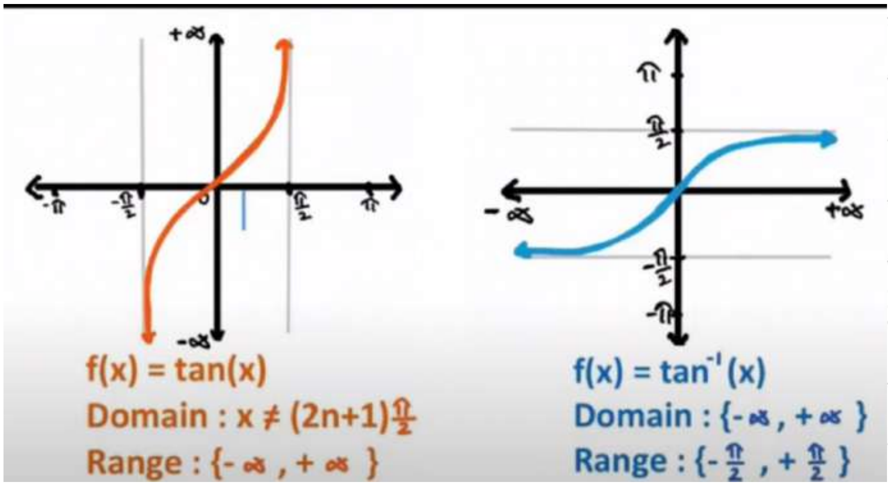
20- $0^{-\infty} = \infty$

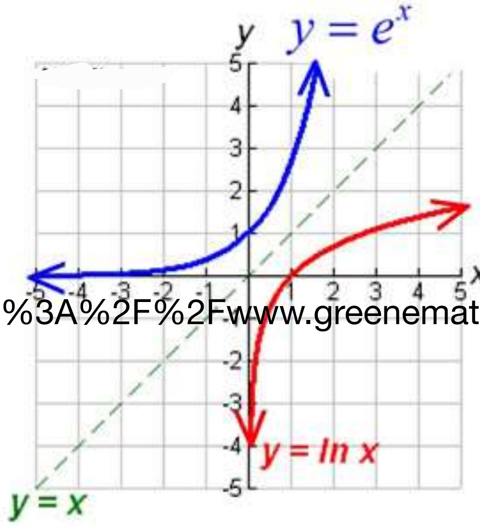
* tan & ln functions

35. $\lim_{x\to\infty} \arctan(e^x) = \tan^{-1}(e^{\infty}) = \tan^{-1}(\infty) = \pi/2$

41. $\lim_{x\to -1^+} \ln(x+1) = \ln(-1^+ + 1) = \ln(0^+) = -\infty$

42. $\lim_{x\to 1^+} \ln(x-1) = \ln(1^+ - 1) = \ln(0^+) = -\infty$





tps%3A%2F%2Fwww.greenemath.com%

2) limit infinity for power and exponential function

evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

---

a) $f(x) = -5x^3$

$$\lim_{x\to-\infty} -5x^3 = -5(-\infty)^3 = -5(-\infty) = \infty$$

$$\lim_{x\to\infty} -5x^3 = -5(\infty)^3 = -5(\infty) = -\infty$$

---

b) $f(x) = 2x^4$

$$\lim_{x\to\infty} 2x^4 = \infty, \quad \lim_{x\to-\infty} 2x^4 = \infty$$

---

c) $f(x) = e^x$

$$\lim_{x\to\infty} e^x = e^{\infty} = \infty, \quad \lim_{x\to-\infty} e^x = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

---

d) $f(x) = \pi^x$ $\quad \lim\limits_{x\to\infty} \pi^x = \pi^{\infty} = \infty, \quad \lim\limits_{x\to-\infty} \pi^x = \frac{1}{\pi^x} = 0$

1.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* ∞ is a number. false

2. **(a)** $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$; **(b)** $\lim\limits_{x \to 0^+} \frac{1}{x} = \infty$; **(c)** $\lim\limits_{x \to 0^+} \ln x = -\infty$

4. If $\lim\limits_{x \to 4} f(x) = \infty$, then the line $x = 4$ is a(n) V. asymptote of the graph of f.

5. **(a)** $\lim\limits_{x \to \infty} \frac{1}{x} = 0$; **(b)** $\lim\limits_{x \to \infty} \frac{1}{x^2} = 0$; **(c)** $\lim\limits_{x \to \infty} \ln x = \infty$

6. *True or False* $\lim\limits_{x \to -\infty} 5 = 0$. False

7. **(a)** $\lim\limits_{x \to -\infty} e^x = 0$; **(b)** $\lim\limits_{x \to \infty} e^x = \infty$; **(c)** $\lim\limits_{x \to \infty} e^{-x} = 0$

In Problems 27–42, find each limit.

27. $\lim\limits_{x \to 2^-} \dfrac{3x}{x-2}$

28. $\lim\limits_{x \to -4^+} \dfrac{2x+1}{x+4}$

33. $\lim\limits_{x \to -3^-} \dfrac{1}{x^2-9}$

34. $\lim\limits_{x \to 2^+} \dfrac{x}{x^2-4}$

---

27. $$\lim_{x \to 2^-} \frac{3x}{x-2} = \frac{3(2)}{2^- - 2} = \frac{6}{0^-} = -\infty$$

28. $$\lim_{x \to -4^+} \frac{2x+1}{x+4} = \frac{-8+1}{-4^+ + 4} = \frac{-7}{0^+} = -\infty$$

33. $$\lim_{x \to -3^-} \frac{1}{x^2-9} = \frac{1}{9^+ - 9} = \infty$$

34. $$\lim_{x \to 2^+} \frac{x}{x^2-4} = \frac{2}{4^+ - 4} = \infty$$

*خطوات حل $\lim\limits_{x\to\pm\infty} f(x)$ مع الكسور:

١) قسمة على أكبر أس في المقام على كل عنصر.

٢) كثيرة الحدود / كسور بسيطة.

إذا عندك بسط ومقام وكلهم حدود عادية (بدون جذر).

$$\bullet \lim_{x\to\infty} \frac{4x+3}{5x^2-1} = \frac{\frac{4x}{x^2}+\frac{3}{x^2}}{\frac{5x^2}{x^2}-\frac{1}{x^2}} =$$

$$\frac{\frac{4}{x}+\frac{3}{x^2}}{5-\frac{1}{x^2}} = \frac{0+0}{5-0} = \frac{0}{5} = 0$$

*إذا أعلى قوة في المقام أكبر $\Rightarrow$ الناتج صفر.

مثال :-

- $$\lim_{x\to\infty} \frac{3-4x^2}{5x-1} = \frac{\frac{3}{x}-\frac{4x^2}{x}}{\frac{5x}{x}-\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{\frac{3}{x}-4x}{5-\frac{1}{x}} = \frac{0-\infty}{5-0} = \frac{-\infty}{5} = -\infty$$

* إذا أعلى قوة في البسط أكبر ⇒ الناتج $\pm\infty$

- $$\lim_{x\to\infty} \frac{4x^2+3}{1-5x^2} = \frac{\frac{4x^2}{x^2}+\frac{3}{x^2}}{\frac{1}{x^2}-\frac{5x^2}{x^2}}$$

$$= \frac{4-\frac{3}{x^2}}{\frac{1}{x^2}-5} = \frac{4-0}{0-5} = \frac{4}{-5}$$

* إذا نفس القوة ⇒ قسم المعاملات

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x\to\infty} \dfrac{2x^2 - 7}{5x^2 + x - 3}$

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2} - \frac{7}{x^2}}{\frac{5x^2}{x^2} + \frac{x}{x^2} - \frac{3}{x^2}}$$

$$\lim_{x\to\infty} \frac{2 - \frac{7}{x^2}}{5 + \frac{1}{x} - \frac{3}{x^2}}$$

$$\frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty} \frac{7}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{3}{x^2}}$$

$$\frac{2 - 0}{5 + 0 - 0} = \frac{2}{5}$$

15–42 Find the limit or show that it does not exist.

16. $\lim_{x\to\infty} \frac{1 - x^2}{x^3 - x + 1}$

$$\lim_{x\to\infty} \frac{\frac{1}{x^3} - \frac{x^2}{x^3}}{\frac{x^3}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}}$$

$$\frac{\lim_{x\to\infty} \frac{1}{x^3} - \lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x^2} + \lim_{x\to\infty} \frac{1}{x^3}}$$

$$\frac{0 - 0}{1 - 0 + 0} = 0$$

In Problems 61–66, find any horizontal or vertical asymptotes of the graph of f.

61. $f(x) = 3 + \frac{1}{x}$

62. $f(x) = 2 - \frac{1}{x^2}$

63. $f(x) = \frac{x^2}{x^2 - 1}$

64. $f(x) = \frac{2x^2 - 1}{x^2 - 1}$

---

61. V.A : $x = 0 \Rightarrow \lim\limits_{x \to 0} \frac{3x+1}{x} = \frac{0+1}{0} = \infty$

H.A : $\lim\limits_{x \to \pm\infty} 3 + \frac{1}{x} = 3 + \frac{1}{\infty} = 3 + 0 = 3$

$x = 0$ is V.A 3 is H.A

---

62. V.A : 0 is V.A

H.A : $\lim\limits_{x \to \pm\infty} \frac{2x^2 - 1}{x^2} = \frac{2}{1} = 2$

$x = 0$ is V.A , $y = 2$ is H.A

15–42 Find the limit or show that it does not exist.

