ثمرة
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Calculus 1 – Limits

(Section 2.2 & 2.3)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

Limits

1- Determine limit from graph

2- Direct Substitution.

3- Squeeze Theorem.

4- Continuity & Discontinuity.

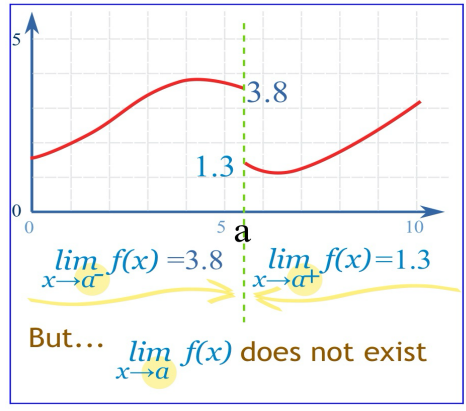
5- Find the values of a & b that make f continues everywhere.

6- IVT.

7- Asymptotes.

8- Definition of Derivative.

$$\lim_{x \to a} f(x)$$



- lim : النهاية
- f(x) : هي الدالة
- $x \to a$: x تقترب من a

الرمز	الوصف
$x \to a^+$	تقترب من a من جهة اليمين
$x \to a^-$	تقترب من a من جهة اليسار
$x \to a$	تقترب من a من الجهتين

1- Determine limit from graph

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

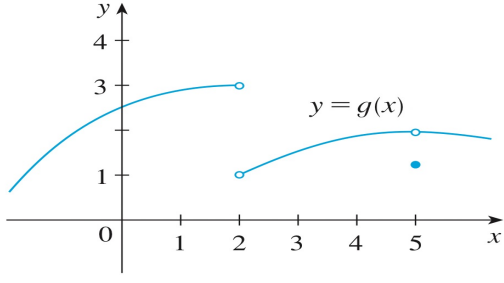
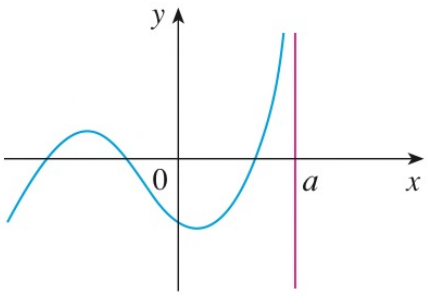
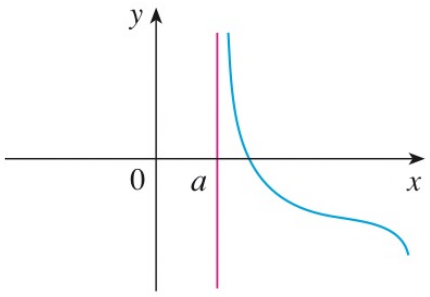


FIGURE 10

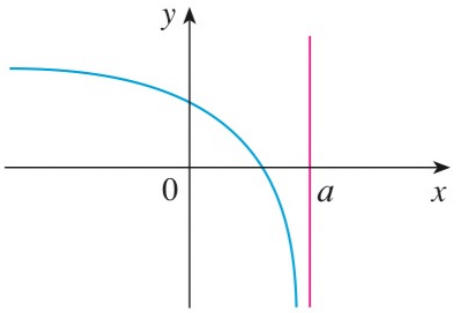
* Vertical Asymptotes



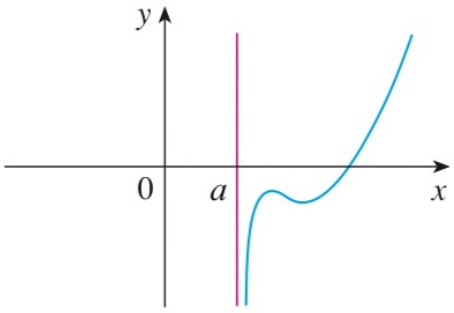
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

a is V.A

Infinite Limits

EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

To indicate the kind of behavior exhibited in Example 8, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

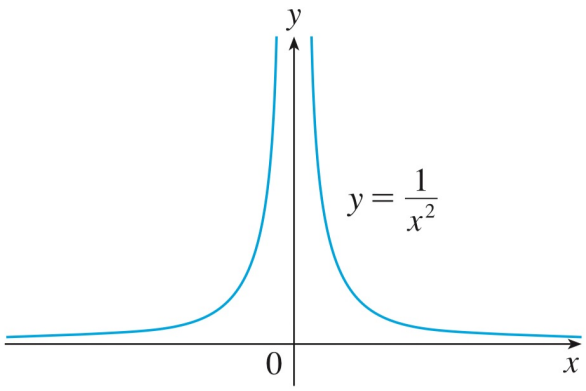


FIGURE 11

EXAMPLE 9 Find $\lim_{x \to 3^+} \dfrac{2x}{x - 3}$ and $\lim_{x \to 3^-} \dfrac{2x}{x - 3}$.

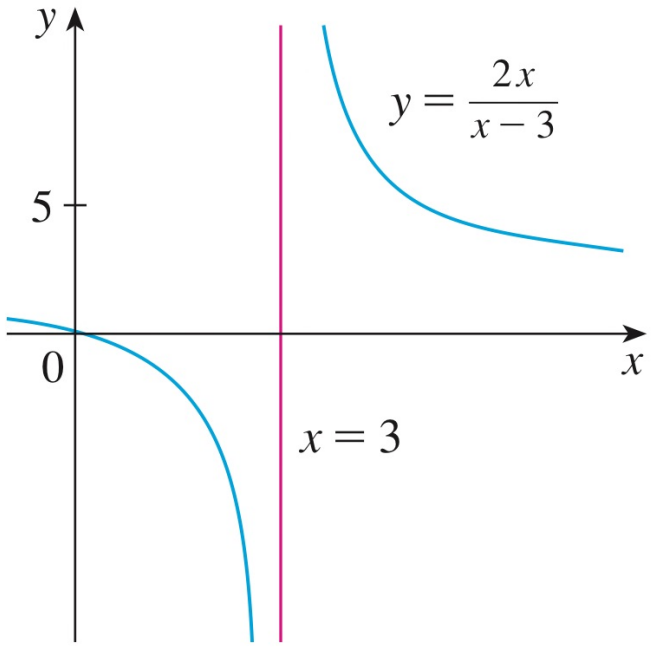


FIGURE 15

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

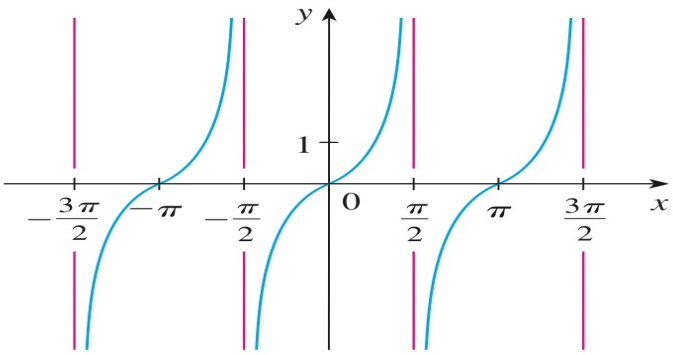
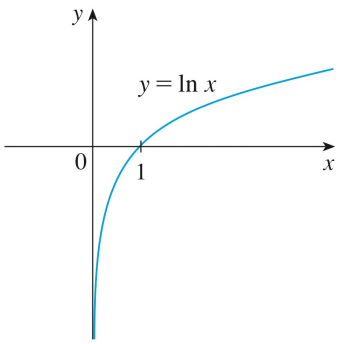


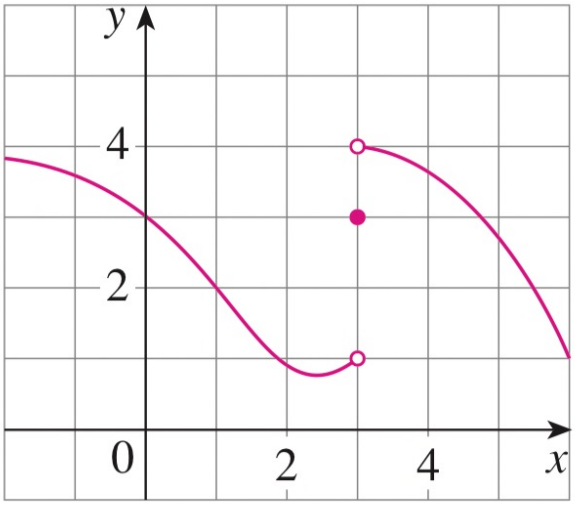
FIGURE 16
$y = \tan x$

$$\lim_{x \to 0^+} \ln x = -\infty$$



$$\lim_{x \to 2^+} \ln(x-2) = \ln(0^+) = -\infty$$

$x = 2$ is V.A



5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

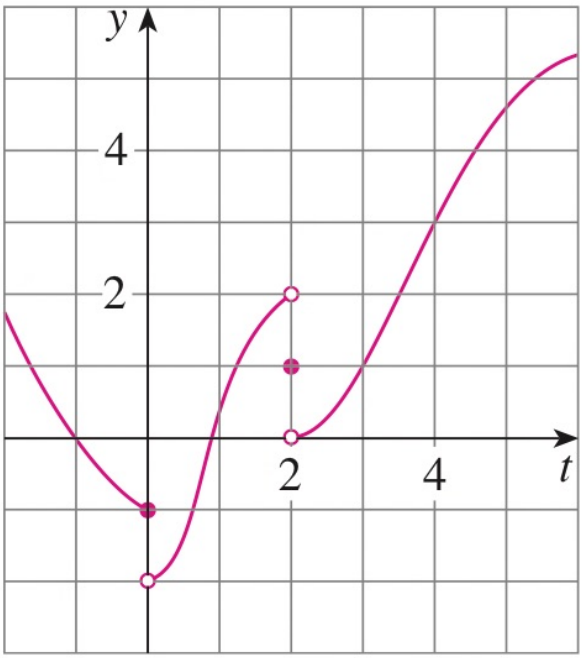
(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$

7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$



(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

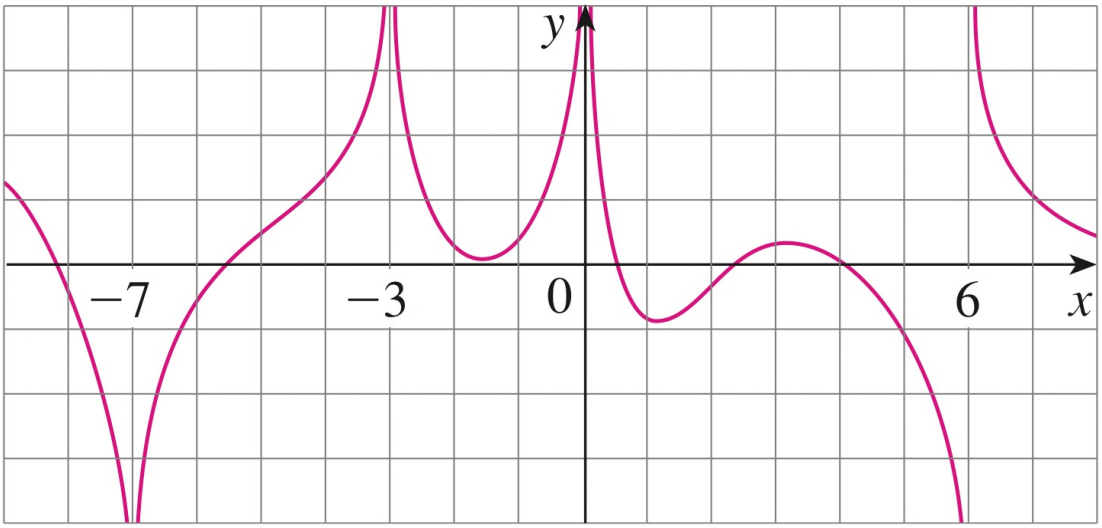
(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$

9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

(f) The equations of the vertical asymptotes.



(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

1. [$2.5 \times 4 = 10$ pts.] Use the given graph of f to evaluate each of the following limits, if it exists.

