UAEU College of Science كلية العلوم
Department of Mathematical Sciences

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United Arab Emirates University
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Math 105
Calculus 1

Final Exam - Spring2017
May 18, 2017

Student Name:................................ **Instructor: ....**

ID number:...................................... **Section number:..................**

Serial number :..................................

Please write your answers in detail and show your work to get full mark. Giving short answers might cause you losing points.

1. (4 points) Evaluate the following limits

(a) $\lim_{x\to 0} \ln(x)\sin(x) = \ln(0)\sin(0) = \infty \cdot 0$ → This limit has the indeterminate form $\infty \cdot 0$

$f \cdot g = \dfrac{f}{1/g}$ $\qquad \underset{f}{\ln(x)} \quad \underset{g}{\sin(x)}$

$\lim_{x\to 0} \ln(x)\sin(x) = \lim_{x\to 0} \dfrac{\ln(x)}{\frac{1}{\sin(x)}} = \dfrac{\ln(0)}{\frac{1}{\sin(0)}} = \dfrac{\infty}{\infty}$ → This limit has the indeterminate form $\frac{\infty}{\infty}$, so we use L'Hospital's Rule

$\lim_{x\to 0} \dfrac{\ln(x)}{\frac{1}{\sin(x)}} = \lim_{x\to 0} \dfrac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\frac{1}{\sin x}} = \lim_{x\to 0} \dfrac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = \lim_{x\to 0} \dfrac{-\sin^2 x}{x\cos x} = \lim_{x\to 0} \dfrac{-\sin x \cdot \sin x}{x\cos x} = \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{-\sin x}{\cos x}$

$= \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{-\sin x}{\cos x} = \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \lim_{x\to 0} \dfrac{-\sin x}{\cos x} = \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \lim_{x\to 0} -\tan x = 1 \cdot 0 = 0$

$\lim_{x\to 0} \ln(x)\sin(x) = 0$

(b) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \left(1 + \frac{1}{\infty}\right)^\infty = (1+0)^\infty = 1^\infty$ → This limit has the indeterminate form 1^∞

Let $y = \left(1 + \frac{1}{x}\right)^x$

$\ln y = \ln\left(1 + \frac{1}{x}\right)^x$

$\ln y = x\ln\left(1 + \frac{1}{x}\right)$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right)$

$\lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right) = \infty \ln 1 = \infty \cdot 0$ This limit has the indeterminate form $\infty \cdot 0$

$\lim_{x\to\infty} \dfrac{\ln\left(1 + \frac{1}{x}\right)}{1/x} = \dfrac{0}{0}$ This limit has the indeterminate form $\frac{\infty}{\infty}$, so we use L'Hospital's Rule

$\lim_{x\to\infty} \dfrac{\frac{d}{dx}\ln\left(1 + \frac{1}{x}\right)}{\frac{d}{dx}\frac{1}{x}} = \lim_{x\to\infty} \dfrac{\frac{-1/x^2}{1 + 1/x}}{-1/x^2} = \lim_{x\to\infty} \dfrac{\cancel{\frac{-1}{x^2}}}{\left(1 + \frac{1}{x}\right)\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} \dfrac{1}{1 + \frac{1}{x}} = \dfrac{1}{1 + \frac{1}{\infty}} = \dfrac{1}{1+0} = \dfrac{1}{1} =$

$\lim_{x\to\infty} \ln y = 1$ $\qquad y = e^{\ln y}$ $\qquad \lim_{x\to\infty} y = e^{\lim_{x\to\infty} \ln y}$

$\lim_{x\to\infty} y = e^1$ $\qquad \boxed{\lim_{x\to\infty} y = e}$

2. (3 points) For what value of the constant a is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^2 + 2a & \text{if } \ x < 2 \\ xa^2 & \text{if } \ x \geq 2 \end{cases}$$