34. $\lim_{x \to -\infty} \frac{1 + x^6}{x^4 + 1}$

$$\lim_{x \to -\infty} \frac{\frac{1}{x^4} + \frac{x^6}{x^4}}{\frac{x^4}{x^4} + \frac{1}{x^4}} = \frac{\frac{1}{x^4} + x^2}{1 + \frac{1}{x^4}}$$

$$= \frac{0 + (-\infty)^2}{1 + 0} = \infty$$

37. $\lim\limits_{x\to\infty} \frac{1-e^x}{1+2e^x}$

38. $\lim\limits_{x\to\infty} \frac{\sin^2 x}{x^2+1}$

$$\lim_{x\to\infty} \frac{\frac{1}{e^x} - \frac{e^x}{e^x}}{\frac{1}{e^x} + \frac{2e^x}{e^x}} = \lim_{x\to\infty} \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 2}$$

$$= \frac{0-1}{0+2} = -\frac{1}{2}$$

---

38. $\lim\limits_{x\to\infty} \frac{\sin^2 x}{x^2+1}$ $= \frac{\sin^2 \infty}{\infty + 1} = \frac{\sin \infty}{\infty}$

∴ Sandwich theorem

$\sin^2 x$ ↙ لأن زوجي

$$0 \leq \sin^2 x \leq 1$$

$$\frac{0}{x^2+1} \leq \frac{\sin^2 x}{x^2+1} \leq \frac{1}{x^2+1} = 0 \leq \frac{\sin^2 x}{x^2+1} \leq \frac{1}{x^2+1}$$

$$\because \lim_{x\to\infty} \frac{0}{x^2+1} = \frac{0}{\infty} = 0 , \lim_{x\to\infty} \frac{1}{x^2+1} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{x\to\infty} \frac{\sin^2 x}{x^2+1} = 0$$ by S.T

1. Evaluate the following the limit and conclude if it has vertical or horizontal asymptotes.

$$\lim_{x\to\infty} \frac{x^3 - 9x + 1}{3x^2 - 2x - 15}$$

$$\lim_{x\to\infty} \frac{x^2\left(\frac{x^3}{x^2} - \frac{9x}{x^2} + \frac{1}{x^2}\right)}{x^2\left(\frac{3x^2}{x^2} - \frac{2x}{x^2} - \frac{15}{x^2}\right)}$$

$$\lim_{x\to\infty} \frac{x^2\left(x - \frac{9}{x} + \frac{1}{x^2}\right)}{x^2\left(3 - \frac{2}{x} - \frac{15}{x^2}\right)}$$

$$= \frac{\left(\infty - \frac{9}{\infty} + \frac{1}{\infty}\right)}{\left(3 - \frac{2}{\infty} - \frac{15}{\infty}\right)}$$

$$= \frac{\infty - 0 + 0}{3 - 0 - 0} = \frac{\infty}{3} = \infty$$

15–42 Find the limit or show that it does not exist.

20. $\lim_{t \to \infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

$$= \lim_{t \to \infty} \frac{t - t^{3/2}}{2t^{3/2} + 3t - 5}$$

$\sqrt{t} = t^{1/2}$

$t \cdot t^{1/2} = t^{3/2}$

$$\lim_{t \to \infty} \frac{\frac{t}{t^{3/2}} - \frac{t^{3/2}}{t^{3/2}}}{\frac{2t^{3/2}}{t^{3/2}} + \frac{3t}{t^{3/2}} - \frac{5}{t^{3/2}}}$$

$$\lim_{t \to \infty} \frac{\frac{1}{t^{1/2}} - 1}{2 + \frac{3}{t^{1/2}} - \frac{5}{t^{3/2}}}$$

$$\frac{\frac{1}{\infty} - 1}{2 + \frac{3}{\infty} - \frac{5}{\infty}} = \frac{0 - 1}{2 + 0 - 0} = -\frac{1}{2}$$

٢) مع جذر التربيعي

* طلع أكبر قوة تحت الجذر

- مثال $\sqrt{x^2} = |x|$

$$|x| = x = \infty$$

$$|x| = -x \Rightarrow -\infty$$

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 + x}}{4x - 1} = \frac{\sqrt{x^2(3 + \frac{1}{x})}}{4x - 1}$$

$$\Rightarrow \frac{|x|\sqrt{3 + \frac{1}{x}}}{4x - 1} \Rightarrow \frac{x\sqrt{3 + \frac{1}{x}}}{x(4 - \frac{1}{x})}$$

$$= \lim_{x \to \infty} \frac{\sqrt{3 + \frac{1}{x}}}{4 - \frac{1}{x}} = \frac{\sqrt{3 + 0}}{4 - 0} = \frac{\sqrt{3}}{4}$$

$$f(x) = \lim_{x \to \infty} \frac{3x - 2}{\sqrt{4x^2+5}}$$

$$\lim_{x \to \infty} \frac{3x-2}{\sqrt{4x^2+5}} = \frac{x(3x-2)}{\sqrt{x^2\left(4+\frac{5}{x^2}\right)}}$$

$$= \lim_{x \to \infty} \frac{x\left(3-\frac{2}{x}\right)}{|x|\sqrt{4+\frac{5}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{x\left(3-\frac{2}{x^{\cancel{2}}}\right)}{x\sqrt{4+\frac{5}{x^2}}}$$

$$\Rightarrow \frac{3-\frac{2}{\infty}}{\sqrt{4+\frac{5}{\infty}}} = \frac{3-0}{\sqrt{4+0}} = \frac{3}{\sqrt{4}}$$

$$= \frac{3}{2}$$

$$f(x) = \lim_{x \to -\infty} \frac{3x-2}{\sqrt{4x^2+5}}$$

$$\lim_{x \to \infty} \frac{3x-2}{\sqrt{4x^2+5}} = \frac{x(3x-2)}{\sqrt{x^2(4+\frac{5}{x^2})}}$$

$$= \lim_{x \to \infty} \frac{x(3-\frac{2}{x})}{|x|\sqrt{4+\frac{5}{x^2}}}$$

$$= \lim_{x \to \infty} \frac{x(3-\frac{2}{x^2})}{-x\sqrt{4+\frac{5}{x^2}}}$$

$$\Longrightarrow \frac{3-\frac{2}{\infty}}{-\sqrt{4+\frac{5}{\infty}}} = \frac{3-0}{-\sqrt{4+0}} = \frac{3}{-\sqrt{4}}$$

$$= -\frac{3}{2}$$

15–42 Find the limit or show that it does not exist.

27. $\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+x}-3x}{1} * \frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}$$

$$\lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}$$

$$\lim_{x\to\infty} \frac{x}{\left(\sqrt{x^2\left(\frac{9x^2}{x^2}+\frac{x}{x^2}\right)}+3x\right)} = \frac{x\left(\frac{x}{x}\right)}{|x|\sqrt{\left(9+\frac{1}{x}\right)}+\frac{3x}{x}}$$

$$\lim_{x\to\infty} \frac{\cancel{x}\,(1)}{\cancel{x}\sqrt{9+\frac{1}{x}}+3}$$

$$\lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{\sqrt{9+0}+3}$$

$$\lim_{x\to\infty} \frac{1}{\sqrt{9+0}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Determine the V.A & H.A

a. $y = \dfrac{3x+1}{x+2}$

. vertical Asy.

$x+2 \Rightarrow \quad x = -2$

$$\lim_{x \to -2} \frac{3x+1}{x+2} = \frac{3(-2)+1}{-2+2} = \frac{-6+1}{0} = -\infty$$

$\therefore x = -2$ is V.A

- horizontal Asy

1) $\displaystyle\lim_{x \to -\infty} \frac{3x+1}{x+2} = \frac{3x}{x} = 3$

2) $\displaystyle\lim_{x \to +\infty} \frac{3x+1}{x+2} = \frac{3x}{x} = 3$

$\therefore y = 3$ is H.A

b) $Y = \dfrac{x^3+5}{(x+2)(x+3)}$

.vertical Asy .

$(x+2)(x+3) \Rightarrow x=-2\,,\, x=-3$

$$\lim_{x \to -2} \frac{x^3+5}{(x+2)(x+3)} = \frac{(-2)^3+5}{(-2+2)(-2+3)} = -\infty$$

$$\lim_{x \to -3} \frac{x^3+5}{(x+2)(x+3)} = \frac{(-3)^3+5}{(-3+2)(-3+3)} = -\infty$$

horizontal Asy

1) $(x+3)(x+2) = x^2+2x+3x+6$

$= x^2+5x+6$

$$\lim_{x \to -\infty} \frac{x^3+5}{x^2+5x+6} = \frac{x^3}{x^2} = x = -\infty$$

No H.A

$$\lim_{x \to +\infty} \frac{x^3+5}{x^2+5x+6} = \frac{x^3}{x^2} = x = +\infty$$

c. $Y = 3 + \frac{3x^2+x+1}{x^2-4}$

.vertical Asy.