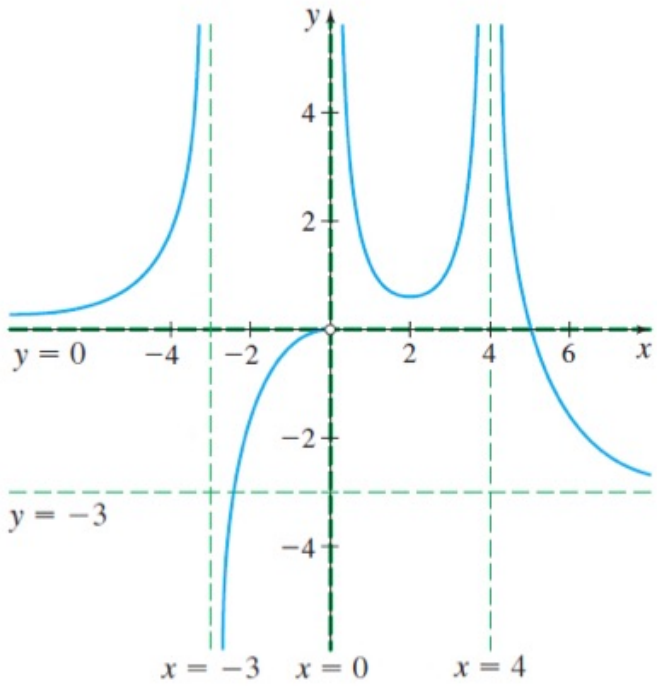


Figure 1: The graph of $y = f(x)$.

(a) $\lim_{x \to -3} f(x) =$

(b) $\lim_{x \to 0^-} f(x) =$

(c) $\lim_{x \to 0^+} f(x) =$

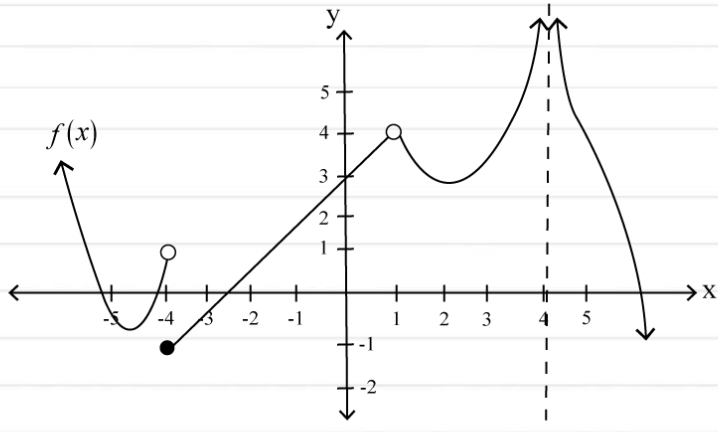
(d) $\lim_{x \to -\infty} f(x) =$

(a) $\lim_{x \to -3} f(x)$).

(b) $\lim_{x \to 0^-} f(x) =$.

(c) $\lim_{x \to 0^+} f(x) =$

(d) $\lim_{x \to -\infty} f(x) =$.



(a) $\lim_{x \to -4^-} f(x) =$ (b) $\lim_{x \to -4^+} f(x) =$ (c) $\lim_{x \to -4} f(x) =$

(d) $\lim_{x \to 1^-} f(x) =$ (e) $\lim_{x \to 1^+} f(x) =$ (f) $\lim_{x \to 1} f(x) =$

(g) $\lim_{x \to 4^-} f(x) =$ (h) $\lim_{x \to 4^+} f(x) =$ (i) $\lim_{x \to 4} f(x) =$

Calc

ملخص :-

١- حدد $x = a$: شوف الخط العامودي عند هذه القيمة.

٢- افحص من اليسار و اليمين :-

* Continuous * Jump

* Asymptote * Hole

2- Direct Substitution.

Solution:-

$5 , \frac{0}{2} = 0 , \frac{1}{0} = \pm\infty , -2 , \frac{1}{3}$

---

1) $\lim_{x \to 3} (2x + 1) = 2(3) + 1 = 7$

2) $\lim_{x \to -2} (x^2 - 4) = (-2)^2 - 4 = 0$

3) $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \frac{0}{0} \Rightarrow ??$

4) $\lim_{x \to \infty} x^{-2} \cos\left(\frac{1}{x}\right) = \infty \cdot \cos\infty$

$\Rightarrow ??$

قيمة غير معرفة

إذا عوضنا وطلع الناتج indeterminate forms :

Determinate-Indeterminate Forms Table	
Indeterminate Forms	Determinate Forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty/\pm\infty$	$-\infty - \infty = -\infty$
$\infty(-\infty)$	$0^{\infty} = 0$
$0(\infty)$	$0^{-\infty} = \infty$
0^0	$(\infty)\cdot(\infty) = \infty$
1^{∞}	
∞^0	
$\sin\infty$	S.T
$\cos\infty$	

a) $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$

b) $\lim_{x \to 5} \frac{x - 5}{x^2 - 6x + 5} = \frac{0}{0}$

c) $\lim_{x \to 4} \dfrac{x-4}{\sqrt{x}-2}$

---

d) $\lim_{x \to 4} \dfrac{x-55}{x^2-8x+16}$

e) $\lim_{h \to 0} \dfrac{(h-3)^2 - 9}{h}$

f) $\lim_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \dfrac{0}{0}$

g) $\lim_{x \to -3^+} \dfrac{x + 2}{x + 3} = \dfrac{-3 + 2}{-3 + 3}$

h) $\lim_{x \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$

---

i) $\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$

J) $\lim_{x \to 4} \dfrac{\sqrt{3} - \sqrt{x-1}}{x-4}$

4. [10 pts.] Evaluate the limit $\lim_{x \to 3} \dfrac{x-3}{2-\sqrt{x+1}}$, if it exists.

* piecewise function

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

* ملاحظة :

limit $x \to a^+$ مطلق a عينك على ١)

٢) صفر المطلق

K) $\lim_{x \to -2^-} \frac{|x+2|}{x+2}$

---

2) $\lim_{x \to 0} \frac{|x-4|}{x-4}$

10. $\lim_{v \to 4^+} \dfrac{4 - v}{|4 - v|}$

2) $\lim_{x \to -2} \dfrac{5|x+2|}{x+2}$

3) $\lim_{x \to 6^-} \left(|x-6| - 3\right)$

Evaluate the following limits:

(a) $f(x) = \begin{cases} \frac{x^2-x-6}{|x+2|}, & \text{if } x \neq -2, \\ 5, & \text{if } x = -2. \end{cases}$ $\lim_{x \to -2} f(x)$

(b) $f(x) = \frac{1}{|e^x - e^{-x}|}$. $\lim_{x \to 0} f(x)$

Evaluate each of the following limits, if it exists.

(a) $\lim_{x \to 1} \dfrac{|x|(x-1)}{x^3 - x}$:

2) Squeeze theorem

نستخدم squeeze theorem مع الدوال المثلثية لما يطلع لي الناتج مثلاً :

$0 \cdot e^{\sin\infty}$ ..... , $0 \sin\infty$, $0\cos\infty$, $\infty \sin\infty$

Reminder :-

$$-1 \leq \sin\theta \leq 1$$

$$-1 \leq \cos\theta \leq 1$$

EXAMPLE 1 Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$. $= 0 \sin\infty$

a) $\lim_{x \to \infty} x^{-6} \sin\left(\frac{2019}{x^2}\right)$ $= \infty^{-6} \sin \infty$

b) $\lim_{x \to 1} \left[x^4 + (x^2 - 2x + 1) \sin\left(\frac{1}{x-1}\right) \right]$

c) $\lim_{x \to 2} (x^2 - 4x + 4) \cos\left(\frac{2}{x-2}\right)$

d) $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin \frac{\pi}{x}}$

e) $\lim_{x \to \infty} \frac{\cos x}{e^{2x}}$

1. Evaluate the limits if they exist.

f) $\lim_{x \to 0} x^2 \cos \frac{1}{x^2}$

b) Use the Squeeze Theorem to evaluate the Following limits:-

$$\lim_{x \to 0} \left(\frac{x}{2+x} \right)^2 \sin \left(\frac{2+x}{x} \right)$$

Evaluate each of the following limits, if it exists.

(b) $\lim_{x \to 2+} \sqrt{x-2} \cos\left(\frac{1}{x-2}\right)$.

✱ Squeeze theorem + inequity

If $2x - 1 \le f(x) \le x^2 - 2x + 3$ for $x \ge 0$, find $\lim_{x \to 2} f(x)$.

5. [10 pts.] If $e^x \leqslant f(x) \leqslant \dfrac{4x^2+1}{x^2}$, find $\lim\limits_{x\to 0} x^4 f(x)$, if it exists.

---

6. [10 pts.] Let $f(x)$ be a function which satisfies

$$5x - 6 \leq f(x) \leq x^2 + 3x - 5 \text{ for all } x \geq 0.$$

Find $\lim\limits_{x\to 1} f(x)$.

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Calculus 1 – Limits

(Section 2.4)

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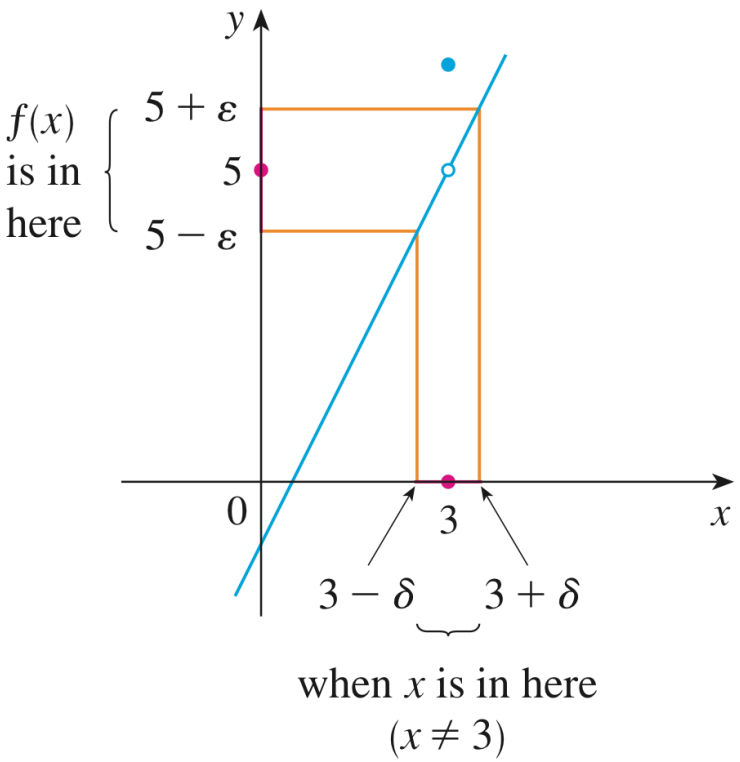
1 Use the ϵ-δ definition of a limit

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$



حل عن طريق δ و ε :-

1) For any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

2) $|f(x) - L| < \varepsilon$ افرض المطلوب

3) $|x - c|$ حولها لشكل

4) $\delta \Leftarrow \varepsilon$ حدد قيمة δ بدلالة

5) $\therefore$ By definition $\lim_{x \to c} f(x) = L$

15–18 Prove the statement using the ε, δ definition of a limit

15. $\lim_{x \to 3} \left(1 + \frac{1}{3}x\right) = 2$

15–18 Prove the statement using the ε, δ definition of a limit

18. $\lim_{x \to -2} (3x + 5) = -1$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x \to 1} \dfrac{2 + 4x}{3} = 2$

19–32 Prove the statement using the ε, δ definition of a limit.

20. $\lim_{x \to 10} \left(3 - \frac{4}{5}x\right) = -5$

19–32 Prove the statement using the ε, δ definition of a limit.

23. $\lim_{x \to a} x = a$

1. [10 pts.] Use the (ϵ, δ)–definition of the limit to show that $\lim_{x \to 1} f(x) = 3$, where $f(x) = 2x + 1$.

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Calculus 1 – Continuity

(Section 2.5)

For Contact and Support:

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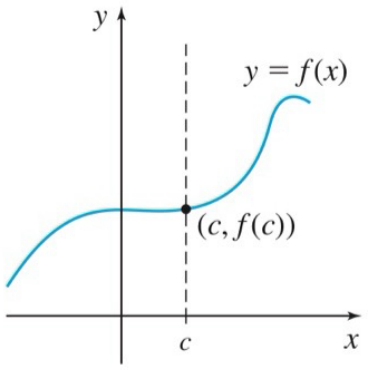
3) Continuity and discontinuity

a) $f(a)$ is defined.

b) $\lim_{x \to a} f(x)$ exist.

c) $\lim_{x \to a} f(x) = f(a)$.



(a) $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

DEFINITION Continuity at a Number

A function f is **continuous at a number** c if the following three conditions are met:

- $f(c)$ is defined (that is, c is in the domain of f)
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

If *any one* of these three conditions is not satisfied, then the function is **discontinuous at** c.

* Where the function Continuous :-

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

* أي دالة فوق فهي مستمرة في مجالها

* الحل = حدد الـ Domain ، وبعدين نقول إن الدالة مستمرة (Cont..) في هذا المجال .