$x^2 + 2a$ $\quad$ xa^2

2

$$\lim_{x \to 2^-} x^2 + 2a = \lim_{x \to 2^+} xa^2$$

$$(2)^2 + 2a = (2)\, a^2$$

$$x^2 + 2a = xa^2$$

$$(2)^2 + 2a = (2)\, a^2$$

$$4 + \frac{2a}{2a} = \frac{2a^{\cancel{2}}}{2a}$$

$$\frac{4}{2} = \frac{2a}{2}$$

$$a = 2$$

3. (3 points) Use **THE DEFINITION OF THE DERIVATIVE** to find $f'(2)$, of the function $f(x) = \sqrt{x-1}$ and determine the tangent line of this function at $x = 2$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$f(x) = \sqrt{x-1}$$

$$f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$f(2+h) = \sqrt{(2+h)-1} = \sqrt{h+1}$$

$$f'(2) = \lim_{h \to 0} \frac{\sqrt{h+1} - 1}{h} \cdot \frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$

$$\lim_{h \to 0} \frac{(\sqrt{h+1})^2 - 1^2}{h \cdot (\sqrt{h+1}+1)}$$

$$\lim_{h \to 0} \frac{h+1-1}{h \cdot (\sqrt{h+1}+1)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{h+1}+1)}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{h+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$f'(2) = \frac{1}{2}$$

4. (6 points) Find $\frac{dy}{dx}$ (Do not simplify)

(a) $y = \left(\frac{2\sqrt{x}}{x+1}\right)^2 = \frac{(2\sqrt{x})^2}{(x+1)^2} = \frac{4x}{(x+1)^2}$

$$\left(\frac{U}{V}\right)' = \frac{U'V - UV'}{V^2}$$

$$f'(x) = \frac{4(x+1)^2 - 4x \cdot 2(x+1)(1)}{((x+1)^2)^2}$$

$$f'(x) = \frac{4(x+1)^2 - 8x(x+1)}{(x+1)^4}$$

$U = 4x$

$U' = 4$

$V = (x+2)^2$

$V' = 2(x+2)\cdot 1$

(b) $y + 2\cos(xy) = x + 1$

Note

$\frac{d}{dx}(\cos(xy))$ chain Rule

$= -\sin(xy)\frac{d}{dx}(xy)$

$y' - 2\sin(xy) \cdot \frac{d}{dx}(xy) = 1 + 1$

$y' - 2\sin(xy) \cdot \underbrace{y + xy'} = 2$

Product Rule

$(UV)' = U'V + UV'$

$y' + xy' = 2 + 2\sin(xy)y$

$y'(1+x) = 2 + 2y\sin(xy)$

$$y' = \frac{2 + 2y\sin(xy)}{x+1}$$

(c) $y = \int_{2}^{x^2} (\cos(t) + \ln(tan(t)))dt$

$$\frac{dy}{dx} = \frac{d}{dx} \int_{2}^{x^2} \cos t + \ln(\tan t)\, dt$$

$$\frac{dy}{dx} = \left(\cos(x^2) + \ln(\tan x^2)\right) \frac{dx^2}{dx}$$

$$= \left[\cos(x^2) + \ln(\tan x^2)\right] 2x$$

5. (3 points) A piece of wire 24 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How much wire should be used for the square in order to maximize the total area?

6. (3 points) Use the Intermediate Value Theorem to verify that $f(x) = 6x^7 - 12x + 1$ has a zero in the interval $[-2, 0]$.

* $f(x)$ is continuous on $[-2, 0]$ because it's a polynomial

$f(a) \le N \le f(b)$

so there is a number c in $[a, b]$ where $f(c) = N$

$f(a) = f(-2) = 6(-2)^7 - 12(-2) + 1 = -743$

$f(b) = f(0) = 6(0)^7 - 12(0) + 1 = 1$

$\Rightarrow -743 \le N \le 1$

$N = 0$ because we want to find the zeros

the equation $f(x) = 6x^7 - 12x + 1$ has at least one solution c in the interval $(-2, 0)$

7. (3 points) Find approximation to $\sqrt{8.999}$

$a = 9$ $dx = -0.001$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(a) = f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$

