Calculus A

Chapter 4: Application of Differentiation

Sections: 4.3 How Derivatives Affect the Shape of a Graph

A+

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* Increaing / Decreasing Test:

a) If $f'(x) > 0$ on an interval I, then f is an increasing (↗) funtion on I.

b) If $f'(x) < 0$ on an interval I, then f is an decreasing (↘) funtion on I.

عشان تعرف إذ الدالة قاعدة تتزايد ولا تتناقص بفترة معينة

عشان تعرف إذا
الدالة لها local
max و local
min

* The First Derivative Test:

Suppose that c is a critical number of a continuous function f.

a) If f' changes from + to − at c, then f has a local maximum at c.

b) If f' changes from − to + at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local extrema at c.

عشان تعرف إذا
الدالة مقعرة لأعلى
أو لأسفل

* Concavity Test:

a) If $f''(x) > 0$ for all x in interval I, then the graph of f is concave upward (CU) on I.

b) If $f''(x) < 0$ for all x in interval I, then the graph of f is concave downward (CD) on I.

* Curve Sketching Guidelines:

1. Domain: find the domain of the function f, denoted by D_f.

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يبيهم

2. intercepts: find y-intercept $(0, f(0))$ and x-intercept $(x, 0)$.
3. Symmetry:
   a) if $f(-x) = f(x)$ for all $x \in D_f$, then f is even function and the curve is symmetric about the y-axis. That is, $(a, b) \mapsto (-a, b)$. Take f to be x^2, x^4, or $cos(x)$ as an example.
   b) if $f(-x) = -f(x)$ for all $x \in D_f$, then f is odd function and the curve is symmetric about the origin. That is, $(a, b) \mapsto (-a, -b)$. Take f to be x, x^3, or $sin(x)$ as an example.
4. Asymptotes: Horizontal asymptotes (H.A.) and Vertical asymptotes (V.A.).
5. Increasing/Decreasing Test: interval on which f is increasing or decreasing.
6. Local extrema: find all max. and min. local extrema.
7. Concavity and Inflection Points: find where the curve of f is CU or CD and find all of the inflection points.
8. Sketch: use all of the previous information to sketch the curve of f.

Definition - Domain of a function:

The domain of a function is the set of all real numbers for which the function is well-defined.

Remark:

The domain may be stated explicitly by giving a specific interval that is a subset of the real domain of the function, for example, if we write

i. The domain of a polynomial is $\mathbb{R}$.

ii. The domain of a rational function $f(x) = \frac{P(x)}{Q(x)}$; where $p(x)$ and $Q(x)$ are polynomials, and $Q(x) \neq 0$; is the set of all real numbers except where $Q(x) = 0$. That is; the domain of a rational function is $\mathbb{R}/\{Q(x) = 0\}$.

iii. The domain of $\sqrt[n]{x}$, where n is an **even** integer, is $x \geq 0$.

iv. The domain of $\sqrt[n]{x}$, where n is an **odd** integer, is $\mathbb{R}$.

V) The domain of $\ln x$ $\log x$ is $(0, \infty)$

Vi) The domain e^x, $\sin x$, $\cos x$ is $\mathbb{R}$

$\mathbb{R} = (-\infty, \infty)$

find the domain

9. $f(x) = x^3 - 3x^2 - 9x + 4$

$$f(x) = \frac{x^3}{x^2 + 1}.$$

$$y = \frac{2x^2}{x^2 - 1}.$$

$$y = 1/(1 + e^{-x})$$

$$y = (1 + e^x)^{-2} = \frac{1}{(1 + e^x)^2}$$

$\therefore 1 + e^x \neq 0$

$e^x \neq -1$

$\therefore \mathbb{R}$

find the domain

$f(x) = \frac{e^x}{1 - e^{x-1}}.$

$y = \ln(4 - x^2).$

$y = \ln(x^2 - 4)$

find the domain

$$f(x) = \frac{x^2+2x+1}{x^2-4x+4}$$

$$f(x) = \ln(x^2+1).$$

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

مو شرط كل دالة لها intercept مع axis ممكن تتقاطع وممكن لا

ex:

$$f(x) = \frac{x^2+2x+3}{x-2}$$

for y-int " $x = 0$ "