$x^2-4 = (x-2)(x+2)$

$\Rightarrow x=-2, x= 2$

$\lim_{x \to 2} 3 + \frac{3x^2+x+1}{x^2-4} = 3 + \infty = \infty$

$\lim_{x \to -2} 3 + \frac{3x^2+x+1}{x^2-4} = 3 + \infty = \infty$

$\therefore x=2, \ x=-2$ are V.A

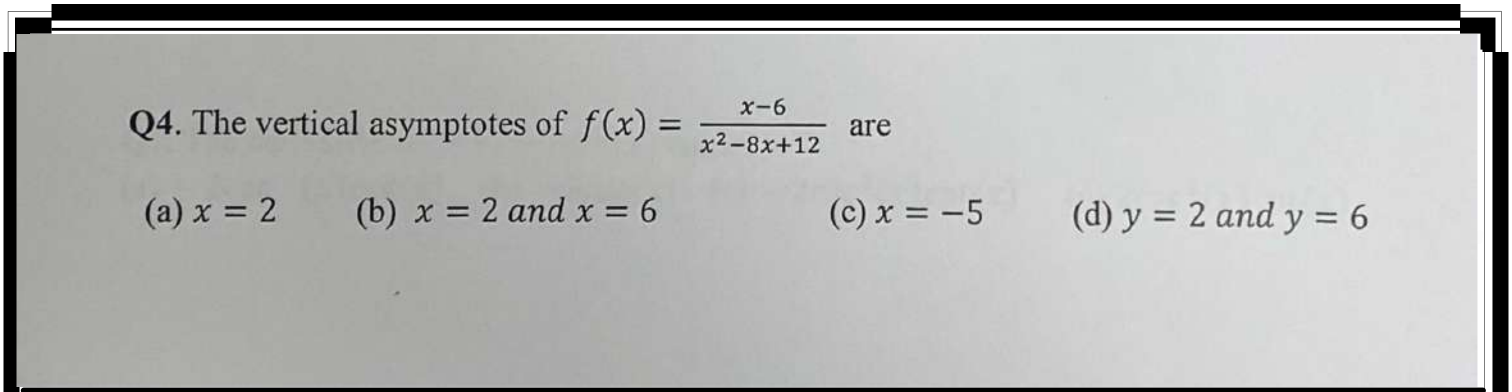
horizontal Asy

$\lim_{x \to \pm\infty} \left(3 + \frac{3x^2+x+1}{x^2-4}\right) = 3 + \frac{3x^2}{x^2}$

$= 3 + \frac{3}{1} = 3+3 = 6$

$= \lim_{x \to +\infty} = 6$

$Y = 6$ is H.A



Q4. The vertical asymptotes of $f(x) = \frac{x-6}{x^2-8x+12}$ are

(a) $x = 2$ (b) $x = 2$ *and* $x = 6$ (c) $x = -5$ (d) $y = 2$ *and* $y = 6$

$$x^2 - 8x + 12 = 0$$

$$(x-6)(x-2) = 0$$

$$x = 2 \quad , \quad x = 6$$

a) $\lim\limits_{x \to 2} \dfrac{x-6}{(x-6)(x-2)} = \dfrac{1}{x-2} = \dfrac{1}{0} = \infty$

b) $\lim\limits_{x \to 6} \dfrac{x-6}{(x-6)(x-2)} = \dfrac{1}{x-2} = \dfrac{1}{4}$

$x = 2$ is only V.A

1) Computing limits at infinity

Determine the horizental asymptotes

a) $f(x) = 5 - \frac{2}{x^2}$

$$\lim_{x \to \pm\infty} \left(5 - \frac{2}{x^2} \right) = 5 - \frac{2}{\pm\infty} = 5 - 0 = 5$$

$\therefore$ $y = 5$ is horizental asymptotes

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b) $f(x) = \frac{\sin x}{x}$

$$\lim_{x \to \infty} \frac{\sin x}{x} = \frac{\sin \infty}{\infty} \quad \therefore \text{S.T}$$

$$-1 \leq \sin x \leq 1 \Rightarrow \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\because \lim_{x \to \infty} \frac{-1}{x} = \lim_{x \to \infty} \frac{1}{x} = 0$$

$$\therefore \text{by S.T} \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$\therefore$ $y = 0$ is horizental asymptotes

c) $\tan^{-1}(x)$

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$\therefore$ $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$ are H.A

$$\frac{a/b}{c/d} \qquad x^3 \cdot x^3 = x^6$$

25. $$\lim_{x\to\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \frac{\lim\limits_{x\to\infty}\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to\infty}(2/x^3-1)} \qquad \left[\text{since } x^3=\sqrt{x^6} \text{ for } x>0\right]$$

$$= \frac{\lim\limits_{x\to\infty}\sqrt{1/x^6+4}}{\lim\limits_{x\to\infty}(2/x^3) - \lim\limits_{x\to\infty}1} = \frac{\sqrt{\lim\limits_{x\to\infty}(1/x^6) + \lim\limits_{x\to\infty}4}}{0-1}$$

$$= \frac{\sqrt{0+4}}{-1} = \frac{2}{-1} = -2$$

26. $$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \frac{\lim\limits_{x\to-\infty}-\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to-\infty}(2/x^3-1)} \qquad \left[\text{since } x^3=-\sqrt{x^6} \text{ for } x<0\right]$$

$$= \frac{\lim\limits_{x\to-\infty}-\sqrt{1/x^6+4}}{2\lim\limits_{x\to-\infty}(1/x^3) - \lim\limits_{x\to-\infty}1} = \frac{-\sqrt{\lim\limits_{x\to-\infty}(1/x^6) + \lim\limits_{x\to-\infty}4}}{2(0)-1}$$

$$= \frac{-\sqrt{0+4}}{-1} = \frac{-2}{-1} = 2$$

27. $$\lim_{x\to-\infty}\frac{2x^5-x}{x^4+3} = \lim_{x\to-\infty}\frac{(2x^5-x)/x^4}{(x^4+3)/x^4} = \lim_{x\to-\infty}\frac{2x-1/x^3}{1+3/x^4}$$

$$= -\infty \text{ since } 2x-1/x^3 \to -\infty \text{ and } 1+3/x^4 \to 1 \text{ as } x\to-\infty$$

29. $$\lim_{t\to\infty}\left(\sqrt{25t^2+2}-5t\right) = \lim_{t\to\infty}\left(\sqrt{25t^2+2}-5t\right)\left(\frac{\sqrt{25t^2+2}+5t}{\sqrt{25t^2+2}+5t}\right) = \lim_{t\to\infty}\frac{(25t^2+2)-(5t)^2}{\sqrt{25t^2+2}+5t}$$

$$= \lim_{t\to\infty}\frac{2}{\sqrt{25t^2+2}+5t} = \lim_{t\to\infty}\frac{2/t}{\left(\sqrt{25t^2+2}+5t\right)/t}$$

$$= \lim_{t\to\infty}\frac{2/t}{\sqrt{25+2/t^2}+5} \qquad [\text{since } t=\sqrt{t^2} \text{ for } t>0]$$

$$= \frac{0}{\sqrt{25+0}+5} = 0$$

30. $$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right) = \lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right] = \lim_{x\to-\infty}\frac{(4x^2+3x)-(2x)^2}{\sqrt{4x^2+3x}-2x}$$

$$= \lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x} = \lim_{x\to-\infty}\frac{3x/x}{\left(\sqrt{4x^2+3x}-2x\right)/x}$$

$$= \lim_{x\to-\infty}\frac{3}{-\sqrt{4+3/x}-2} \qquad \left[\text{since } x=-\sqrt{x^2} \text{ for } x<0\right]$$

$$= \frac{3}{-\sqrt{4+0}-2} = -\frac{3}{4}$$

EXAMPLE 5 Compute $\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right)$. $= \infty - \infty$

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x}$$

Notice that the denominator of this last expression $\left(\sqrt{x^2+1} + x\right)$ becomes large as $x \to \infty$ (it's bigger than x). So

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} = 0$$

EXAMPLE 6 Evaluate $\lim_{x \to 2^+} \arctan\left(\dfrac{1}{x-2}\right)$.

SOLUTION If we let $t = 1/(x-2)$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the second equation in (4), we have

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t \to \infty} \arctan t = \frac{\pi}{2}$$ ■

$$\lim_{x \to \infty} \frac{e^x + 300}{10^x + 3^x}$$

$$\lim_{x \to \infty} \frac{\frac{e^x}{10^x} + \frac{300}{10^x}}{\frac{10^x}{10^x} + \frac{3^x}{10^x}}$$

$$\lim_{x \to \infty} \frac{\frac{e^x}{10^x} + \frac{300}{10^x}}{1 + \frac{3^x}{10^x}}$$

$$= \frac{0 + 0}{1 + 0} = 0$$

39. $\lim_{x\to\infty} (e^{-2x} \cos x)$

$= e^{-\infty} \cos \infty = \frac{1}{e^{\infty}} \cos \infty = 0 \cos \infty$

using squeeze theorem

$-1 \leqslant \cos x \leqslant 1$

$-e^{-2x} \leqslant e^{-2x} \cos x \leqslant e^{-2x}$

$\therefore \lim_{x\to\infty} -e^{-2x} = -e^{-\infty} = -\frac{1}{e^{\infty}} = -\frac{1}{\infty}$

$= 0$

$\lim_{x\to\infty} e^{-2x} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$\therefore \lim_{x\to\infty} e^{-2x} \cos x = 0$ by S.T

3. [10 + 10 = 20 pts.] Let $f(x) = \dfrac{x^2 - 4x}{|x|(x+2)}$.

(a) Find and classify the discontinuities of f as removable, infinite or jump.

(b) Find the horizontal and vertical asymptotes of the graph of f, if any.

(a) Find and classify the discontinuities of f as removable, infinite or jump.

f is discontinuous at $\boxed{x = -2, 0}$.

$\boxed{\text{For } x = -2}$: since $\boxed{\displaystyle\lim_{x \to -2^{\pm}} \frac{x^2 - 4x}{|x|(x+2)} = \pm\infty}$, $\boxed{f \text{ has an } \underline{\text{infinite}} \text{ discontinuity at } x = -2}$.

$\boxed{\text{For } x = 0}$: since $\boxed{\displaystyle\lim_{x \to 0^{\pm}} \frac{x^2 - 4x}{|x|(x+2)} = \lim_{x \to 0^{\pm}} \frac{x(x-4)}{\pm x(x+2)} = \mp 2}$, $\boxed{f \text{ has a } \underline{\text{jump}} \text{ discontinuity at } x = 0}$.