$dy = f'(9)\, dx$

$dy = \dfrac{1}{6}(-0.001)$

$dy = \dfrac{-1}{6000}$

$f(a) = f(9) = \sqrt{9} = 3$

$f(a + dx) = f(a) + dy$

$f(9 - 0.001) = f(9) + dy$

$f(8.999) = 3 + \dfrac{-1}{6000}$

$f(8.999) \approx 2.99983$

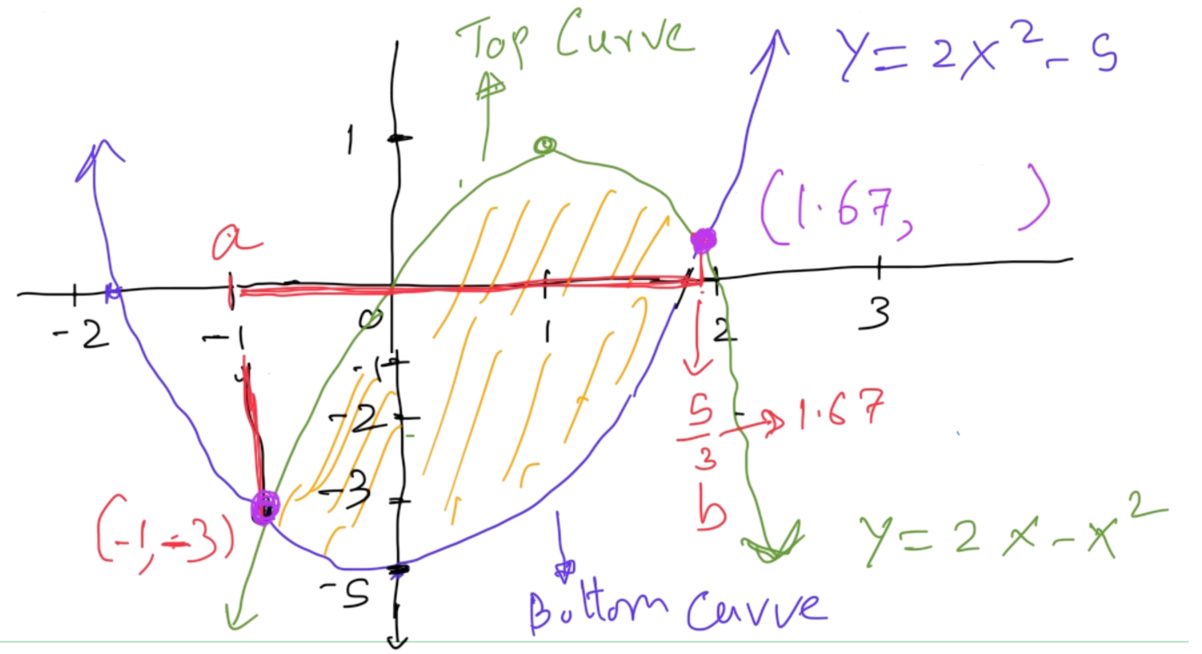
8. (3 points) Find the area between the two curves $y = 2x^2 - 5$ and $y = 2x - x^2$

1- Find the points of . ersection :-

$y_1 = y_2 \qquad 2x^2 - 5 = 2x - x^2 \Rightarrow 3x^2 - 2x - 5 = 0$

$x_1 = -1 \qquad x_2 = \frac{5}{3} \approx 1.67$

Graphs will be given in the exam.



3- write the Area as integration

$$A = \int_a^b (\text{Top Curve} - \text{Bottom Curve})\, dx$$

$$A = \int_{-1}^{\frac{5}{3}} (2x - x^2) - (2x^2 - 5)\, dx$$

$$A = \int_{-1}^{\frac{5}{3}} 2x - x^2 - 2x^2 + 5 \, dx = \int_{-1}^{\frac{5}{2}} (2x - 3x^2 + 5)\, dx$$

$$A = \left(x^2 - x^3 + 5x\right)\Big]_{-1}^{\frac{5}{3}} = \frac{256}{27} \approx 9.5$$

9. (6 points) Evaluate the following integrals

(a) $\int (x-1)(\sqrt{x}+1)dx$

$$\int (x-1)(x^{\frac{1}{2}}+1)\,dx = \int x^{\frac{3}{2}} + x - x^{\frac{1}{2}} - 1 \; dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^2}{2} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - x + C$$