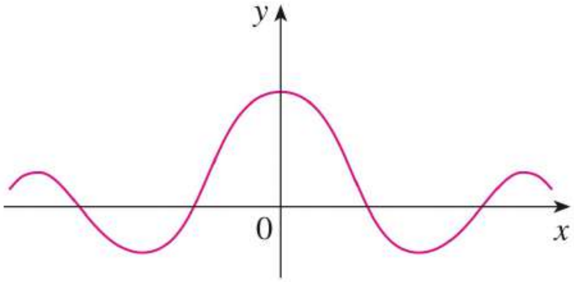
C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

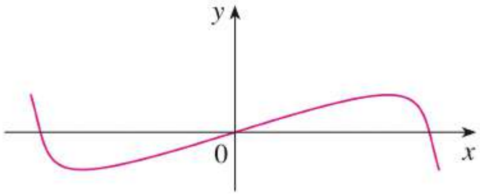
if $f(-x) = f(x)$

Then symmetric about y-axis



(a) Even function: reflectional symmetry

If $f(-x) = -f(x)$, Then it's symmetric about the origin



(b) Odd function: rotational symmetry

If neither of the above, then there is no symmetry.

$$f(x) = \frac{x^3 + 3}{x^2 - 4x}$$

$$f(-x) = \frac{(-x)^3 + 3}{(-x)^2 - 4(-x)}$$

$$= \frac{-x^3+3}{x^2+4x}$$

Examples (Even Functions):	***Examples (Odd Functions):***
$x^2 - 2$	$x^3 - x$
5	$\sqrt[5]{x}$
$x^2\lvert x\rvert$	$x^3\lvert x\rvert$
$\frac{x^4+1}{3x^8}$	$\frac{x^2+5}{x^3+2x}$
$\frac{x^3-2x}{x^5}$	$\frac{x^3-x^9}{x^4}$

<u>Example</u>: **Determine whether each function is Even, Odd or Neither.**

1. $f(x) = x^5 + x$

2. $f(x) = 1 - x^4$

3. $f(x) = 2x - x^2$

4. $f(x) = |x| + 2$

5. $f(x) = 3$

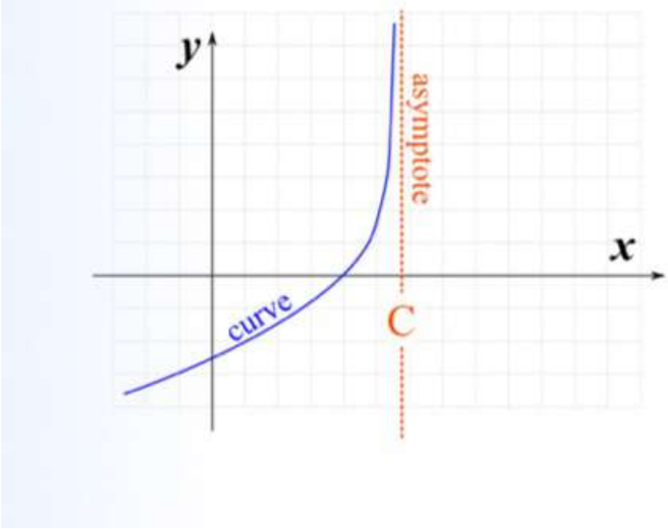
6. $f(x) = \frac{x}{x^2 + x^6}$

7. $f(x) = \frac{x^2}{x^4 + x}$

D. Asymptotes

Vertical Asymptotes

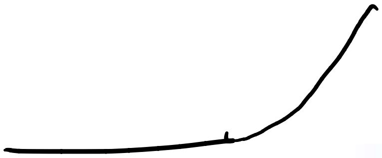
شلون نطلع V.A



١) نطلع الارقام الي تخلي الدالة غير معرفة مثلا أصفار المقام

٢) كل رقم أطلعه ! لازم تدرس الlimit من اليمين أو اليسار

٣) إذا طلع لك الجواب $\pm\infty$ يعني العدد الي طلعته يعتبر V.A



It is a Vertical Asymptote when:

as x approaches some constant value **c** (from the left or right) then the curve goes towards infinity (or −infinity).