(b) Find the horizontal and vertical asymptotes of the graph of f, if any.

Since $\boxed{\displaystyle\lim_{x \to \pm\infty} \frac{x^2 - 4x}{|x|(x+2)} = \pm 1}$, $\boxed{\text{the lines } \boxed{y = \pm 1} \text{ are horizontal asymptotes of the graph of } f}$.

Also since $\boxed{\displaystyle\lim_{x \to -2^{\pm}} \frac{x^2 - 4x}{|x|(x+2)} = \pm\infty}$, $\boxed{\text{the line } \boxed{x = -2} \text{ is a vertical asymptotes of the graph of } f}$.

1. $(10+10=20$ pts) Evaluate each of the following limits, if it exists.

(b) $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2+2}\right)$.

$$=\lim_{x\to-\infty}\frac{\left(x+\sqrt{x^2+2}\right)\left(x-\sqrt{x^2+2}\right)}{x-\sqrt{x^2+2}}=\lim_{x\to-\infty}\frac{(x^2-(x^2+2))}{x-\sqrt{x^2+2}}=0.$$

3. (10 + 5 = 15 pts) Let $f(x)=\dfrac{2x^2+3}{x\sqrt{x^2+4}}$.

(a) Find the horizontal asymptotes if the graph of f, if any.

(b) Find the vertical asymptotes if the graph of f, if any.

(a) $f(x)=\dfrac{2x^2+3}{x\sqrt{x^2+4}}=\dfrac{2x^2+3}{x\,|x|\,\sqrt{1+4/x^2}}$

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}\frac{2x^2+3}{\pm x^2\sqrt{1+4/x^2}}=\lim_{x\to\pm\infty}\frac{2+3/x^2}{\pm\sqrt{1+4/x^2}}=\pm2.$$

Thus, the lines $y=\pm2$ are horizontal asymptotes to the curve $y=f(x)$.

(b) $\lim\limits_{x\to0^\pm}f(x)=\lim\limits_{x\to0^\pm}\dfrac{2x^2+3}{x\sqrt{x^2+4}}=\pm\infty$: The line $x=0$ is a vertical asymptote to the curve $y=f(x)$.

2. [10+10=20 pts.] Let $f(x) = \dfrac{3x^2+10}{x\sqrt{x^2+1}}$.

(a) Find the horizontal asymptotes of the graph of f, if any.

(b) Find the vertical asymptotes of the graph of f, if any.

(a) Find the horizontal asymptotes of the graph of f, if any.

Since $\lim\limits_{x\to\pm\infty} \dfrac{3x^2+10}{x\sqrt{x^2+1}} = \lim\limits_{x\to\pm\infty} \dfrac{x^2(3+10/x^2)}{x|x|\sqrt{1+1/x^2}} = \lim\limits_{x\to\pm\infty} \pm\dfrac{\cancel{x^2}(3+10/x^2)}{\cancel{x^2}\sqrt{1+1/x^2}} = \boxed{\pm 3}$.

$\boxed{\text{Therefore, the lines } y=3 \text{ and } y=-3 \text{ are horizontal asymptotes of the graph of } f}$.

(b) Find the vertical asymptotes of the graph of f, if any.

Since $\lim\limits_{x\to 0^\pm} \dfrac{3x^2+10}{x\sqrt{x^2+1}} = \boxed{\pm\infty}$.

$\boxed{\text{Therefore, the line } x=0 \text{ is a vertical asymptote of the graph of } f}$.

∞

$$|x| \begin{cases} \to x & x \geq 0 \\ \to -x & x < 0 \end{cases}$$

$-\infty$

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9. [10+10 = 20 pts.] Let $f(x) = \dfrac{x|x+1|}{x^2-1}$.

(a) Find the horizontal asymptotes of the graph of f, if any.

(b) Find the vertical asymptotes of the graph of f, if any.

(a) We have $\lim\limits_{x\to\pm\infty} \dfrac{x|x+1|}{x^2-1} = \lim\limits_{x\to\pm\infty} \dfrac{\pm x(x+1)}{x^2-1} = \lim\limits_{x\to\pm\infty} \dfrac{\pm(x^2+x)}{x^2-1} = \boxed{\pm 1}$.

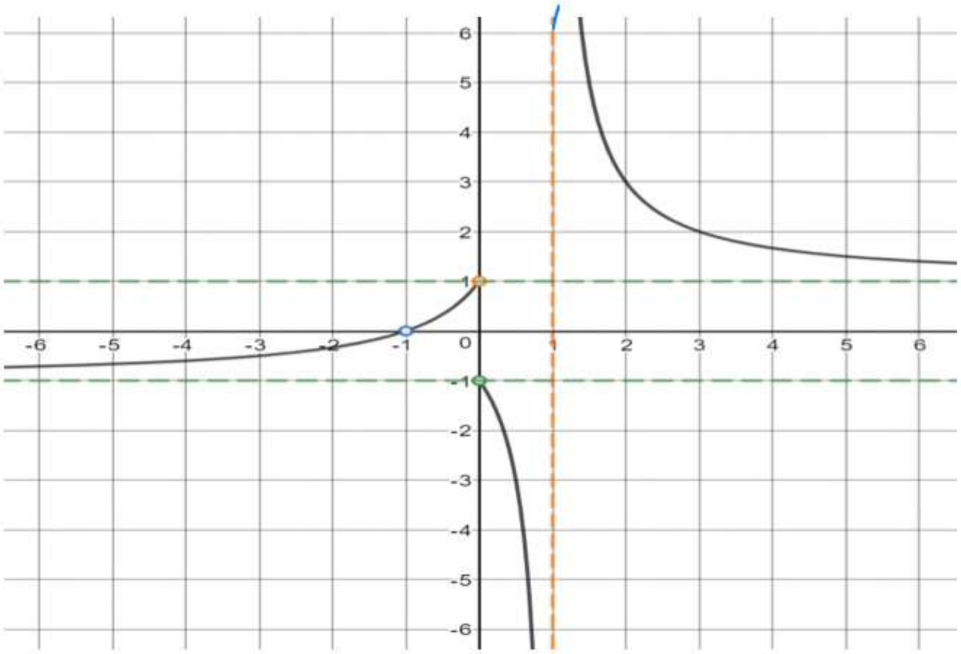
Therefore, the lines $\boxed{y = \pm 1}$ are horizontal asymptotes to the curve $y = f(x)$.

(b) We have $\lim\limits_{x\to 1^\pm} \dfrac{x|x+1|}{x^2-1} = \lim\limits_{x\to 1^\pm} \dfrac{x(x+1)}{(x-1)(x+1)} = \boxed{\pm\infty}$.

Note that $\lim\limits_{x\to -1^\pm} \dfrac{x|x+1|}{x^2-1} = \lim\limits_{x\to -1^\pm} \dfrac{\pm x(x+1)}{(x-1)(x+1)} = \lim\limits_{x\to -1^\pm} \dfrac{\pm x}{(x-1)} = \boxed{\pm 1/2}$.

Therefore, the line $\boxed{x = 1}$ is a vertical asymptote to the curve $y = f(x)$.

1. [$5 \times 4 = 20$ pts.] Use the given graph of f to select the correct answer.



(I) $\lim_{x\to 0} f(x) =$

(a) 1 (b) 0 (c) ∞ (d) -1 [✓ (e) None of the mentioned]

(II) $\lim_{x\to 1^+} f(x) =$

[✓ (a) $+\infty$] (b) $-\infty$ (c) 1 (d) 0 (e) None of the mentioned

(III) The graph of f has

a) no vertical asymptotes

[✓ b) only one vertical asymptote]

c) only one horizontal asymptote

d) no horizontal asymptotes

e) None of the above

(IV) The function f has

a) a jump discontinuity at $x = 1$

b) a removable discontinuity at $x = 0$

c) a jump discontinuity at $x = -1$

[✓ d) a removable discontinuity at $x = -1$]

e) None of the above

1. [3 + 3 + 3 + 3 + 3 + 5 + 5 = 25 pts.] Let

$$f(x) = \begin{cases} e^x - \dfrac{3}{x-7} + 5, & \text{if } x \leqslant 1, \\ \ln(x-1) + 3, & \text{if } 1 < x \leqslant 3 \\ \dfrac{x^2+1}{x^3+1}, & \text{if } x > 3. \end{cases}$$

(a) Find $\lim_{x\to 1} f(x)$, if it exists.

We have $\lim_{x\to 1^+} f(x) = -\infty$, so $\lim_{x\to 1} f(x)$ does not exist.

(b) Find $\lim_{x\to 3} f(x)$, if it exists.

We have $\lim_{x\to 3^+} f(x) = \dfrac{10}{28}$ and $\lim_{x\to 3^-} f(x) = \ln 2 + 3$, so $\lim_{x\to 3} f(x)$ does not exist.

(c) Find $\lim_{x\to -\infty} f(x)$, if it exists.

We have $\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} e^x - \dfrac{3}{x-7} + 5 = 5$.

(d) Find $\lim_{x\to \infty} f(x)$, if it exists.

We have $\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \dfrac{x^2+1}{x^3+1} = 0$.

(e) Find, if any, all vertical asymptotes of the graph of f.

The only vertical asymptote of the graph of f is $x = 1$ because $\lim_{x\to 1^+} f(x) = -\infty$

(f) Find and classify, if any, all the points of discontinuity.