تذكر :-

١) المقام ≠ ٠ .

٢) داخل جذر الزوجي ≥ ٠ .

٣) داخل اللوغاريتم ln( ) ، log( ) .

٤) الدوال المجمعة :

إذا في أكثر من شرط (كسر ، جذر ، لوغاريتم) ناخذ التقاطع بين الشروط .

EXAMPLE 6 Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

On what interval are the following function continuous

$$f(x) = \sqrt{x} + \frac{2x}{x^2+1}$$

On what interval are the following function continuous

Evaluate $\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 9 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

28. $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$

I. The function $f(x) = \dfrac{3^x}{(x-2)\sqrt{1-x^2}}$ is continuous on the interval

a) $(-\infty, \infty)$.

✓ b) $(-1, 1)$.

c) $(-1, 2)$.

c) $(-1, 2)$.

d) $(1, 2)$.

e) None of the above.

١) شروط الاستمرارية عند $x = a$

1. $f(a)$ مُعرفة

2. $\lim_{x \to a^-} f(x)$ و $\lim_{x \to a^+} f(x)$ exist.

3. $\lim_{x \to a} f(x) = f(a)$

* الخلاصة : قيمة الدالة = نهاية اليمين = نهاية اليسار عند النقطة .

* طريقة حل سؤال الاستمرارية.

١) حدد نقطة الشك .

٢) احسب $\lim_{x \to a^-} f(x)$ و $\lim_{x \to a^+} f(x)$

٣) احسب $f(a)$

٤) إن وجد مجهول / مجاهيل : ساوي النهايات ببعض أو مع الصورة لتحصل ع معادلة / معادلتين.

Show that f is Continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 1 \\ \ln x & \text{if } x > 1 \end{cases}$$

$\lim_{x \to 1^-} f(x) = f(1)$

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$$

For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

$f(2) = \lim_{x \to 2^+} f(x)$

$$f(x) = \begin{cases} ax^2 + 3, & \text{if } x \leq -1 \\ 3x + 4, & \text{if } x > -1 \end{cases}$$

$f(x)$ Continuous, $a = ??$

2. (3 points) For what value of the constant a is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^2 + 2a & \text{if } x < 2 \\ xa^2 & \text{if } x \geq 2 \end{cases}$$

Q5. Determine the value of the constant a that makes the function f continuous on the real numbers $\Re$ $\quad f(x) = \begin{cases} \frac{x^2-8x+15}{x-5} & if\ x \neq 5 \\ 2a & if\ x = 5 \end{cases}$

(a) -2 (b) 1 (c) -1 (d) 2

*طريقة حل معادلتين بالمجهولين :-

الطريقة الأولى :-

١) رتب المعادلتين.

٢) وحد معامل أحد المجهولين بضرب المعادلة / معادلتين.

٣) اجمع / اطرح المعادلتين لإلغاء مجهول واحد.

٤) احسب المجهول المتبقي.

٥) عوض به في إحدى المعادلتين لإيجاد المجهول الآخر.

٦) تحقق شريطاً بالتعويض في الشرط الأصلي.

الطريقة الثانية ، التعويض (Subsitution) :-

١) من إحدى المعادلتين عزل مجهولاً (مثلاً = b) .

٢) عوض في المعادلة التالية .

٣) احسب المجهول الأول .

٤) عوض به لإيجاد المجهول الثاني .

٥) تحقق بالتعويض في الشروط الأصلية .

2. For what value of a and b is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^3 - 6a & \text{if} \quad x < -3 \\ 3 & \text{if} \quad x = -3 \\ bx - a & \text{if} \quad x > -3 \end{cases}$$

48. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$

2. For what value of a and b is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^3 - 6a + 3 & \text{if} \quad x < -5 \\ 3 & \text{if} \quad x = -5 \\ bx - a + 2 & \text{if} \quad x > -5 \end{cases}$$

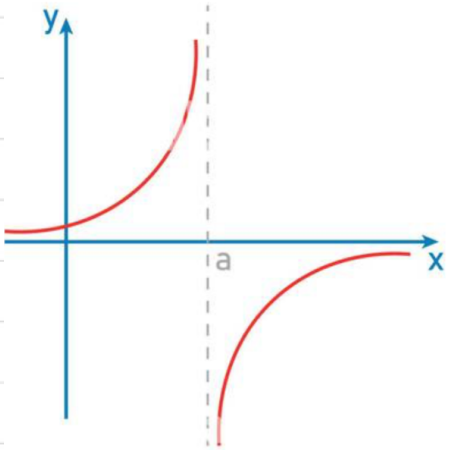
* finding function value using limits & continuity

4. Suppose f and g are continuous functions such that $g(2) = 3$ and

$$\lim_{x \to 2} [3f(x) + f(x)g(x)] = 39.$$

Find $f(2)$?

Types of discontinuity



Infinite Discontinuity

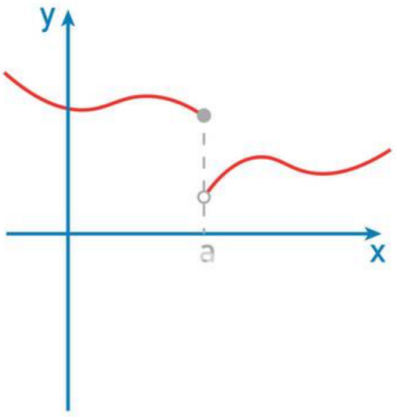
$\lim_{x \to a^+} f(x) = \pm\infty$

or

$\lim_{x \to a^-} f(x) = \pm\infty$

النهاية إلى خط مقارب عامودي لي انفنتي

* infinite discontinuity

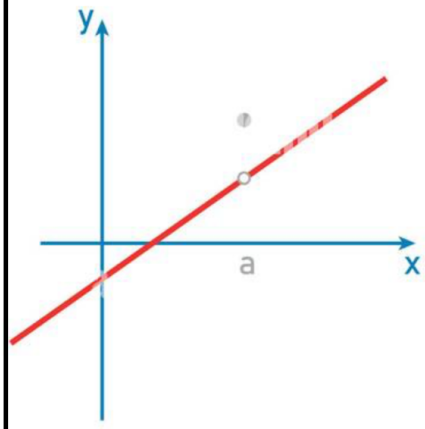


Jump Discontinuity

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

نهاية اليمين ≠ نهاية اليسار

* Jump discontinuity



Removable Discontinuity

$\lim_{x \to a} f(x)$ exist

$\lim_{x \to a^+} f(x) =$

$\lim_{x \to a^+} f(x)$

$\lim_{x \to a} f(x) \neq f(a)$

النهاية موجودة لكن غير معرفة

* Removable discontinuity

b) Find and classify the discontinuities of f as removable, infinite or jump.

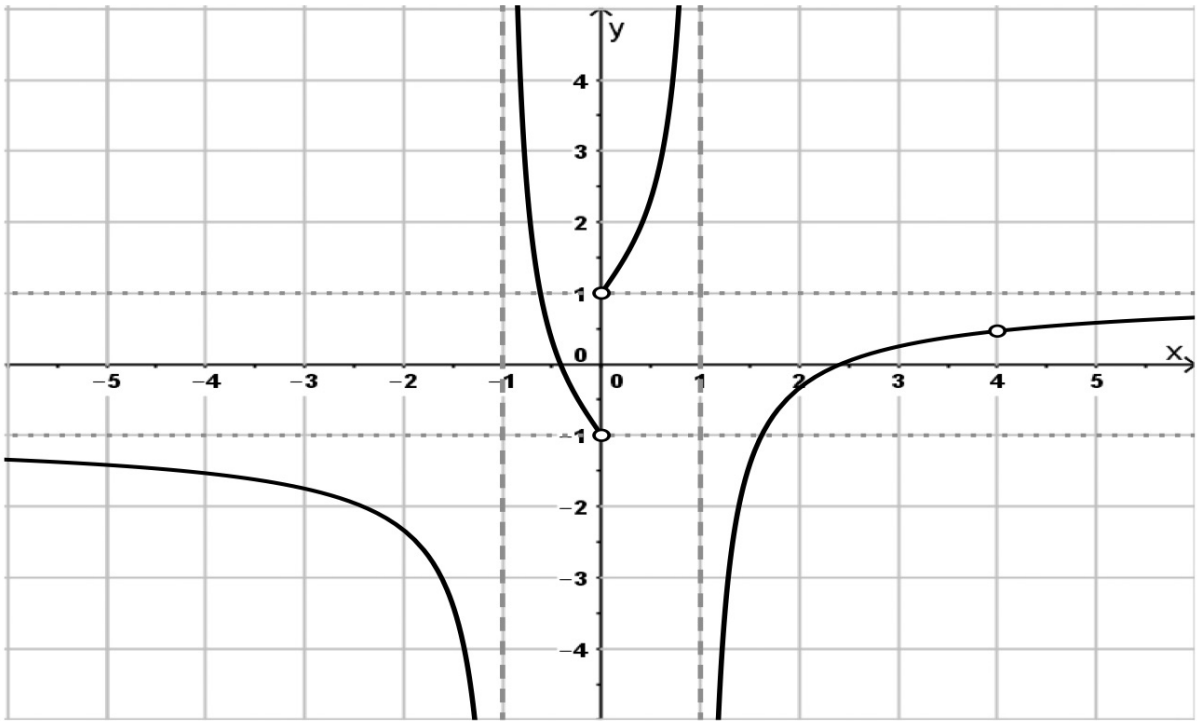
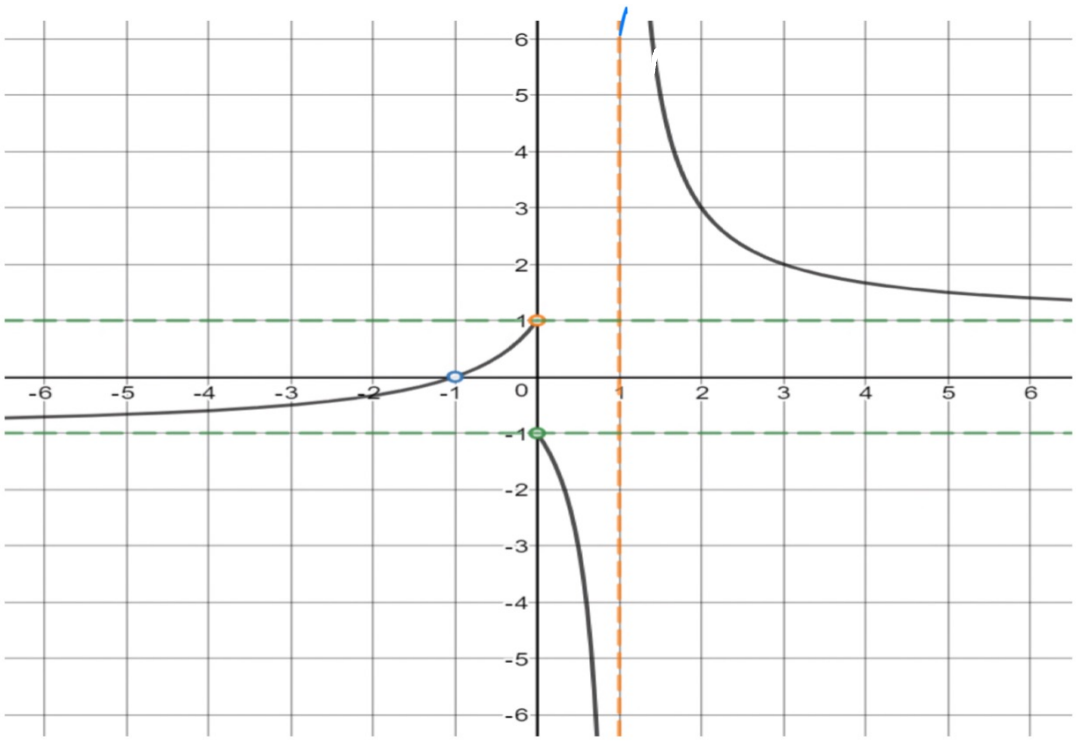


Figure 1: The graph of $y = f(x)$.

1. [$5 \times 4 = 20$ pts.] Use the given graph of f to select the correct answer.