$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} - x + C$$

(b) $\cancel{\int_1^\infty \frac{1}{x^{3/2}}dx}$

(c) $\int \frac{\sin(t)}{\cos^2(t)} dt =$

Put $U = \cos(t)$

$dU = -\sin(t)\,dt$

$\Rightarrow -dU = \sin(t)\,dt$

$$\int \frac{\sin(t)\,dt}{\cos^2(t)} = \int \frac{1}{\cos^2(t)} \cdot \sin(t)\,dt$$

$$= \int \frac{1}{U^2}(-dU) = -\int U^{-2}\,dU$$

$$= -\frac{U^{-2+1}}{-2+1} + C$$

$$= -\frac{U^{-1}}{-1} + C \Rightarrow \frac{1}{U} + C$$

$$\Rightarrow \frac{1}{\cos x} + C$$

10. (6 points) Consider the function f defined by

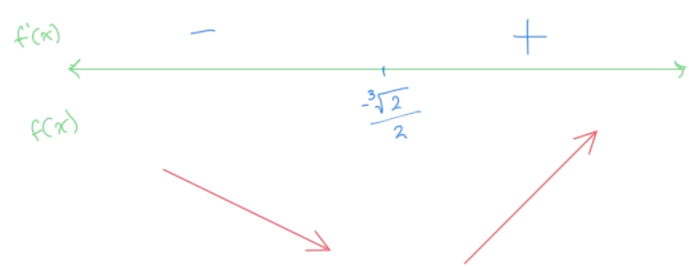
$$f(x) = x^4 + x - 9$$

(a) Find the intervals on which f is increasing and decreasing

$f(x) = x^4 + x - 9$

$f'(x) = 4x^3 + 1 = 0$

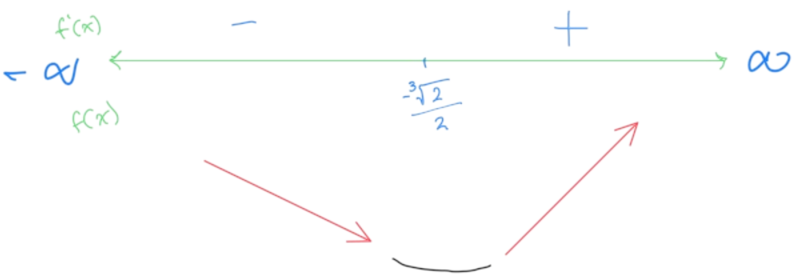
$x = -\frac{\sqrt[3]{2}}{2} \approx -0.63$



f is increasing on $\left(-\frac{\sqrt[3]{2}}{2}, \infty\right)$

f is decreasing on $\left(-\infty, -\frac{\sqrt[3]{2}}{2}\right)$

(b) Find the local minimum and maximum values of f



Local minimum :- $\left(\frac{-\sqrt[3]{2}}{2}, -9.5\right)$

Local maximum :- NIL

(c) Find the inflection points

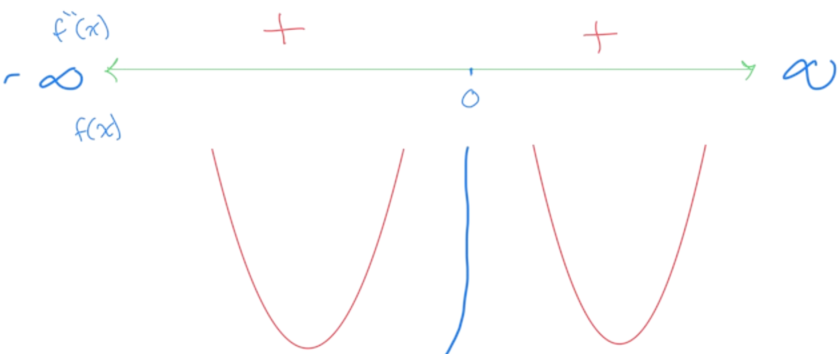
$f'(x) = 4x^3 + 1$

$f''(x) = 12x^2 = 0$

$x = 0$

$X = 0$ is NOT an inflection Points. see next page.

(d) Find the interval on which f is concave up and concave down



f is concaved upward on $(-\infty, 0) \cup (0, \infty)$

$x = 0$ is NOT an inflection point as $f''(x)$ does not change the sign.

x, y intercepts

$y = x^4 + x - 9$

$0 = x^4 + x - 9$ | $y = (0)^4 + (0) - 9$

$x_1 \approx -1.81$ $x_2 \approx 1.65$ | $y = -9$

(e) Sketch the graph of the function f