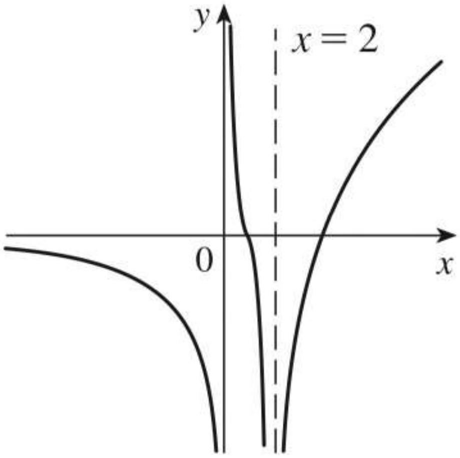
The function has infinite discontinuity at $x = 1$ and jump discontinuity at $x = 3$.

(g) Find, if any, all horizontal asymptotes of the graph of f.

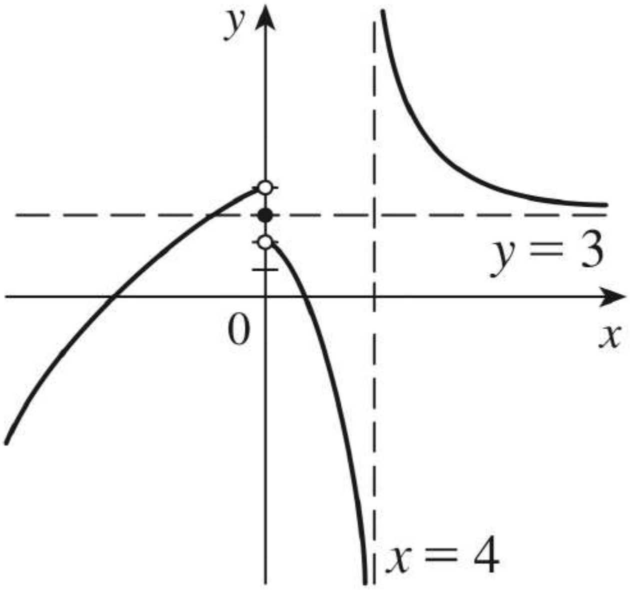
The graph of f has two horizontal asymptotes $y = 0, 5$.

5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

7. $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 0$,
$\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$



9. $f(0) = 3$, $\lim_{x\to 0^-} f(x) = 4$, $\lim_{x\to 0^+} f(x) = 2$,
$\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to 4^+} f(x) = \infty$,
$\lim_{x\to\infty} f(x) = 3$



3. [10+10 = 20 pts.] Let $f(x) = \dfrac{\sqrt{x^2+5}}{2x+3}$. Find the horizontal and vertical asymptotes of the graph of f.

Since $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{\sqrt{x^2+5}}{2x+3} = \lim_{x\to\pm\infty} \dfrac{\sqrt{x^2(1+5/x^2)}}{x(2+3/x)} = \lim_{x\to\pm\infty} \dfrac{\pm x\sqrt{(1+5/x^2)}}{x(2+3/x)} = \boxed{\pm 1/2}$.

$\boxed{\text{Therefore, the lines } y=-1/2 \text{ and } y=1/2 \text{ are horizontal asymptotes of the graph of } f}$.

Also since $\lim_{x\to(-3/2)^\pm} f(x) = \lim_{x\to(-3/2)^\pm} \dfrac{\sqrt{x^2+5}}{2x+3} = \boxed{\pm\infty}$.

$\boxed{\text{Therefore, the line } x=-3/2 \text{ is a vertical asymptote of the graph of } f}$.

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37. $\lim\limits_{x\to\infty} \dfrac{1-e^{x}}{1+2e^{x}} =$

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37. $\lim\limits_{x\to\infty} \dfrac{1-e^{x}}{1+2e^{x}} = \lim\limits_{x\to\infty} \dfrac{(1-e^{x})/e^{x}}{(1+2e^{x})/e^{x}} = \lim\limits_{x\to\infty} \dfrac{1/e^{x}-1}{1/e^{x}+2} = \dfrac{0-1}{0+2} = -\dfrac{1}{2}$

$$\lim_{x\to 0+} \tan^{-1}(\ln x)$$

40. Let $t = \ln x$. As $x \to 0^{+}$, $t \to -\infty$. $\lim\limits_{x\to 0+} \tan^{-1}(\ln x) = \lim\limits_{t\to -\infty} \tan^{-1} t = -\frac{\pi}{2}$

Find the limit.

$$\lim_{x\to\infty} \left(e^{-4x} \cos x\right)$$

36. $\lim\limits_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

$$\lim_{x\to\infty} \frac{\frac{e^{3x}}{e^{3x}} - \frac{e^{-3x}}{e^{3x}}}{\frac{e^{3x}}{e^{3x}} + \frac{e^{-3x}}{e^{3x}}} = \lim_{x\to\infty} \frac{1 - e^{-6x}}{1 + e^{-6x}}$$

$$\frac{1 - e^{-\infty}}{1 + e^{-\infty}} = \frac{1 - 0}{1 + 0} = 1$$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$ لأن

1. Evaluate the limits if they exist.

(a) $$\lim_{x \to \infty} \frac{(2x^2+1)(4x^2+4x+2)}{(x^2+2)^2}.$$

$$\lim_{x \to \infty} \frac{(2x^2)(4x^2)}{(x^2)^2} = \frac{8x^4}{x^4}$$

$$= 8$$

ثمرة

Thamrah

Kuwait University

Calculus 1 – Definition of Derivative

(Section 2.7 & 2.8)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

جدول الحالات في المستقيمات :-

الشرط / القانون	الحالة
$m=0$ $f'(x)=0$	المماس الأفقي (Horizontal Tangent).
$m=\infty$ أو $f'(a)$ غير معرف غير موجود	المماس العامودي (vertical Tangent).
$m_1 = m_2$	مستقيمان متوازيان (parallel Lines)
$m_1 . m_2 = -1$ or $m_1 = -\frac{1}{m_2}$	مستقيمان متعامدان (perpendicular Lines)
$y_1 - y_2 = m(x - x_1)$	معادلة مستقيم من الميل و النقطة.
$m = \frac{y_2 - y_1}{x_2 - x_1}$	ميل مستقيم من نقطتين
$y = mx + b$	معادلة مستقيم بصيغة الميل الأجزاء المقطوعة.
$y - y_1 = \frac{1}{-m}(x - x_1)$	معادلة المستقيم العامودي

خطوات عامة لحل مسائل المماس و معادلة المستقيم :-

١ـ تحديد المعطيات :

- الدالة $f(x)$.
- النقطة $P\left(\underset{\overset{\downarrow}{x}}{a}, \underset{\overset{\downarrow}{y}}{f(a)}\right)$ إذا كانت معطاة .

٢ـ إيجاد ميل المماس (Slope) :

- باستخدام التعريف $m=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$
- أو باستخدام صيغة ال h :

$$m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

٣ـ كتابة معادلة المماس :

- استخدم صيغة النقطة والميل :

$$y-f(a)=m(x-a)$$

٤ـ ترتيب المعادلة :

- بسط المعادلة لتكون بالصورة المطلوبة

Use the **definition of the derivative** to find

إذا طلب منك تحل المشتقة بالتعريف لازم تحل بإحدى
الطرق التالية (كيفك أي طريقة)

1) $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$

2) $m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$

Other symbols of Derivatives

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

1) $y - y_1 = m(x_1 - x)$ المطلوب

2) $f(x) = y = x^2$ المعادلة

3) $P(1, 1)$ النقاط

4) $f(1) = 1^2 = 1$

$$m = f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$ الميل

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1}$$

$$\lim_{x \to 1} x + 1 = 2$$

$$y - 1 = 2(x - 1) \Rightarrow y - 1 = 2x - 2$$

or $y = 2x - 1$ معادلة المستقيم

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

SOLUTION Let $f(x) = 3/x$. Then, by Equation 2, the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to $$x + 3y - 6 = 0$$

EXAMPLE 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

SOLUTION From Definition 4 we have

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\
&= 2a - 8
\end{aligned}$$

3. (a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$

(i) using Definition 1 (ii) using Equation 2

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(b) Find an equation of the tangent line in part (a).

$$i)\ \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = m$$

$$f(x) = 4x - x^2$$

$$f(a) = f(1) = 4(1) - (1)^2 = 3$$

$$\therefore \lim_{x \to 1} \frac{4x - x^2 - 3}{x-1} = \frac{-(x^2-4x+3)}{x-1}$$

$$\lim_{x \to 1} \frac{-(x-3)(x-1)}{x-1} = \lim_{x \to 1} -(x-3) = -(-2) = 2$$

$$ii)\ \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = m$$

$$f(a+h) = f(1+h) = 4(1+h) - (1+h)^2$$

$$= 4 + 4h - (h^2 + 2h + 1) = 4 + 4h - h^2 - 2h - 1$$

$$= -h^2 + 2h + 3$$

⇒ Continue

$f(a) = f(1) = 4(1) - (1)^2 = 3$

$$\therefore \lim_{h\to 0} \frac{-h^2 + 2h + 3 - (3)}{h}$$

$$= \lim_{h\to 0} \frac{-h^2 + 2h}{h} = \frac{h(-h+2)}{h}$$

$$\lim_{h\to 0} -h + 2 = 2$$

---

b) equation of tangent line is

$y - y_1 = m(x - x_1)$

at $(1,3)$ "given" & from part a $m=2$

$2 = \frac{y-3}{x-1} \Rightarrow y - 3 = 2(x-1)$

$\Rightarrow y - 3 = 2x - 2 \Rightarrow y = 2x - 2 + 3$

$\therefore y = 2x + 1$

7. [10 pts.] Use the **definition of the derivative** to find $f'(3)$ of $f(x)=\sqrt{x-2}+3$.