(I) $\lim_{x \to 0} f(x) =$

(a) 1 (b) 0 (c) ∞ (d) -1 (e) None of the mentioned

(II) $\lim_{x \to 1^+} f(x) =$

(a) $+\infty$ (b) $-\infty$ (c) 1 (d) 0 (e) None of the mentioned

---

(IV) The function f has

a) a jump discontinuity at $x = 1$

b) a removable discontinuity at $x = 0$

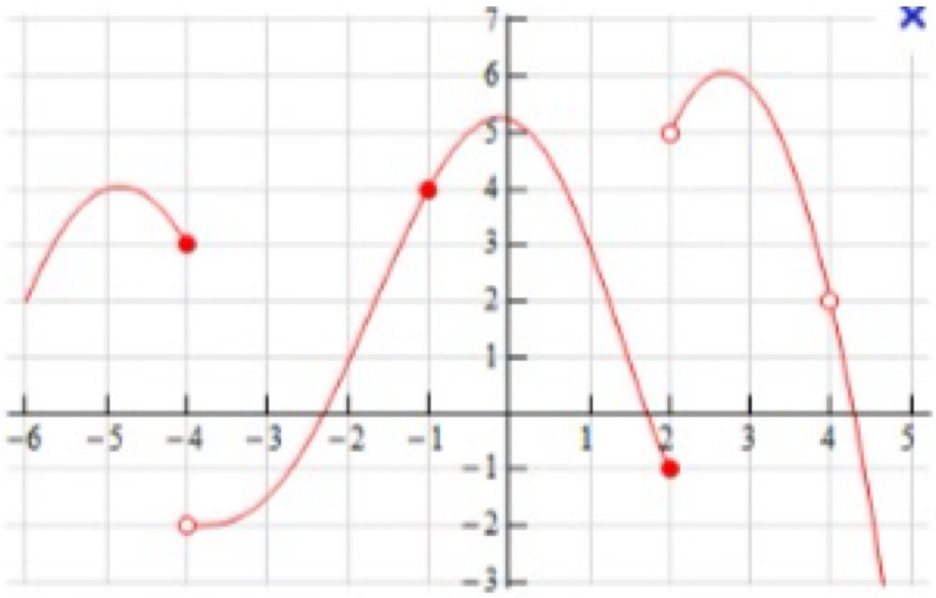
c) a jump discontinuity at $x = -1$

d) a removable discontinuity at $x = -1$

e) None of the above

The graph of function f(x) is given below. Find the following limits. If limit does not exist, write DNE.

b) Find and classify the discontinuities of f as removable, infinite or jump.



1) $f(4) + \lim_{x \to 4} f(x) =$

2) $\lim_{x \to -4^-} f(x) + \lim_{x \to 4} f(x) =$

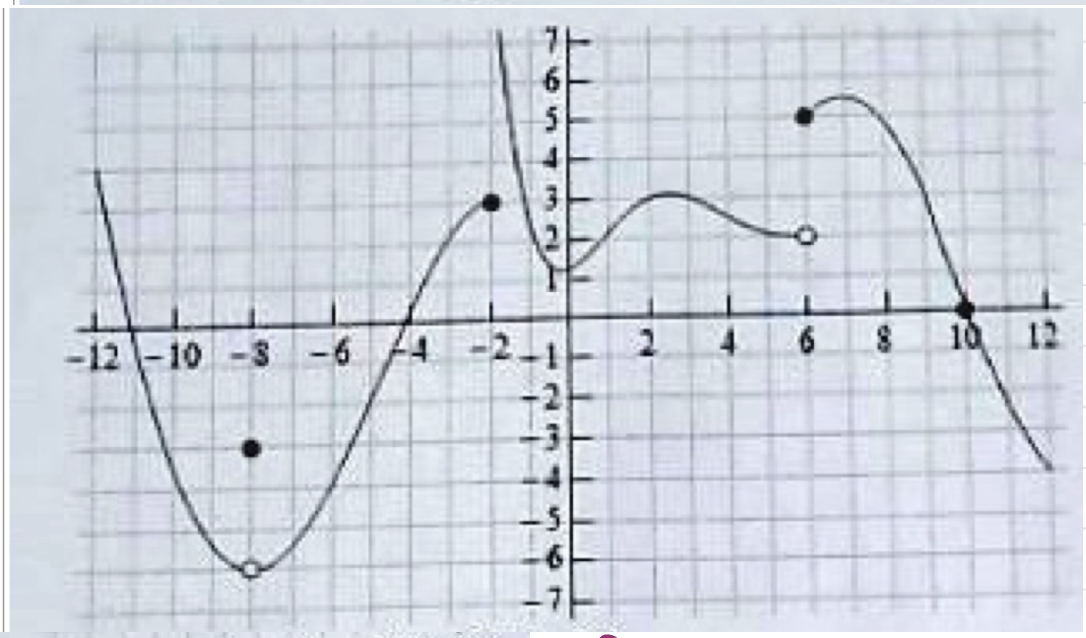
3) $\lim_{x \to 2^-} f(x) =$

4) $\lim_{x \to 4^+} f(x) + 2 \lim_{x \to 2^+} f(x)$

5) $f(-1) + 2 \lim_{x \to -4^+} f(x) =$

b) f is discontinuous at $x =$

6. Answer the following questions using the graph of $f(x)$ shown below.



$\lim_{x \to -2^-} f(x) =$

$\lim_{x \to -2^+} f(x) =$

a. Find the value of $f(-8)$. $f(-8) =$

b. Find the following limits. If a limit does not exist explain why.

1) $\lim_{x \to -8} f(x) =$

2) $\lim_{x \to 6^-} f(x) =$

3) $\lim_{x \to 6^+} f(x) =$

4) $\lim_{x \to 6} f(x) =$

c) Find and classify the discontinuities of f as removable, infinite or jump.

f is discontinuous at $x =$

* أنواع الـ Discontinuity

1) Removable :-

$$f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

2) Jump

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

3) Infinite (vertical Asy)

$$f(x) = \lim_{x \to 3} \frac{1}{x - 3}$$

خطوات حل الـDiscontinuity

١) افحص المقام إذا كان = ٥ .

٢) إذا دالة مقطعية (piece wise)

افحص limit من اليمين و اليسار

٣) حدد إذا Cont أو disCont و النوع

Jump or Removable or infinite

Classify the discontinuities

$$f(x) = \frac{x^2 - 2x}{|x|(x^2 - x - 2)}$$

2. (20 pts) Let $f(x) = \begin{cases} \dfrac{x^2+x-6}{x+3}, & \text{if } x < 0 \\ \dfrac{x+1}{(x-1)^2}, & \text{if } x \geq 0 \end{cases}$

Find all the points of discontinuity of f. Classify each discontinuity as removable, jump, or infinite.

---

6. [$10 + 5 = 15$ pts.] Let

$$f(x) = \begin{cases} x^2 - \ln(x+e), & \text{if } -1 < x \leqslant 0, \\ e^x - \sin x, & \text{if } x > 0. \end{cases}$$

(a) Find and classify the discontinuities of f as removable, infinite, or jump.

(b) Does $\lim_{x\to 0}(f(x))^2$ exist? explain why or why not?

17–22 Explain why the function is discontinuous at the given number a.

18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$ $\qquad a = -2$

17–22 Explain why the function is discontinuous at the given number a.

19. $f(x) = \begin{cases} x + 3 & \text{if } x \leqslant -1 \\ 2^x & \text{if } x > -1 \end{cases}$ $\qquad a = -1$

17–22 Explain why the function is discontinuous at the given number a.

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$

5. [10 pts.] Determine whether f is continuous at $x = 1$, where:

$$f(x) = \begin{cases} 2^x, & \text{if } x \geq 1 \\ \dfrac{1 - x^2}{|1 - x|}, & \text{if } x < 1. \end{cases}$$

4. [2 + 8 = 10 pts.] Define $f(x) = \tan^{-1}\left(\frac{1}{x-2}\right)$ for $x \neq 2$.

(a) Explain why f is discontinuous at $x = 2$.

(b) Classify the type of this discontinuity.

3. [10 + 10 = 20 pts.] Let $f(x) = \dfrac{x^2 - 4x}{|x|(x+2)}$.

(a) Find and classify the discontinuities of f as removable, infinite or jump.

Q2. [5+5=10 pts.] Let $f(x) = \begin{cases} x^2 - 3 & \text{if } x \neq 3 \\ 3 & \text{if } x = 3. \end{cases}$

(a) Determine whether f is continuous at $x = 3$.

Q1. [5+5=10 pts.] Let $f(x) = 3 + \ln(2x + 5)$.

(a) Find the interval(s) where f is continuous.

---

5. [10 pts.] Find a value for the constant A, if any, that makes the function f continuous at $x = 1$, where

$$f(x) = \begin{cases} \dfrac{\sqrt{x+1} - \sqrt{2x}}{x-1} & \text{if } x \geqslant 0,\ x \neq 1 \\ \\ A & \text{if } x = 1. \end{cases}$$

EXAMPLE 5 Determining Whether a Function Is Continuous on a Closed Interval

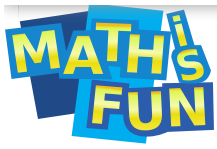
Is the function $f(x) = \sqrt{4 - x^2}$ continuous on the closed interval $[-2, 2]$?

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

15. $f(x) = x + \sqrt{x - 4}\,, \quad [4, \infty)$

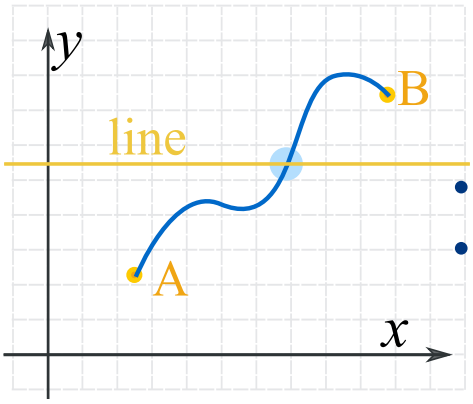
15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

16. $g(x) = \dfrac{x - 1}{3x + 6}, \quad (-\infty, -2)$



Intermediate Value Theorem

The idea behind the Intermediate Value Theorem is this:



When we have **two points** connected by a continuous curve:

- one point below the line
- the other point above the line

... then there will be **at least one place** where the curve crosses the line!

Well **of course** we must cross the line to get from A to B!

Now that you know the **idea**, let's look more closely at the details.

Continuous

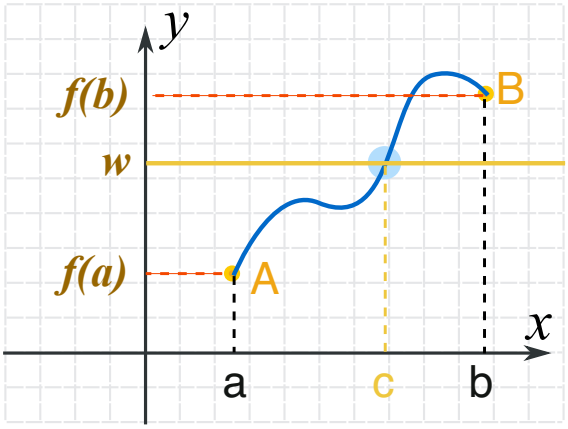
The curve must be **continuous** ... no gaps or jumps in it.

Continuous is a special term with an exact definition in calculus, but here we will use this simplified definition:

we can draw it without lifting our pen from the paper

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Here is the Intermediate Value Theorem stated more formally:



When:

- The curve is the function y = f(x),
- which is continuous on the interval [a, b],
- and w is a number between f(a) and f(b),

Then ...

> ... there must be at least one value **c** within [a, b] such that **f(c) = w**

In other words the function y = f(x) at some point must be w = f(c)

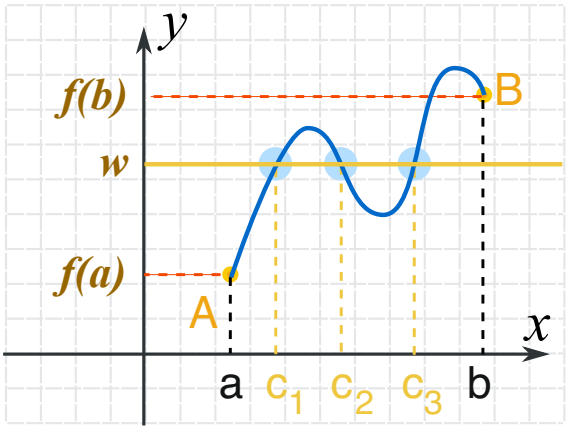
Notice that:

- **w** is between f(a) and f(b), *which leads to ...*
- **c** must be between a and b

At Least One

It also says "at least one value c", which means we **could** have more.

Here, for example, are 3 points where f(x)=w:



How Is This Useful?

١) أتأكد المعادلة صفرية
continuous ٢) أتأكد أن
٣) إذا عطاني فترة رح أعوض فيها
٤) حَوله الاحتمالات

Whenever we can show that:

- there is a point above some line
- and a point below that line, and
- that the curve is continuous,

we can then safely say "yes, there is a value somewhere **in between** that is on the line".

