Midterm Exam - Solution

1. Evaluate the limits if they exist.

(a) $\lim\limits_{x\to\infty} \dfrac{(2x^2+1)(4x^2+4x+2)}{(x^2+2)^2}$.

$$= \lim_{x\to\infty} \frac{(2x^2)(4x^2)}{(x^2)^2}$$

$$= \lim_{x\to\infty} \frac{8x^4}{x^4} = \boxed{8}$$

(b) $\lim\limits_{x\to 0} x^2 \cos \dfrac{1}{x^2}$.

Since

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1 \quad , x \neq 0,$$

$$-x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2, \quad x \neq 0.$$

Since $\lim\limits_{x\to 0} -x^2 = 0$ and $\lim\limits_{x\to 0} x^2 = 0$,

by ST

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = \boxed{0}$$

2. Find $\frac{dy}{dx}$.

(a) $y = \dfrac{1+\cos x}{1-\sin x}$ (Do not simplify)

$$\frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - (1+\cos x)(-\cos x)}{(1-\sin x)^2}$$

(b) $y = \tan^{-1}(e^x+1)$. (Do not simplify)

$$\frac{dy}{dx} = \frac{e^x}{1+(e^x+1)^2}$$

(c) $y = x^{\sec x}$

$$\ln y = \ln x^{\sec x} = (\sec x) \ln x$$

$$\frac{y'}{y} = \sec x \tan x \ln x + \frac{\sec x}{x}$$

$$y' = \left[\sec x \tan x \ln x + \frac{\sec x}{x}\right] x^{\sec x}$$

(d) $x^2 - y^2 = 5x + 4y$.

$$2x - 2yy' = 5 + 4y'$$

$$-2yy' - 4y' = 5 - 2x$$

$$[-2y-4]\, y' = 5 - 2x$$

$$\boxed{y' = \frac{5-2x}{-2y-4}}$$

where s is in meters and t is in seconds.

3. The equation of the motion of a particle is $s(t) = \frac{3}{2}t^2 - \frac{1}{6}t^3$, Find the velocity $v(t)$ and acceleration $a(t)$ at $t = 2$ seconds.

$$v(t) = 3t - \frac{1}{2}t^2$$

$$v(2) = 6 - 2 = 4 \text{ m/s}$$

$$a(t) = 3 - t$$

$$a(2) = 3 - 2 = 1 \text{ m/s}^2$$

4. Suppose f and g are continuous functions such that $g(2) = 3$ and

$$\lim_{x\to 2} [3f(x) + f(x)g(x)] = 39.$$

Find $f(2)$?

Since f and g are cts,

$$39 = \lim_{x\to 2}\left[3f(x) + f(x)g(x)\right]$$

$$= 3f(2) + f(2)g(2)$$

$$= 3f(2) + 3f(2)$$

$$= 6f(2)$$

$$\Rightarrow f(2) = \frac{39}{6} = \boxed{\frac{13}{2}}$$

5. Use the Intermediate Value Theorem to show that there is a solution of the equation

$$x - \cos(x) = 0$$

on the interval (0,1).

Let $f(x) = x - \cos x$. Then,

1) f is cts on $[0,1]$,

2) $f(0)\cdot f(1) = (-1)(0.46) < 0$.

Then, by IMVT, there exists $c \in (0,1)$ such that

$$f(c) = 0$$