We have

a+h a

$$\begin{aligned} f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{\sqrt{3+h-2}+3-\sqrt{3-2}-3}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{3+h-2}-1}{h} = \lim_{h\to 0}\frac{(3+h-2)-1}{h\left(\sqrt{3+h-2}+1\right)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{3+h-2}+1} = \frac{1}{2} \end{aligned}$$

---

7. [10 pts.] Use the definition of the derivative to find $f'(1)$ of $f(x)=x|x|$.

We have $f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x|x|-f(1)}{x-1}$
$\lim_{x\to 1} x+1=2.$

$$|x| \begin{cases} \rightarrow \boxed{x \quad x\geq 0} \\ -x \quad x<0 \end{cases}$$

$$= \lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1=2.$$

---

6. (10 pts) Let $f(x)=\sqrt{2x+1}$. Use the definition of the derivative to find $f'(4)$.

$$\begin{aligned} f'(4) = \lim_{x\to 4}\frac{f(x)-f(4)}{x-4} &= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4} = \lim_{x\to 4}\frac{2x+1-9}{(x-4)\left(\sqrt{2x+1}+3\right)} \\ &= \lim_{x\to 4}\frac{2(x-4)}{(x-4)\left(\sqrt{2x+1}+3\right)} = \lim_{x\to 4}\frac{2}{\sqrt{2x+1}+3} = \frac{1}{3}. \end{aligned}$$

5. [10 pts.] Let $f(x) = \dfrac{2}{x+1}$. Use the definition of the derivative to find $f'(0)$.

$$\boxed{f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{2}{x+1}-2}{x} = \lim_{x\to 0} \frac{2-2x-2}{x(x+1)} = \lim_{x\to 0} \frac{-2\cancel{x}}{\cancel{x}(x+1)} = -2}.$$

6. [10 pts.] Let $f(x) = \sqrt{x-1}$. Use the definition of the derivative to find $f'(5)$.

$$\boxed{f'(5) = \lim_{x\to 5} \frac{f(x)-f(5)}{x-5} = \lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5} = \lim_{x\to 5} \left(\frac{\sqrt{x-1}-2}{x-5}\right)\left(\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}\right)}$$

i.e., $\boxed{f'(5) = \lim_{x\to 5} \dfrac{(x-1)-4}{(x-5)(\sqrt{x-1}+2)} = \lim_{x\to 5} \dfrac{1}{\sqrt{x-1}+2} = 1/4}$.

5–8 Find an equation of the tangent line to the curve at the given point.

6. $y = x^3 - 3x + 1, \quad (2, 3)$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x) = x^3 - 3x + 1$

$f(2) = f(2) = 2^3 - 3(2) + 1 = 3$

$f(x) - f(2) = x^3 - 3x + 1 - (3)$

$= x^3 - 3x - 2$

$$\therefore \lim_{x \to 2} \frac{x^3 - 3x - 2}{x - 2} =$$

توهقت ؟

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$\longrightarrow$

5–8 Find an equation of the tangent line to the curve at the given point.

6. $y = x^3 - 3x + 1, \quad (2, 3)$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$\because a = 2$

$f(a+h) = f(2+h) = (2+h)^3 - 3(2+h) + 1$

$= (8 + 6h^2 + 12h + h^3) - 6 - 3h + 1$

$= h^3 + 6h^2 + 9h + 3$

$f(a) = f(2) = 2^3 - 3(2) + 1 = 3$

$f(a+h) - f(a):$

$f(2+h) - f(2) = h^3 + 6h^2 + 9h + 3 - 3$

$= h^3 + 6h^2 + 9h$

$\Rightarrow$

$$\lim_{h \to 0} \frac{h^3 + 6h^2 + 9h}{h} = \frac{h(h^2 + 6h + 9)}{h}$$

$$\therefore \lim_{h \to 0} h^2 + 6h + 9 = 0 + 0 + 9$$

$$= 9$$

equation of tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 9(x - 2)$$

$$y = 9x - 18 + 3$$

$$y = 9x - 15$$

5–8 Find an equation of the tangent line to the curve at the given point.

7. $y = \sqrt{x}, \quad (1, 1)$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad a = 1$$

$$f(x) = \sqrt{x} \quad , \quad f(a) = f(1) = \sqrt{1} = 1$$

$$\therefore \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \times \frac{\sqrt{x} + 1}{\sqrt{x} + 1}$$

$$= \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1}$$

$$\lim_{x \to 1} \frac{1}{\sqrt{1} + 1} = \frac{1}{2}$$

equation of tangent line $Y - Y_1 = m(x - x_1)$

$$Y - 1 = 1/2\,(x - 1) \Rightarrow Y = 1/2\,x - 1/2 + 1$$

$$Y = 1/2\,x + 1/2$$

20. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

---

equation of tangent line:

$y - y_1 = m(x - x_1)$

$\because g(5) = -3$ $\therefore x = 5,\ y = -3$

$(5, 3)$ is the Point

$\because g'(5) = 4$ $\therefore m = 4$

$\therefore y - (-3) = 4(x - 5)$

$\Rightarrow y + 3 = 4x - 20$

$\Rightarrow y = 4x - 23$

31–36 Find $f'(a)$. **33.** $f(t) = \dfrac{2t+1}{t+3}$

$$f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$

$$\because f(t) = \frac{2t+1}{t+3}$$

$$f(a) = \frac{2a+1}{a+3}$$

$$f(t) - f(a) = \frac{2t+1}{t+3} - \frac{2a+1}{a+3}$$

$$= \frac{(2t+1)(a+3) - [(2a+1)(t+3)]}{(a+3)(t+3)}$$

$$= \frac{2at+6t+a+3-(2at+6a+t+3)}{(a+3)(t+3)}$$

$$= \frac{2at+6t+a+3-2at-6a-t-3}{(a+3)(t+3)}$$

$$= \frac{5t-5a}{(a+3)(t+3)} = \frac{5(t-a)}{(a+3)(t+3)}$$

$\Longrightarrow$ cont

$$\therefore \lim_{t \to a} \frac{\frac{5(t-a)}{(a+3)(t+3)}}{t-a}$$

$$\lim_{t \to a} \frac{5(t-a)}{(a+3)(t+3)(t-a)}$$

$$= \lim_{t \to a} \frac{5}{(a+3)(t+3)} = \frac{5}{(a+3)(a+3)}$$

$$\therefore f'(a) = \frac{5}{(a+3)^2}$$

3. (a) (i) Using Definition 1 with $f(x) = x^2 + 3x$ and $P(-1, -2)$, the slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to -1} \frac{(x^2 + 3x) - (-2)}{x - (-1)} = \lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1} = \lim_{x \to -1} \frac{(x + 2)(x + 1)}{x + 1}$$

$$= \lim_{x \to -1} (x + 2) = -1 + 2 = 1$$

(ii) Using Equation 2 with $f(x) = x^2 + 3x$ and $P(-1, -2)$, the slope of the tangent line is

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h} = \lim_{h \to 0} \frac{[(-1 + h)^2 + 3(-1 + h)] - (-2)}{h}$$

$$= \lim_{h \to 0} \frac{1 - 2h + h^2 - 3 + 3h + 2}{h} = \lim_{h \to 0} \frac{h^2 + h}{h} = \lim_{h \to 0} \frac{h(h + 1)}{h} = \lim_{h \to 0} (h + 1) = 1$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(-1) = f'(-1)(x - (-1)) \;\Rightarrow$

$y - (-2) = 1(x + 1) \;\Rightarrow\; y + 2 = x + 1$, or $y = x - 1$.

4. (a) (i) Using Definition 1 with $f(x) = x^3 + 1$ and $P(1, 2)$, the slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{(x^3 + 1) - 2}{x - 1} = \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)\left(x^2 + x + 1\right)}{x - 1}$$

$$= \lim_{x \to 1} \left(x^2 + x + 1\right) = 1^2 + 1 + 1 = 3$$

(ii) Using Equation 2 with $f(x) = x^3 + 1$ and $P(1, 2)$, the slope of the tangent line is

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1 + h)^3 + 1\right] - 2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 + 1 - 2}{h} = \lim_{h \to 0} \frac{h^3 + 3h^2 + 3h}{h} = \lim_{h \to 0} \frac{h(h^2 + 3h + 3)}{h}$$

$$= \lim_{h \to 0} (h^2 + 3h + 3) = 3$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 2 = 3\,(x - 1)$, or $y = 3x - 1$.

5. Using (1) with $f(x) = 2x^2 - 5x + 1$ and $P(3, 4)$ [we could also use Equation (2)], the slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 3} \frac{(2x^2 - 5x + 1) - 4}{x - 3} = \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(2x + 1)(x - 3)}{x - 3}$$

$$= \lim_{x \to 3} (2x + 1) = 2(3) + 1 = 7$$

Tangent line: $y - 4 = 7(x - 3) \;\Leftrightarrow\; y - 4 = 7x - 21 \;\Leftrightarrow\; y = 7x - 17$

2.8 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a fixed number a:

$$\boxed{1} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

$$\boxed{2} \qquad \boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

---

4 Theorem If f is differentiable at a, then f is continuous at a.

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

(Rationalize the numerator.)