Example: is there a solution to $x^5 - 2x^3 - 2 = 0$ between x=0 and x=2?

At x=0:

$$0^5 - 2 \times 0^3 - 2 = \mathbf{-2} < 0$$

At x=2:

$$2^5 - 2 \times 2^3 - 2 = \mathbf{14} > 0$$

Now we know:

- at x=0, the curve is below zero
- at x=2, the curve is above zero

And, being a polynomial, the curve will be continuous,

so **somewhere in between** the curve must cross through y=0

Yes, there is a solution to $x^5 - 2x^3 - 2 = 0$ in the interval [0, 2]

f is cont. on (0, 2) Then
by I.V.T, There exists a c in (0, 2)
such that f(c) = 0 ∴ c is a root

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

EXAMPLE 10 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

53. $x^4 + x - 3 = 0, \quad (1, 2)$

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

54. $\ln x = x - \sqrt{x}, \quad (2, 3)$

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

55. $e^x = 3 - 2x, \quad (0, 1)$

7. [10 pts.] Use the Intermediate Value Theorem to show that the equation

$$e^x + \cos x = 4 \text{ has a real root.}$$

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

56. $\sin x = x^2 - x, \quad (1, 2)$

1. [10 pts.] Let $f(x) = \begin{cases} x, & \text{if } x < a, \\ x^2 - 2, & \text{if } x \geqslant a. \end{cases}$

Find all values of a, if any, for which f is continuous everywhere.

---

6. [10 pts.] Use the Intermediate Value Theorem to show that the equation $\cos x - \ln(x+1) = 0$ has a real root.

---

6. [10 pts.] Suppose that the function f is continuous on $[0, 1]$ and satisfies $0 < f(x) < 1$ for all x in $[0, 1]$. Use the Intermediate Value Theorem to show that the equation $f(x) - x = 0$ has a solution in $(0, 1)$.

8. [$5 + 10 = 15$ pts.] a) State the Intermediate Value Theorem.

b) Show that there is at least one real root of the equation $\cos(\sqrt{x}) = e^x - 2$.

* IvT + intersection

Show that the graphs

$f(x) = -3x^3 - 2x + 1$

$g(x) = 2x^3 - x^2 + 4$

intersect

إذا طلب منك intersect يعني يبي يعرف وين تتقاطع الدالتين.
نفس فكرة حل IVT بس الفرق بالبداية تطرح الدالتين

Q6. [10 pts.] Let $f(x) = 2 + x\sin(x)$ and $g(x) = x^2$. Use the Intermediate Value Theorem to show that the graphs of f and g intersect at least once.

Q6. [10 pts.] Let $f(x) = 2 + x\sin(x)$ and $g(x) = x^2$. Use the Intermediate Value Theorem to show that the graphs of f and g intersect at least once.

ثمرة
Thamrah

Kuwait University

Calculus 1 – Asymptotes

(Section 2.6)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

* Asymptotes

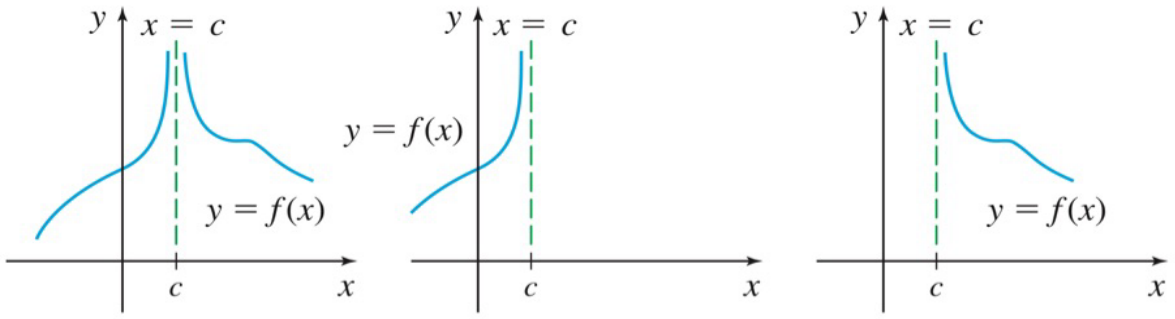
* How determine Asymptotes from graph.

* evaluating $\lim_{x \to \pm\infty} f(x)$.

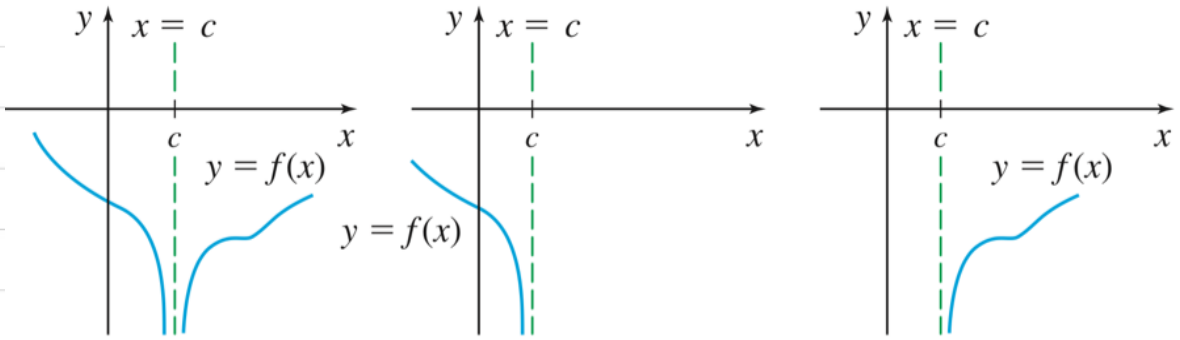
* Find the vertical & horizontal asymptotes "if any".

* How determine asymptotes from graph.

Vertical Asymptotes



(a) $\lim_{x \to c} f(x) = \infty$ (b) $\lim_{x \to c^-} f(x) = \infty$ (c) $\lim_{x \to c^+} f(x) = \infty$



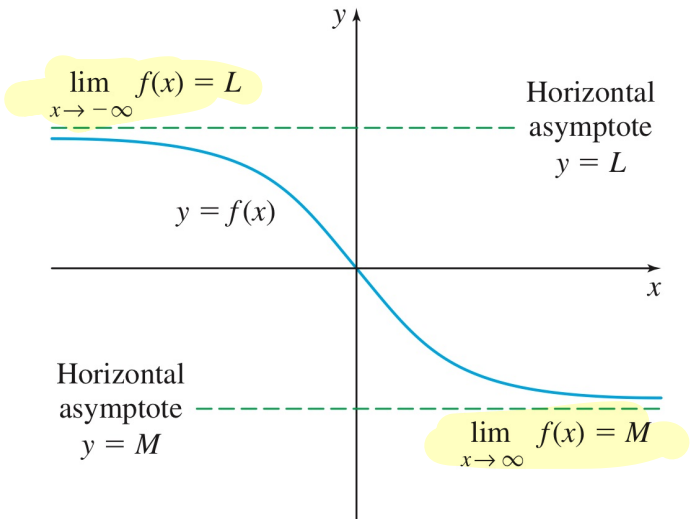
(d) $\lim_{x \to c} f(x) = -\infty$ (e) $\lim_{x \to c^-} f(x) = -\infty$ (f) $\lim_{x \to c^+} f(x) = -\infty$

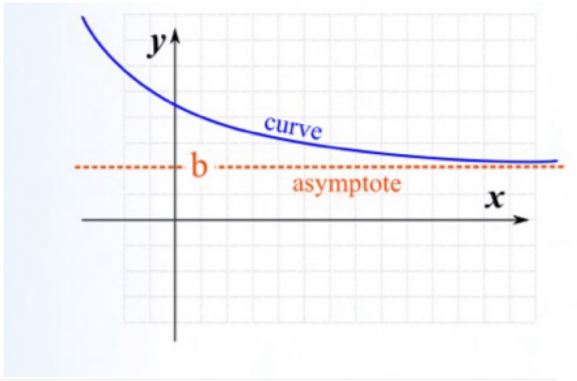
---

* بالرسم: نشوف خط عامودي يقترب منه المنحنى

ـــ الدالة تروح ∞ أو ∞ - .

Horizontal Asymptotes





If $\lim_{x \to -\infty} f(x) = L$

then $y = L$ is H.A

$\lim_{x \to \infty} f(x) = M$

then $y = M$ is H.A

In Problems 9–16, use the accompanying graph of $y = f(x)$.

9. Find $\lim_{x \to \infty} f(x)$.

10. Find $\lim_{x \to -\infty} f(x)$.

11. Find $\lim_{x \to -1^-} f(x)$.

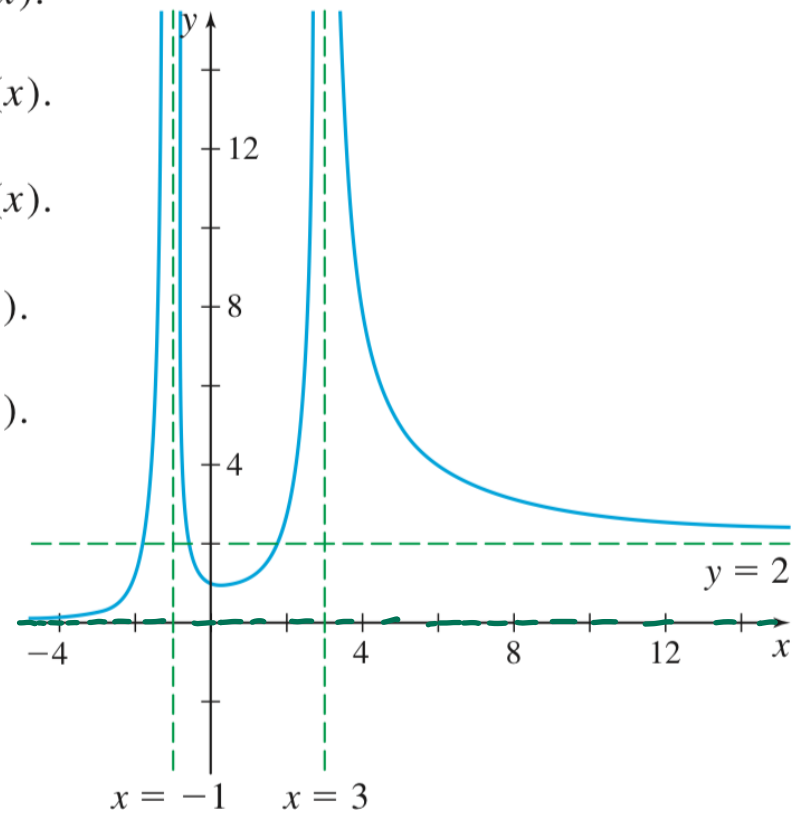
12. Find $\lim_{x \to -1^+} f(x)$.

13. Find $\lim_{x \to 3^-} f(x)$.

14. Find $\lim_{x \to 3^+} f(x)$.

15. Identify all vertical asymptotes.

16. Identify all horizontal asymptotes.



In Problems 17–26, use the accompanying graph of $y = f(x)$.

17. Find $\lim_{x \to \infty} f(x)$.

18. Find $\lim_{x \to -\infty} f(x)$

19. Find $\lim_{x \to -3^-} f(x)$.

20. Find $\lim_{x \to -3^+} f(x)$.

21. Find $\lim_{x \to 0^-} f(x)$.

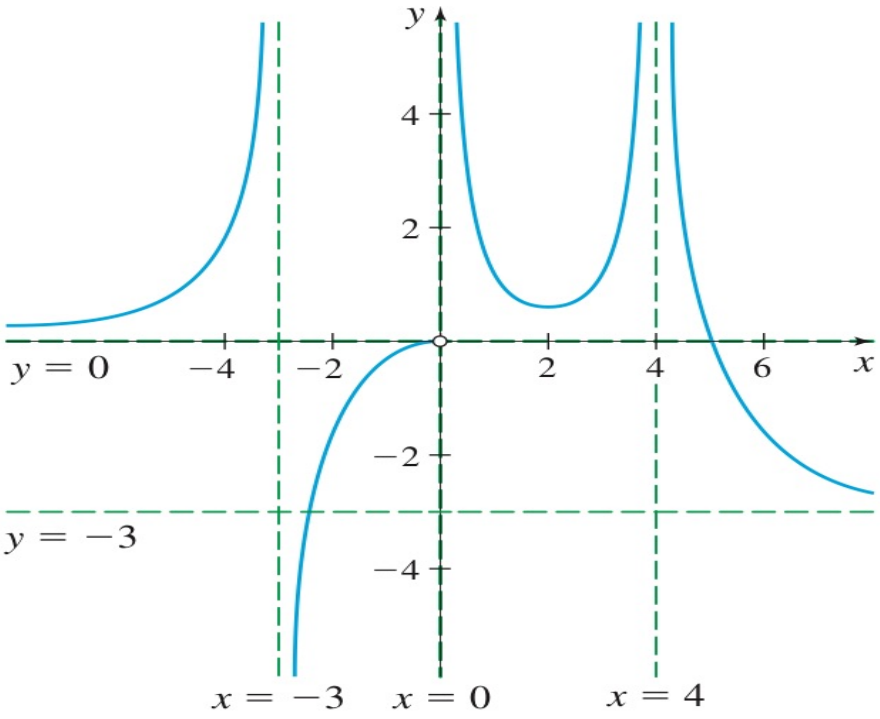
22. Find $\lim_{x \to 0^+} f(x)$.