Horizontal Asymptotes

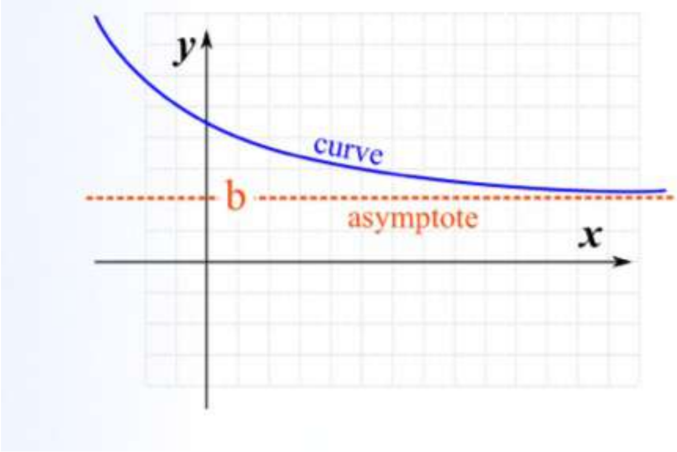
شلون نطلع H.A

١) نطلع $\lim_{x \to \infty} f(x)$

إذا طلع الlimit له قيمة ، إذا هذي القيمة أهيا H.A

٢) نطلع $\lim_{x \to -\infty} f(x)$

إذا طلع الlimit له قيمة ، إذا هذي القيمة أهيا H.A



It is a Horizontal Asymptote when:

as x goes to infinity (or −infinity) the curve approaches some constant value **b**

D. Asymptotes

طريقة الحل أن نقسم على أكبر أس بالمقام

find H.A $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$

a) $f(x) = \dfrac{3x-1}{2x+5}$

درجة الأس البسطا تساوي درجة
الأس في المقام

b) $f(x)=\dfrac{3x^2+2x}{4x^3-5x+7}$ درجة المقام أكبر من البسط

c) $f(x) = \dfrac{3x^2+4x}{x+2}$ درجة البسط أكبر من المقام

$$\lim_{x \to \infty} \frac{e^x + 300}{10^x + 3^x}$$

4) limit infinity for functions involving a radical

evalute $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$

$$f(x) = \frac{3x - 2}{\sqrt{4x^2+5}}$$

What Does f' Say About f?

E. Intervals of Increase or Decrease

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

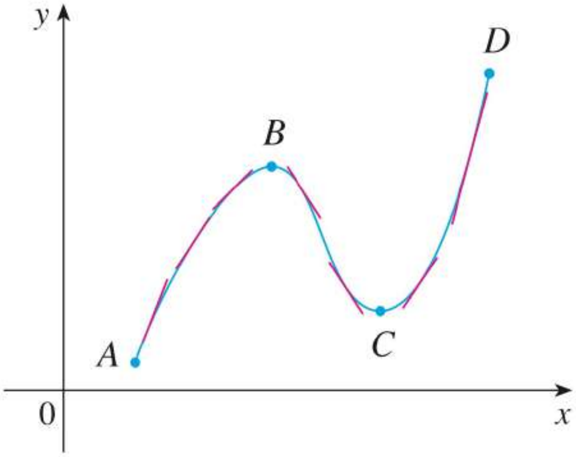


FIGURE 1

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

$c \in$ Domain

if not, we will not take it

Local Extreme Values

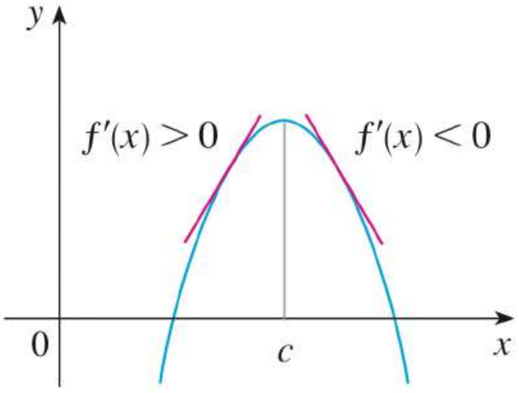
F. Local Maximum and Minimum Values

The First Derivative Test Suppose that c is a critical number of a continuous function f.