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is slightly smaller than the domain of f, which is $[0, \infty)$. ■

EXAMPLE 4 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

SOLUTION

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h} \\
&= \lim_{h\to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{-3h}{h(2+x+h)(2+x)} = \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2}
\end{aligned}
$$

■

21–31 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

26. $g(t) = \dfrac{1}{\sqrt{t}}$

$$\lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = g'(t)$$

$$\lim_{h \to 0} \frac{\dfrac{1}{\sqrt{t+h}} - \dfrac{1}{\sqrt{t}}}{h}$$

$$\lim_{h \to 0} \frac{\dfrac{\sqrt{t}}{\sqrt{t+h}\,\sqrt{t}} - \dfrac{\sqrt{t+h}}{\sqrt{t}\,\sqrt{t+h}}}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}\;h} \cdot \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}$$

$$\lim_{h \to 0} \frac{t - (t+h)}{\sqrt{t+h}\,\sqrt{t}\;h\,(\sqrt{t} + \sqrt{t+h})}$$

$$\lim_{h \to 0} \frac{-h}{\sqrt{t+h}\,\sqrt{t}\,h\,(\sqrt{t}+\sqrt{t+h})}$$

$$\lim_{h \to 0} \frac{-1}{\sqrt{t+h}\,\sqrt{t}\,(\sqrt{t}+\sqrt{t+h})}$$

$$= \frac{-1}{\sqrt{t+0}\,\sqrt{t}\,(\sqrt{t}+\sqrt{t+0})}$$

$$= \frac{-1}{t(\sqrt{t}+\sqrt{t})}$$

$$\therefore g'(t) = \frac{-1}{t(2\sqrt{t})} = \frac{-1}{2t\sqrt{t}}$$

57. Let $f(x) = \sqrt[3]{x}$.

(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.

(b) Show that $f'(0)$ does not exist.

(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at (0, 0).

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a}$$

$$f'(a) = \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{1}{(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \frac{1}{a^{2/3} + (a)^{1/3}(a)^{1/3} + a^{2/3}} = \frac{1}{a^{2/3} + a^{2/3} + a^{2/3}}$$

$$= \frac{1}{3a^{2/3}}$$

b) $f'(0) = \frac{1}{3(0)^{2/3}} = \frac{1}{0} = DNE$

c) $\because f'(0) = DNE$

$\therefore$ The slope "m" is undefined

$\therefore$ $f(x)$ at $(0,0)$ has vertical line

Use the **definition of the derivative** to find

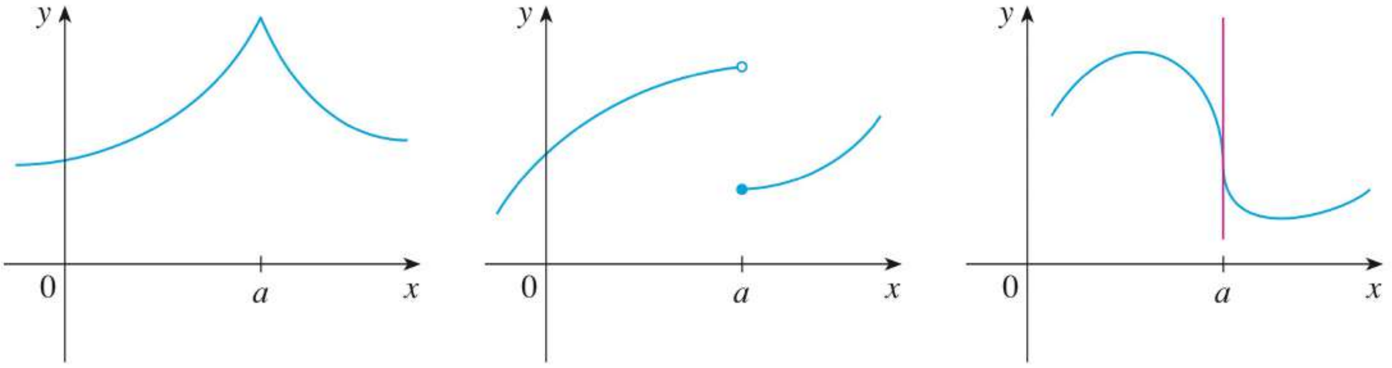
إذا طلب منك تحل المشتقة بالتعريف لازم تحل بإحدى
الطرق التالية (كيفك أي طريقة)

1) $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$

2) $m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$

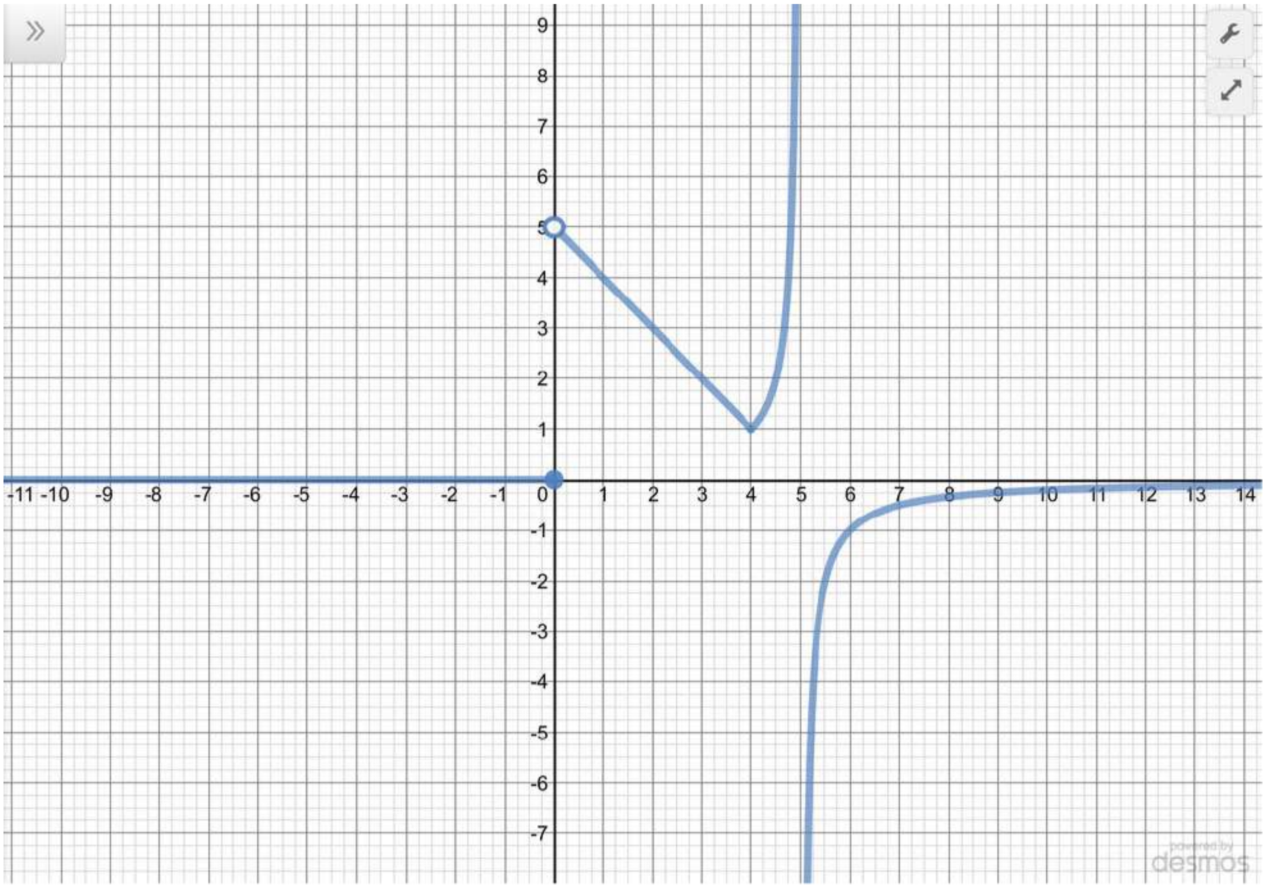
How Can a Function Fail To Be Differentiable?

not differentiable يعني متى تكون



(a) A corner (b) A discontinuity (c) A vertical tangent

الرسمة معطى



(c) Where is f discontinuous? $x = 0, 5$

(d) Where is f not differentiable? $x = 0, 4, 5$

corner

Rules:-

1) $$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{x}{\sin(x)}=1$$

2) $$\lim_{x\to 0}\frac{\tan(x)}{x}=\lim_{x\to 0}\frac{x}{\tan x}=1$$

3) $$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$

شروط استخدام القوانين

١- المفروض لما أعوض دايركت تعطيني الزاوية بصفر

٢- لازم زاوية البسط نفس المقام (وإذا مو نفسها نحاول نوصلها لنفس الشكل)

Remark:-

$$\sin^2 x=(\sin x)^2$$

$$\sin^2 x\neq\sin(x^2)$$

1) $$\lim_{x \to 0} \frac{\sin x}{2x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = 1/2\,(1) = 1/2$$

2) $$\lim_{x \to 0} \frac{\sin(2x)}{x}$$

$$\lim_{x \to 0} \frac{\sin(2x)}{x} \cdot \frac{2}{2}$$

$$2 \lim_{x \to 0} \frac{\sin(2x)}{2x} = 2\,(1) = 2$$

$$3)\ \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$$

$$x = \frac{1}{\frac{1}{x}}$$

$$4)\ \lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}$$

تذكير ! ما يصير
تختصر شي داخل
الدالة المثلثية مع شي
برا الدالة

$$\lim_{x\to 1} \frac{\sin((x-1)(x+1))}{(x-1)} \ , \ \frac{(x+1)}{(x+1)}$$

$$\lim_{x\to 1} (x+1) \cdot \lim_{x\to 1} \frac{\sin(x^2-1)}{x^2-1}$$

$$= 2 \cdot 1 = 2$$

$$\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

When $x \to 0$

$$\frac{1}{x} \longrightarrow \infty$$

$\therefore$ We use sandwitch theorem

ما نقدر نطبق نفس القاعدة الي في المسائل الي قبل
لان الزاوية لما عوضت ما كانت بصفر

EXAMPLE 5 Find $\lim_{x \to 0} \frac{\sin 7x}{4x}$.