23. Find $\lim_{x \to 4^-} f(x)$.

24. Find $\lim_{x \to 4^+} f(x)$.

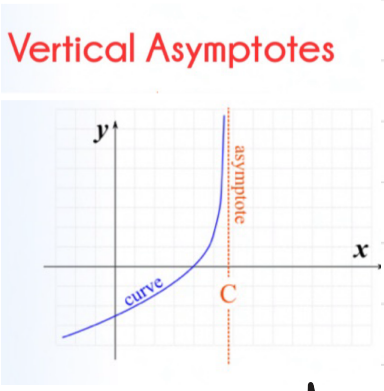
25. Identify all vertical asymptotes.

26. Identify all horizontal asymptotes.



* خطوات التعامل مع Asymptotes

١- Vertical Asymptote



* حل معادلة المقام = ٠

* افحص limit من اليمين و اليسار
إذا راحت لـ ±∞ معناته الدالة عند
هالنقطة vertical Asymptote .

* إذا عطاني رقم إذاً NO V.A

٢- Horizantal Asymptote

* أفحص $\lim_{x \to -\infty} f(x)$ & $\lim_{x \to \infty} f(x)$

* إذا عطاني رقم ، هذا الرقم أهوا H.A .

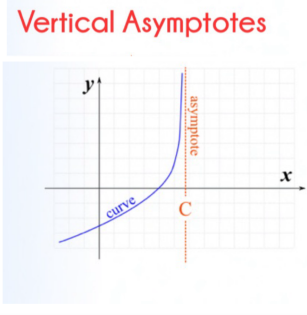
* إذا عطاني ∞ أو -∞ إذاً NO H.A .

* ملخص *

find the vertical & horizontal asymptote "if any".

- **vertical Asymptote :**

١) أصفار المقام ، $x = c$

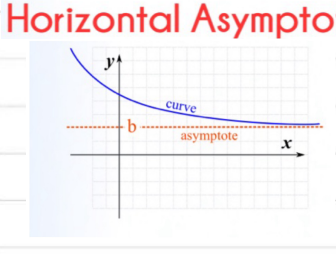


٢) $\lim_{x \to c} f(x) = \infty$

٣) $\therefore c$ is V.A

- **Horizontal Asymptote :**

١) $\lim_{x \to \infty^+} f(x) = c$ or $\lim_{x \to \infty^-} f(x) = -c$



$\therefore c$ & $-c$ are H.A

limit

تذكر أن !

1- $e^{0} = 1$

2- $e^{\infty} = \infty$

3- $e^{-\infty} = 0$

4- $\tan^{-1}(-\infty) = -\frac{\pi}{2}$

5- $\ln 0^{+} = -\infty$

6- $\tan^{-1}(\infty) = \frac{\pi}{2}$

7- $\ln e = 1$

8- $\ln 1 = 0$

9- $\ln \infty = \infty$

10- $\sin 0 = 0$

11- $\cos 0 = 1$

12- $\frac{1}{0} = \mp\infty$

13- $\frac{1}{\pm\infty} = 0$

14- $\frac{0}{\pm\infty} = 0$

15- $\frac{\pm 3}{\infty} = 0$

16- $\frac{0}{\pm 4} = 0$

17- $0^{\infty} = 0$

18- $3^{\infty} = \infty$

19- $(\frac{1}{4})^{\infty} = 0$

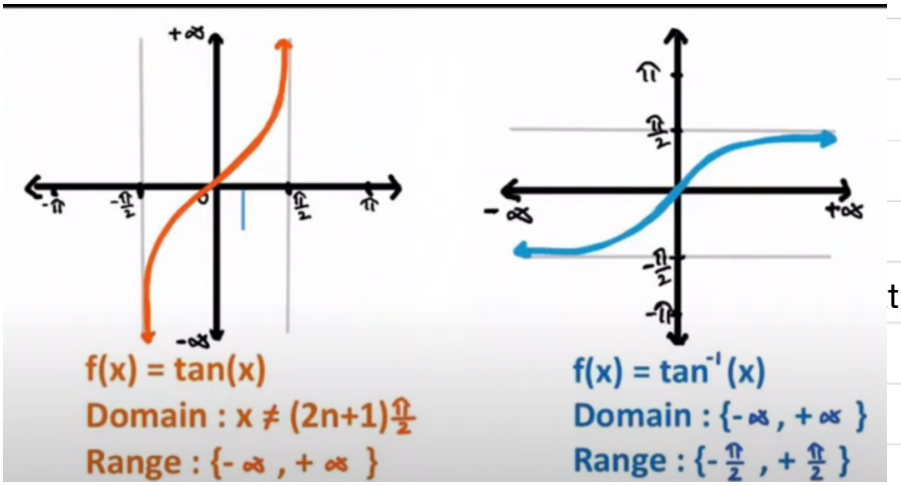
20- $0^{-\infty} = \infty$

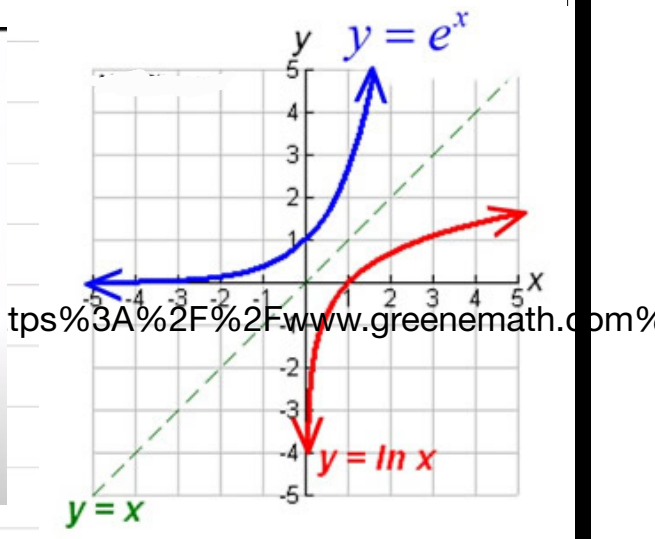
* tan & ln functions

35. $\lim_{x \to \infty} \arctan(e^x)$

41. $\lim_{x \to -1^+} \ln(x+1)$ =

42. $\lim_{x \to 1^+} \ln(x-1)$





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2) limit infinity for power and exponential function

evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

a) $f(x) = -5x^3$

b) $f(x) = 2x^4$

c) $f(x) = e^x$

d) $f(x) = \pi^x$

1.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* ∞ is a number.

2. **(a)** $\lim_{x \to 0^-} \frac{1}{x} =$ ____ ; **(b)** $\lim_{x \to 0^+} \frac{1}{x} =$ ____ ; **(c)** $\lim_{x \to 0^+} \ln x =$ ____

4. If $\lim_{x \to 4} f(x) = \infty$, then the line $x = 4$ is a(n) ____ asymptote of the graph of f.

5. **(a)** $\lim_{x \to \infty} \frac{1}{x} =$ ____ ; **(b)** $\lim_{x \to \infty} \frac{1}{x^2} =$ ____ ; **(c)** $\lim_{x \to \infty} \ln x =$ ____

6. *True or False* $\lim_{x \to -\infty} 5 = 0$.

7. **(a)** $\lim_{x \to -\infty} e^x =$ ____ ; **(b)** $\lim_{x \to \infty} e^x =$ ____ ; **(c)** $\lim_{x \to \infty} e^{-x} =$ ____

In Problems 27–42, find each limit.

27. $\lim_{x \to 2^-} \dfrac{3x}{x - 2}$

28. $\lim_{x \to -4^+} \dfrac{2x + 1}{x + 4}$

33. $\lim_{x \to -3^-} \dfrac{1}{x^2 - 9}$

34. $\lim_{x \to 2^+} \dfrac{x}{x^2 - 4}$

* خطوات حل $\lim_{x \to \pm\infty} f(x)$ مع الكسور:

١) قسمة على أكبر أس في المقام على كل عنصر.

٢) كثيرة الحدود / كسور بسيطة.

إذا عندك بسط ومقام وكلهم حدود عادية (بدون جذر).

- $\lim_{x \to \infty} \frac{4x+3}{5x^2-1} = \frac{\frac{4x}{x^2} + \frac{3}{x^2}}{\frac{5x^2}{x^2} - \frac{1}{x^2}} =$

$$\frac{\frac{4}{x} + \frac{3}{x^2}}{5 - \frac{1}{x^2}} = \frac{0+0}{5-0} = \frac{0}{5} = 0$$

* إذا أعلى قوة في المقام أكبر ⇒ الناتج صفر.

مثال :-

$$\bullet \lim_{x \to \infty} \frac{3 - 4x^2}{5x - 1} = \frac{\frac{3}{x} - \frac{4x^2}{x}}{\frac{5x}{x} - \frac{1}{x}}$$

$$\lim_{x \to \infty} \frac{\frac{3}{x} - 4x}{5 - \frac{1}{x}} = \frac{0 - \infty}{5 - 0} = \frac{-\infty}{5} = -\infty$$

* إذا أعلى قوة في البسط أكبر ⇒ الناتج $\pm \infty$

$$\bullet \lim_{x \to \infty} \frac{4x^2 + 3}{1 - 5x^2} = \frac{\frac{4x^2}{x^2} + \frac{3}{x^2}}{\frac{1}{x^2} - \frac{5x^2}{x^2}}$$

$$= \frac{4 - \frac{3}{x^2}}{\frac{1}{x^2} - 5} = \frac{4 - 0}{0 - 5} = \frac{4}{-5}$$

* إذا نفس القوة ⇐ قسم المعاملات

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x \to \infty} \dfrac{2x^2 - 7}{5x^2 + x - 3}$

15–42 Find the limit or show that it does not exist.

16. $\lim_{x \to \infty} \dfrac{1 - x^2}{x^3 - x + 1}$

In Problems 61–66, find any horizontal or vertical asymptotes of the graph of f.

61. $f(x) = 3 + \dfrac{1}{x}$

62. $f(x) = 2 - \dfrac{1}{x^2}$

63. $f(x) = \dfrac{x^2}{x^2 - 1}$

64. $f(x) = \dfrac{2x^2 - 1}{x^2 - 1}$

15–42 Find the limit or show that it does not exist.

34. $\lim_{x \to -\infty} \dfrac{1 + x^6}{x^4 + 1}$

37. $\lim_{x \to \infty} \dfrac{1 - e^x}{1 + 2e^x}$

38. $\lim_{x \to \infty} \dfrac{\sin^2 x}{x^2 + 1}$

38. $\lim_{x \to \infty} \dfrac{\sin^2 x}{x^2 + 1}$

1. Evaluate the following the limit and conclude if it has vertical or horizontal asymptotes.

$$\lim_{x \to \infty} \frac{x^3 - 9x + 1}{3x^2 - 2x - 15}$$

15–42 Find the limit or show that it does not exist.

20. $\lim_{t \to \infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

٢) مع جذر التربيعي

* طلع أكبر قوة تحت الجذر

مثال - $\sqrt{x^2} = |x|$

$|x| = x = \infty$

$|x| = -x \Rightarrow -\infty$

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+x}}{4x-1} = \frac{\sqrt{x^2\left(3+\frac{1}{x}\right)}}{4x-1}$$

$$\Rightarrow \frac{|x|\sqrt{3+\frac{1}{x}}}{4x-1} \Rightarrow \frac{x\sqrt{3+\frac{1}{x}}}{x\left(4-\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3+\frac{1}{x}}}{4-\frac{1}{x}} = \frac{\sqrt{3+0}}{4-0} = \frac{\sqrt{3}}{4}$$

$$f(x) = \lim_{x \to \infty} \frac{3x - 2}{\sqrt{4x^2 + 5}}$$

$$f(x) = \lim_{x \to -\infty} \frac{3x - 2}{\sqrt{4x^2 + 5}}$$

15–42 Find the limit or show that it does not exist.

27. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x \right)$

Determine the V.A & H.A

a. $y = \frac{3x+1}{x+2}$

b) $y = \dfrac{x^3 + 5}{(x+2)(x+3)}$

c. $y = 3 + \dfrac{3x^2 + x + 1}{x^2 - 4}$

Q4. The vertical asymptotes of $f(x) = \frac{x-6}{x^2-8x+12}$ are

(a) $x = 2$ (b) $x = 2 \; and \; x = 6$ (c) $x = -5$ (d) $y = 2 \; and \; y = 6$

1) Computing limits at infinity

Determine the horizental asymptotes

a) $f(x) = 5 - \frac{2}{x^2}$

b) $f(x) = \frac{\sin x}{x}$

c) $\tan^{-1}(x)$

25. $\lim_{x\to\infty} \dfrac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty} \dfrac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \dfrac{\lim_{x\to\infty} \sqrt{(1+4x^6)/x^6}}{\lim_{x\to\infty} (2/x^3 - 1)}$ $\quad \left[\text{since } x^3 = \sqrt{x^6} \text{ for } x > 0\right]$

$$= \frac{\lim_{x\to\infty} \sqrt{1/x^6 + 4}}{\lim_{x\to\infty} (2/x^3) - \lim_{x\to\infty} 1} = \frac{\sqrt{\lim_{x\to\infty} (1/x^6) + \lim_{x\to\infty} 4}}{0 - 1}$$

$$= \frac{\sqrt{0+4}}{-1} = \frac{2}{-1} = -2$$

26. $\lim_{x\to-\infty} \dfrac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty} \dfrac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \dfrac{\lim_{x\to-\infty} -\sqrt{(1+4x^6)/x^6}}{\lim_{x\to-\infty} (2/x^3 - 1)}$ $\quad \left[\text{since } x^3 = -\sqrt{x^6} \text{ for } x < 0\right]$

$$= \frac{\lim_{x\to-\infty} -\sqrt{1/x^6 + 4}}{2\lim_{x\to-\infty} (1/x^3) - \lim_{x\to-\infty} 1} = \frac{-\sqrt{\lim_{x\to-\infty} (1/x^6) + \lim_{x\to-\infty} 4}}{2(0) - 1}$$

$$= \frac{-\sqrt{0+4}}{-1} = \frac{-2}{-1} = 2$$

27. $\lim_{x\to-\infty} \dfrac{2x^5 - x}{x^4 + 3} = \lim_{x\to-\infty} \dfrac{(2x^5 - x)/x^4}{(x^4+3)/x^4} = \lim_{x\to-\infty} \dfrac{2x - 1/x^3}{1 + 3/x^4}$

$= -\infty$ since $2x - 1/x^3 \to -\infty$ and $1 + 3/x^4 \to 1$ as $x \to -\infty$

29. $\lim_{t\to\infty} \left(\sqrt{25t^2+2} - 5t\right) = \lim_{t\to\infty} \left(\sqrt{25t^2+2} - 5t\right)\left(\dfrac{\sqrt{25t^2+2} + 5t}{\sqrt{25t^2+2} + 5t}\right) = \lim_{t\to\infty} \dfrac{(25t^2+2) - (5t)^2}{\sqrt{25t^2+2} + 5t}$

$$= \lim_{t\to\infty} \frac{2}{\sqrt{25t^2+2} + 5t} = \lim_{t\to\infty} \frac{2/t}{\left(\sqrt{25t^2+2} + 5t\right)/t}$$

$$= \lim_{t\to\infty} \frac{2/t}{\sqrt{25 + 2/t^2} + 5} \qquad [\text{since } t = \sqrt{t^2} \text{ for } t > 0]$$

$$= \frac{0}{\sqrt{25+0}+5} = 0$$

30. $\lim_{x\to-\infty} \left(\sqrt{4x^2+3x} + 2x\right) = \lim_{x\to-\infty} \left(\sqrt{4x^2+3x} + 2x\right)\left[\dfrac{\sqrt{4x^2+3x} - 2x}{\sqrt{4x^2+3x} - 2x}\right] = \lim_{x\to-\infty} \dfrac{(4x^2+3x) - (2x)^2}{\sqrt{4x^2+3x} - 2x}$

$$= \lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+3x} - 2x} = \lim_{x\to-\infty} \frac{3x/x}{\left(\sqrt{4x^2+3x} - 2x\right)/x}$$

$$= \lim_{x\to-\infty} \frac{3}{-\sqrt{4+3/x} - 2} \qquad \left[\text{since } x = -\sqrt{x^2} \text{ for } x < 0\right]$$

$$= \frac{3}{-\sqrt{4+0}-2} = -\frac{3}{4}$$

EXAMPLE 5 Compute $\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x\right)$. $= \infty - \infty$

EXAMPLE 6 Evaluate $\lim_{x \to 2^+} \arctan\left(\frac{1}{x - 2}\right)$.

$$\lim_{x \to \infty} \frac{e^x + 300}{10^x + 3^x}$$

39. $\lim_{x \to \infty} (e^{-2x} \cos x)$

3. [$10 + 10 = 20$ pts.] Let $f(x) = \dfrac{x^2 - 4x}{|x|(x+2)}$.

(a) Find and classify the discontinuities of f as removable, infinite or jump.

(b) Find the horizontal and vertical asymptotes of the graph of f, if any.

1. $(10 + 10 = 20$ pts) Evaluate each of the following limits, if it exists.

(b) $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2}\right)$.

3. (10 + 5 = 15 pts) Let $f(x) = \dfrac{2x^2 + 3}{x\sqrt{x^2 + 4}}$.

(a) Find the horizontal asymptotes if the graph of f, if any.

(b) Find the vertical asymptotes if the graph of f, if any.

2. [10+10=20 pts.] Let $f(x) = \dfrac{3x^2 + 10}{x\sqrt{x^2 + 1}}$.

(a) Find the horizontal asymptotes of the graph of f, if any.

(b) Find the vertical asymptotes of the graph of f, if any.

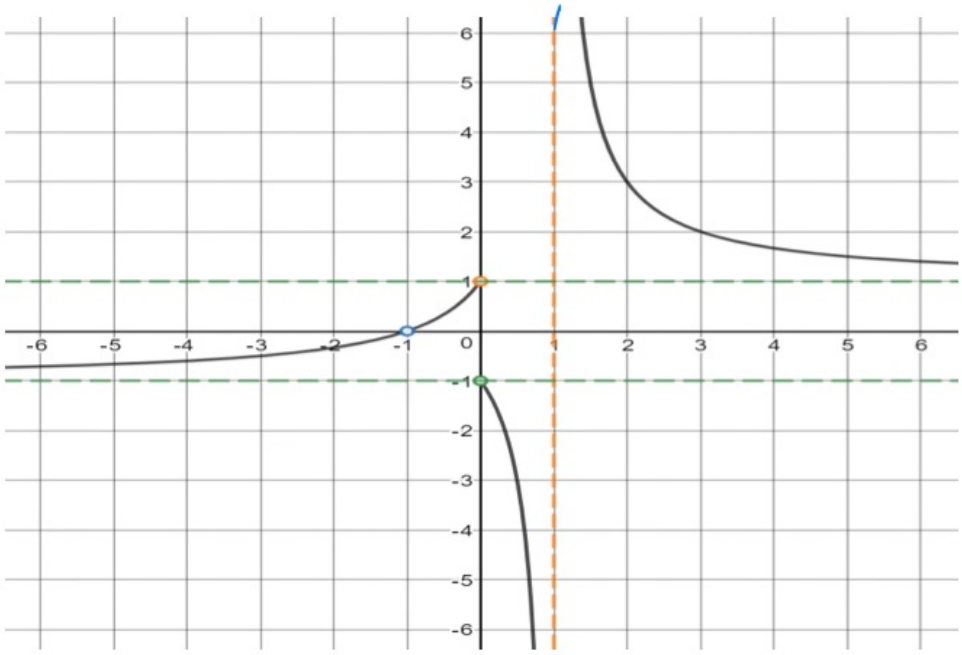
---

9. [10+10 = 20 pts.] Let $f(x) = \dfrac{x|x + 1|}{x^2 - 1}$.

(a) Find the horizontal asymptotes of the graph of f, if any.

(b) Find the vertical asymptotes of the graph of f, if any.

1. [$5 \times 4 = 20$ pts.] Use the given graph of f to select the correct answer.



(I) $\lim_{x \to 0} f(x) =$

(a) 1 (b) 0 (c) ∞ (d) -1 (e) None of the mentioned

(II) $\lim_{x \to 1^+} f(x) =$

(a) $+\infty$ (b) $-\infty$ (c) 1 (d) 0 (e) None of the mentioned

(III) The graph of f has

a) no vertical asymptotes

b) only one vertical asymptote

c) only one horizontal asymptote

d) no horizontal asymptotes

e) None of the above

(IV) The function f has

a) a jump discontinuity at $x = 1$

b) a removable discontinuity at $x = 0$

c) a jump discontinuity at $x = -1$

d) a removable discontinuity at $x = -1$

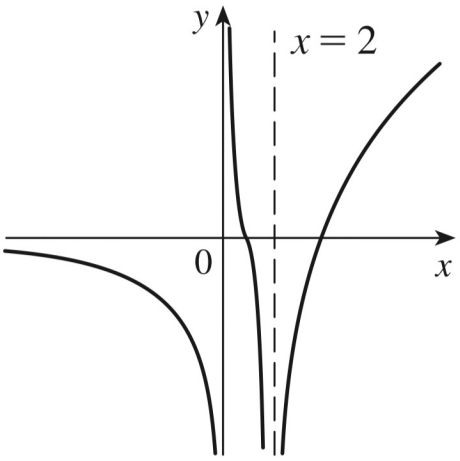
e) None of the above

1. $[3+3+3+3+3+5+5=25$ pts.] Let

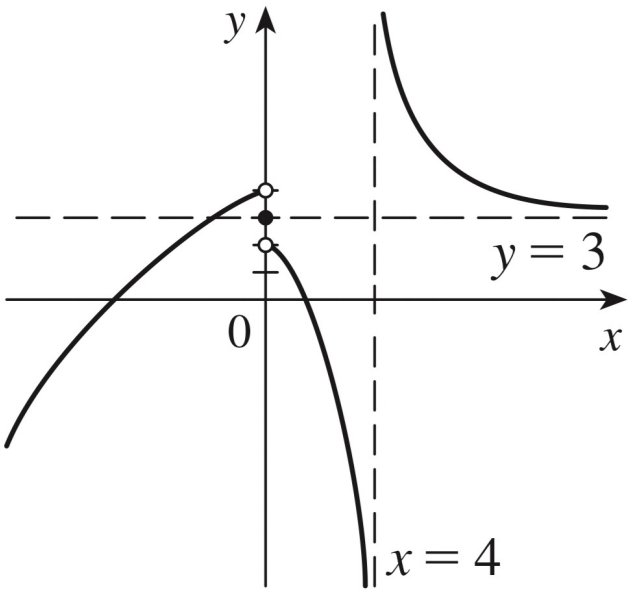
$$f(x)=\begin{cases} e^x-\dfrac{3}{x-7}+5, & \text{if } x\leqslant 1,\\ \ln(x-1)+3, & \text{if } 1<x\leqslant 3\\ \dfrac{x^2+1}{x^3+1}, & \text{if } x>3. \end{cases}$$

5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

7. $\lim_{x \to 2} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$



9. $f(0) = 3$, $\lim_{x \to 0^-} f(x) = 4$, $\lim_{x \to 0^+} f(x) = 2$, $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to 4^-} f(x) = -\infty$, $\lim_{x \to 4^+} f(x) = \infty$, $\lim_{x \to \infty} f(x) = 3$



3. [10+10 = 20 pts.] Let $f(x) = \dfrac{\sqrt{x^2+5}}{2x+3}$. Find the horizontal and vertical asymptotes of the graph of f.

37. $\lim_{x \to \infty} \dfrac{1 - e^x}{1 + 2e^x} =$

$$\lim_{x \to 0+} \tan^{-1}(\ln x)$$

Find the limit.

$$\lim_{x \to \infty} \left(e^{-4x} \cos x \right)$$

36. $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

1. Evaluate the limits if they exist.

(a) $\lim_{x \to \infty} \dfrac{(2x^2+1)(4x^2+4x+2)}{(x^2+2)^2}$.