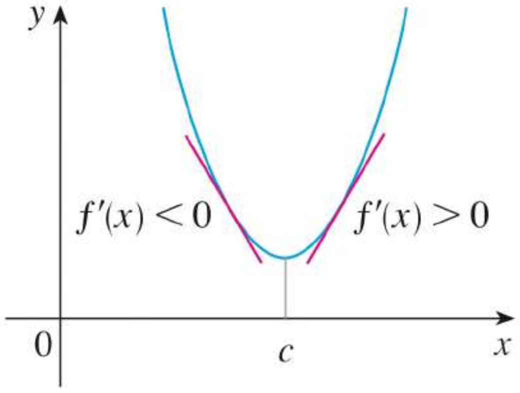
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

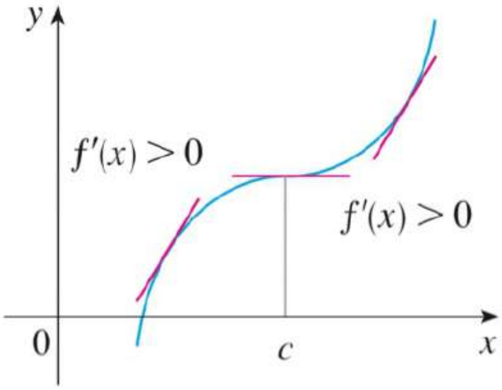
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



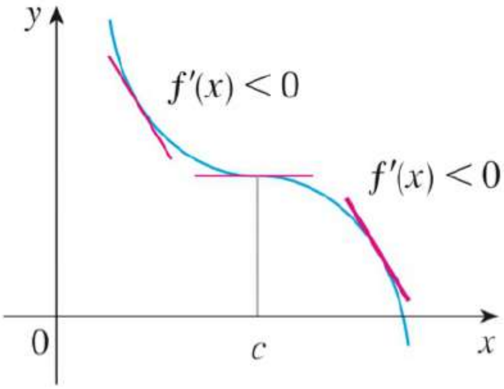
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum



(d) No maximum or minimum

تأكد أن قيمة الc تنتمي للdomain

“C” should $\in$ Domain of the function $f(x)$

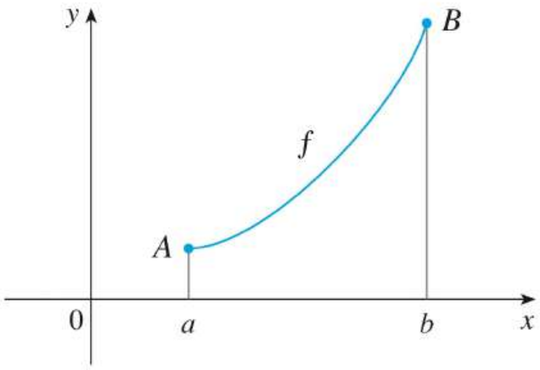
EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

مو مطلوب الرسم في هذا السؤال

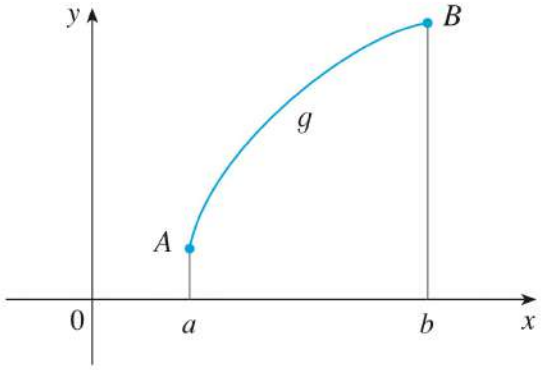
EXAMPLE 2 Find the local minimum and maximum values of the function f in Example 1.

مو مطلوب الرسم في هذا السؤال

What Does f'' Say About f?