SOLUTION In order to apply Equation 2, we first rewrite the function by multiplying and dividing by 7:

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

If we let $\theta = 7x$, then $\theta \to 0$ as $x \to 0$, so by Equation 2 we have

$$\lim_{x \to 0} \frac{\sin 7x}{4x} = \frac{7}{4} \lim_{x \to 0}\left(\frac{\sin 7x}{7x}\right)$$

$$= \frac{7}{4} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4}$$

■

EXAMPLE 6 Calculate $\lim_{x \to 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x:

$$\lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x \cos x}{\sin x}$$

$$= \lim_{x \to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x \to 0} \cos x}{\lim_{x \to 0} \dfrac{\sin x}{x}}$$

$$= \frac{\cos 0}{1}$$ (by the continuity of cosine and Equation 2)

$$= 1$$

39–50 Find the limit.

39. $\lim_{x\to 0} \frac{\sin 5x}{3x}$ $= \frac{\sin 0}{0} = \frac{0}{0}$

$$\lim_{x\to 0}\left(\frac{\sin 5x}{3x} \cdot \frac{5}{5}\right) = \frac{5}{3} \lim_{x\to 0} \frac{\sin 5x}{5x}$$

$$= \frac{5}{3} \cdot (1) = \frac{5}{3}$$

40. $\lim_{x\to 0} \frac{\sin x}{\sin \pi x}$ $= \frac{\sin 0}{\sin 0} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{\sin x}{1} \cdot \frac{1}{\sin \pi x}$$

$$\lim_{x\to 0}\left(\frac{\sin x}{\sin \pi x} \cdot \frac{x}{x} \cdot \frac{\pi}{\pi}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x} \cdot \frac{\pi x}{\sin \pi x} \cdot \frac{1}{\pi}\right)$$

$$= (1 \cdot 1)\left(\frac{1}{\pi}\right) = \frac{1}{\pi}$$

39–50 Find the limit.

42. $\displaystyle \lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta}$

$$\lim_{\theta \to 0} \left(\frac{\cos\theta - 1}{\sin\theta} \cdot \frac{\theta}{\theta} \right)$$

$$\lim_{\theta \to 0} \left(\frac{\cos\theta - 1}{\theta} \cdot \frac{\theta}{\sin\theta} \right) = 0 \cdot 1 = 0$$

43. $\displaystyle \lim_{x \to 0} \frac{\sin 3x}{5x^3 - 4x}$

$$\lim_{x \to 0} \left(\frac{\sin 3x}{5x^3 - 4x} \cdot \frac{3x}{3x} \right)$$

$$\lim_{x \to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3\cancel{x}}{\cancel{x}(5x^2 - 4)} \right) = 1 \cdot \frac{3}{5(0) - 4} = -\frac{3}{4}$$

39–50 Find the limit.

44. $\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2}$

$$\lim_{x\to 0}\left(\frac{\sin 3x}{x}\cdot\frac{\sin 5x}{x}\right)\left(\frac{3}{3}\cdot\frac{5}{5}\right)$$

$$\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{\sin 5x}{5x}\cdot 3\cdot 5\right)$$

$$= 3\cdot 5\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{\sin 5x}{5x}\right)$$

$$= 15\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{\sin 5x}{5x}$$

$$= 15\ (1)\ (1) = 15$$

54. (a) Evaluate $\lim\limits_{x\to\infty} x \sin \frac{1}{x}$. $= 1$

(b) Evaluate $\lim\limits_{x\to 0} x \sin \frac{1}{x}$. $\Rightarrow$ By S.T $= 0$

c) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x}{x} = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

I. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x}{2x} =$

a) $\frac{1}{2}$.

b) $\boxed{\frac{1}{\pi}}$.

c) 1.

d) None of the above.

I. $\lim\limits_{x\to 0} \frac{\cos^2 x - 1}{x^2} =$

a) 1.

b) 0.

[✓ c) -1]

d) ∞.

e) None of the above.

39–50 Find the limit.

46. $\lim_{x \to 0} \csc x \sin(\sin x)$

$$\lim_{x \to 0} \frac{1}{\sin x} \sin(\sin x)$$

$$\lim_{x \to 0} \frac{\sin(\sin x)}{\sin x} = 1$$

---

4. [5 + 5 = 10 pts.] Let $f(x) = \dfrac{\sin(x-1)}{x^3-1}$.
Show that f is discontinuous at $x = 1$, and classify the discontinuity as removable, jump, or infinite.

We have $\displaystyle\lim_{x\to 1} \frac{\sin(x-1)}{x^3-1} = \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} \lim_{x\to 1} \frac{1}{(x^2+x+1)} = \boxed{1/3}$.

Since f is undefined at $x = 1$, therefore f has a removable discontinuity at 1.

ما يصير اختصر أي شي داخل الدالة مع برا الدالة مثلا الي داخل الln والlog. والمثال هذا
داخل الدالة المثلثية مع الي برًا ما يصير !!

1. [10 pts.] Evaluate: $\lim\limits_{x\to 2} \dfrac{\sin(x-2)}{x^2+x-6}$.

$$\lim_{x\to 2} \frac{\sin(x-2)}{x^2+x-6} = \lim_{x\to 2} \frac{\sin(x-2)}{(x+3)(x-2)} = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)} \times \lim_{x\to 2} \frac{1}{(x+3)} = \boxed{\frac{1}{5}}.$$

---

3. [10 pts.] Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\sin(x^2)+2x}{x}$, if it exits.

We have $\lim\limits_{x\to 0} \dfrac{\sin(x^2)+2x}{x} = \lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x} + \lim\limits_{x\to 0} \dfrac{2x}{x} = 0 + 2 = 2.$

III. $\lim\limits_{x\to 0} \dfrac{\sin(x-1)}{x-1} =$ ← 280

a) 1.

b) $-\sin 1$.

c) -1.

[✓ d) $\sin 1$.]

e) None of the above.

47. $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{2\theta^2}$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta^2} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}$$

$$\lim_{\theta \to 0} \frac{\cos^2\theta - 1}{(2\theta^2)(\cos\theta + 1)} = \frac{-\sin^2\theta}{(2\theta^2)(\cos\theta + 1)}$$

$$= -\frac{1}{2} \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta + 1}$$

$$= -\frac{1}{2} \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)^2 \frac{1}{\cos 0 + 1}$$

$$= -\frac{1}{2} (1) \left(\frac{1}{1+1}\right)$$

$$= -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$$

Evaluate the limit $\lim\limits_{\theta\to 0} \dfrac{\cos\theta - 1}{3\theta^2}$, if it exits.

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{3\theta^2} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}$$

$$\lim_{\theta\to 0} \frac{\cos^2\theta - 1}{(3\theta^2)(\cos\theta + 1)} = \frac{-\sin^2\theta}{(3\theta^2)(\cos\theta + 1)}$$

$$\frac{-1}{3} \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2} \cdot \lim_{\theta\to 0} \frac{1}{\cos\theta + 1}$$

$$= \frac{-1}{3} \lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right)^2 \cdot \frac{1}{\cos 0 + 1}$$

$$= \frac{-1}{3} (1)^2 \cdot \left(\frac{1}{2}\right)$$

$$= \frac{-1}{3} \cdot \frac{1}{2} = -\frac{1}{6}$$

$$\lim_{x \to 0} x \cot(\pi x)$$

$$= \lim_{x \to 0} x \frac{1}{\tan \pi x} = 0 \frac{1}{\tan 0} = \frac{0}{0}$$

$$\lim_{x \to 0} x \frac{1}{\tan \pi x} \cdot \frac{\pi}{\pi} = \lim_{x \to 0} \frac{\pi x}{\pi \tan \pi x}$$

$$= \frac{1}{\pi} \lim_{x \to 0} \frac{\pi x}{\tan \pi x} = \frac{1}{\pi} \cdot (1) = \frac{1}{\pi}$$

III. $\lim\limits_{x \to 0} x \cot(\pi x) =$

(A) π.

(B) $-\pi$.

✓ **(C)** $\dfrac{1}{\pi}$.

(D) $-\dfrac{1}{\pi}$.

(E) None of the above.

39–50 Find the limit.

48. $\lim_{x\to 0}\frac{\sin(x^2)}{x}$ $= \lim_{x\to 0}\frac{\sin(x^2)}{x} * \frac{x}{x}$

$= \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot x = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot\lim_{x\to 0} x$

$= 1 * 0 = 0$

---

49. $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$ $= -\sqrt{2}$

$$\frac{1-\frac{\sin x}{\cos x}}{\sin x-\cos x} = \frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\sin x-\cos x} = \frac{\frac{\cos x-\sin x}{\boxed{\cos x}}}{\sin x-\cos x} = \boxed{\frac{\cos x-\sin x}{\cos x(\sin x-\cos x)}}$$

لو نلاحظ أقدر أشطب البسط مع
الي بالمقام بس الفرق الوحيد
الاشارة فراح آخذ السالب عامل
مشترك بالبسط عشان أشتبه
المقام

$$= \lim_{x\to\frac{\pi}{4}}\frac{-\cancel{(\sin x-\cos x)}}{\cos x\cancel{(\sin x-\cos x)}} = \lim_{x\to\frac{\pi}{4}} = \frac{-1}{\cos x} = \frac{-1}{\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

---

50. $\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$ $= \lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1}\frac{1}{x+2}\cdot\lim_{x\to 1}\frac{\sin(x-1)}{x-1}$

$= \frac{1}{3}\cdot 1 = \frac{1}{3}$