ثمرة
Thamrah

Kuwait University

Calculus 1 – Definition of Derivative

(Section 2.7 & 2.8)

For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

جدول الحالات في المستقيمات :-

الشرط / القانون	الحالة
$m=0 \quad f'(x)=0$	المماس الأفقي (Horizontal Tangent).
$m=\infty$ أو $f'(a)$ غير معرف غير موجود	المماس العامودي (Vertical Tangent).
$m_1 = m_2$	مستقيمان متوازيان (parallel Lines)
$m_1 \cdot m_2 = -1$ or $m_1 = -\frac{1}{m_2}$	مستقيمان متعامدان (perpendicular Lines)
$y_1 - y_2 = m(x - x_1)$	معادلة مستقيم من الميل و النقطة.
$m = \frac{y_2 - y_1}{x_2 - x_1}$	ميل مستقيم من نقطتين
$y = mx + b$	معادلة مستقيم بصيغة الميل الأجزاء المقطوعة.
$y - y_1 = \frac{1}{-m}(x - x_1)$	معادلة المستقيم العامودي

خطوات عامة لحل مسائل المماس و معادلة المستقيم :-

١- تحديد المعطيات :

- الدالة $f(x)$.
- النقطة $P(a, f(a))$ إذا كانت معطاة .

٢- إيجاد ميل المماس (slope) :

- باستخدام التعريف $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
- أو باستخدام صيغة الـ h :

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

٣- كتابة معادلة المماس :

- استخدم صيغة النقطة والميل :

$$y - f(a) = m(x - a)$$

٤- ترتيب المعادلة :

- بسط المعادلة لتكون بالصورة المطلوبة

Use the **definition of the derivative** to find

إذا طلب منك تحل المشتقة بالتعريف لازم تحل بإحدى
الطرق التالية (كيفك أي طريقة)

1) $$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

2) $$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$$

Other symbols of Derivatives

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

EXAMPLE 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

3. (a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$

(i) using Definition 1 (ii) using Equation 2

يعني حل بالطريقتين

(b) Find an equation of the tangent line in part (a).

7. [10 pts.] Use the **definition of the derivative** to find $f'(3)$ of $f(x) = \sqrt{x-2} + 3$.

7. [10 pts.] Use the definition of the derivative to find $f'(1)$ of $f(x) = x|x|$.

6. (10 pts) Let $f(x) = \sqrt{2x+1}$. Use the definition of the derivative to find $f'(4)$.

5. [10 pts.] Let $f(x) = \dfrac{2}{x+1}$. Use the definition of the derivative to find $f'(0)$.

6. [10 pts.] Let $f(x) = \sqrt{x-1}$. Use the definition of the derivative to find $f'(5)$.

5–8 Find an equation of the tangent line to the curve at the given point.

6. $y = x^3 - 3x + 1, \quad (2, 3)$

5–8 Find an equation of the tangent line to the curve at the given point.

7. $y = \sqrt{x}, \quad (1, 1)$

20. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

31–36 Find $f'(a)$.

33. $f(t) = \dfrac{2t + 1}{t + 3}$

$f(x) = x^2 + 3x$ and $P(-1, -2)$, the slope of the tangent line is

$f(x) = x^3 + 1$ and $P(1, 2)$, the slope of the tangent line is

$f(x) = 2x^2 - 5x + 1$ and $P(3, 4)$ the slope of the tangent line is

2.8 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a fixed number a:

1 $$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

2 $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

---

4 Theorem If f is differentiable at a, then f is continuous at a.

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

EXAMPLE 4 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

21–31 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

26. $g(t) = \dfrac{1}{\sqrt{t}}$

57. Let $f(x) = \sqrt[3]{x}$.

(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.

(b) Show that $f'(0)$ does not exist.

(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$.

Use the **definition of the derivative** to find

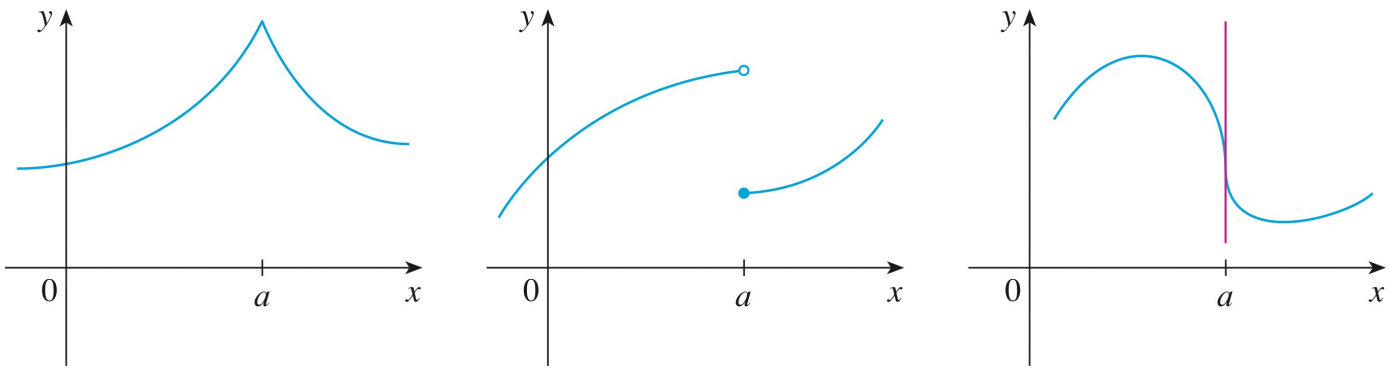
إذا طلب منك تحل المشتقة بالتعريف لازم تحل بإحدى
الطرق التالية (كيفك أي طريقة)

1) $$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

2) $$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$$

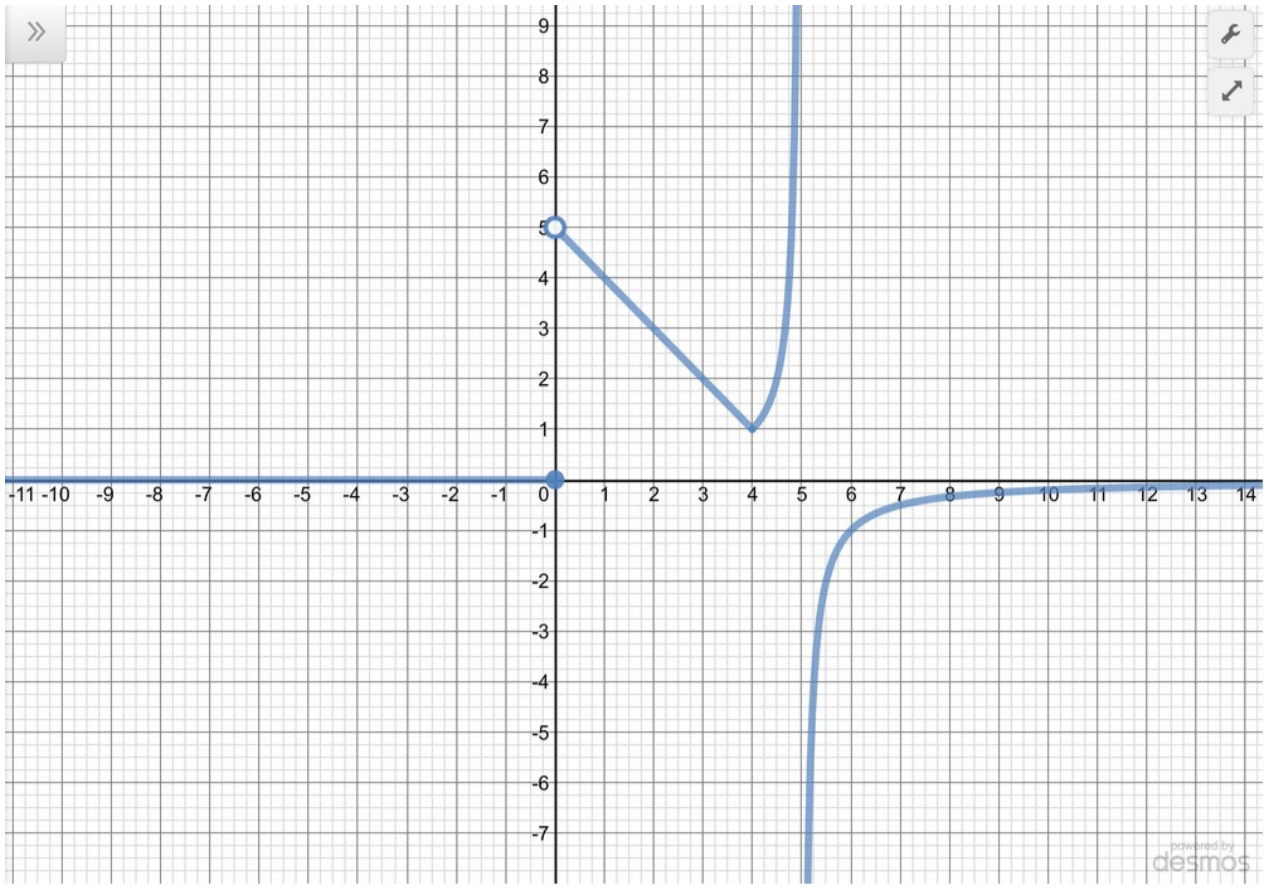
How Can a Function Fail To Be Differentiable?

not differentiable يعني متى تكون



(a) A corner (b) A discontinuity (c) A vertical tangent

الرسمة معطى



(c) Where is f discontinuous?

(d) Where is f not differentiable?

Rules:-

1) $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x}{\sin(x)} = 1$

2) $\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{x}{\tan x} = 1$

3) $\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{\cos(x)-1}{x} = 0$

شروط استخدام القوانين

١- المفروض لما أعوض دايركت تعطيني الزاوية بصفر

٢- لازم زاوية البسط نفس المقام (وإذا مو نفسها نحاول نوصلها لنفس الشكل)

Remark:-

$\sin^2 x = (\sin x)^2$

$\sin^2 x \neq \sin(x^2)$

1) $\lim_{x \to 0} \frac{\sin x}{2x}$

2) $\lim_{x \to 0} \frac{\sin(2x)}{x}$

3) $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

4) $\lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1}$

تذكير ! ما يصير تختصر شي داخل الدالة المثلثية مع شي برا الدالة

$$\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

EXAMPLE 5 Find $\lim_{x \to 0} \frac{\sin 7x}{4x}$.

EXAMPLE 6 Calculate $\lim_{x \to 0} x \cot x$.

39–50 Find the limit.

39. $\lim_{x \to 0} \dfrac{\sin 5x}{3x}$

40. $\lim_{x \to 0} \dfrac{\sin x}{\sin \pi x}$

39–50 Find the limit.

42. $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\sin\theta}$

43. $\lim_{x \to 0} \dfrac{\sin 3x}{5x^3 - 4x}$

39–50 Find the limit.

44. $\lim_{x \to 0} \dfrac{\sin 3x \sin 5x}{x^2}$

54. (a) Evaluate $\lim_{x\to\infty} x \sin\frac{1}{x}$.

(b) Evaluate $\lim_{x\to 0} x \sin\frac{1}{x}$.

c) $\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{x}$

I. $\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{2x} =$

a) $\frac{1}{2}$.

b) $\frac{1}{\pi}$

c) 1.

d) None of the above.

I. $\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2} =$

a) 1. b) 0. c) -1

d) ∞. e) None of the above.

39–50 Find the limit.

46. $\lim_{x \to 0} \csc x \sin(\sin x)$

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4. [5 + 5 = 10 pts.] Let $f(x) = \dfrac{\sin(x-1)}{x^3 - 1}$.
Show that f is discontinuous at $x = 1$, and classify the discontinuity as removable, jump, or infinite.

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1. [10 pts.] Evaluate: $\lim_{x \to 2} \dfrac{\sin(x-2)}{x^2+x-6}$.

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3. [10 pts.] Evaluate the limit $\lim_{x \to 0} \dfrac{\sin(x^2)+2x}{x}$, if it exits.

III. $\lim_{x \to 0} \dfrac{\sin(x-1)}{x-1} =$

a) 1.

b) $-\sin 1$.

c) -1.

d) $\sin 1$.

e) None of the above.

47. $\lim_{\theta \to 0} \dfrac{\cos \theta - 1}{2\theta^2}$

Evaluate the limit $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{3\theta^2}$, if it exits.

$\lim_{x \to 0} x \cot(\pi x)$

III. $\lim_{x \to 0} x \cot(\pi x) =$

(A) π.

(B) $-\pi$.

(C) $\frac{1}{\pi}$.

(D) $-\frac{1}{\pi}$.

(E) None of the above.

39–50 Find the limit.

48. $\lim_{x \to 0} \dfrac{\sin(x^2)}{x}$

49. $\lim_{x \to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x}$

50. $\lim_{x \to 1} \dfrac{\sin(x - 1)}{x^2 + x - 2} =$