(a)



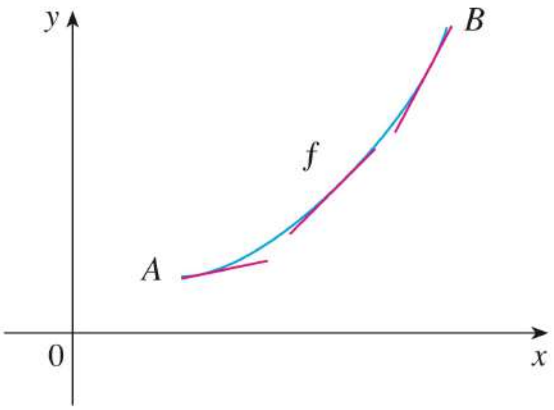
(b)

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

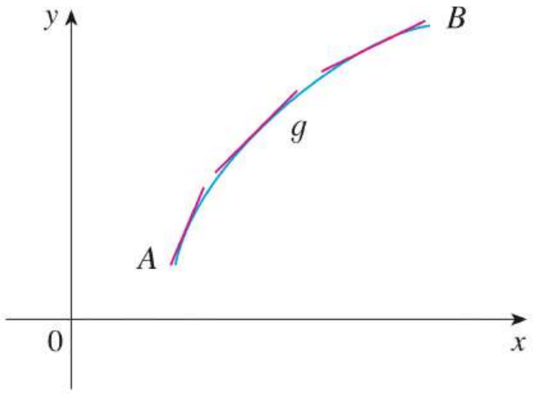
Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



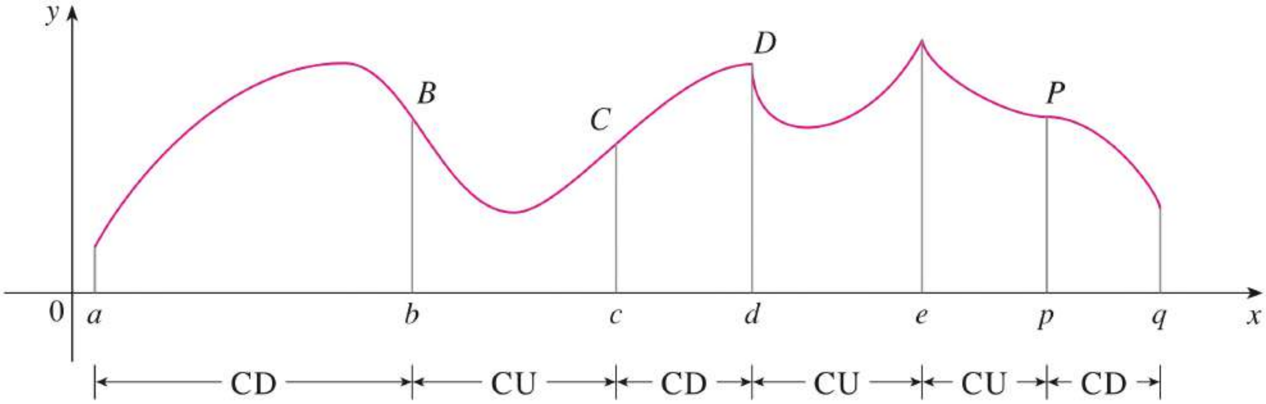
(a) Concave upward



(b) Concave downward

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).



(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

9. $f(x) = x^3 - 3x^2 - 9x + 4$

Calculus A

Chapter 4: Application of Differentiation

Sections: 4.5 Summary of Curve Sketching

A+

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* Increaing / Decreasing Test:

a) If $f'(x) > 0$ on an interval I, then f is an increasing (↗) funtion on I.

b) If $f'(x) < 0$ on an interval I, then f is an decreasing (↘) funtion on I.

* The First Derivative Test:

Suppose that c is a critical number of a continuous function f.

a) If f' changes from + to − at c, then f has a local maximum at c.

b) If f' changes from − to + at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local extrema at c.

* Concavity Test:

a) If $f''(x) > 0$ for all x in interval I, then the graph of f is concave upward (CU) on I.

b) If $f''(x) < 0$ for all x in interval I, then the graph of f is concave downward (CD) on I.

* The Second Derivative Test:

Suppose that f'' is continuous near a number c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

* Curve Sketching Guidelines:

1. Domain: find the domain of the function f, denoted by D_f.

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2. intercepts: find y-intercept $(0, f(0))$ and x-intercept $(x, 0)$.
3. Symmetry:
   a) if $f(-x) = f(x)$ for all $x \in D_f$, then f is even function and the curve is symmetric about the y-axis. That is, $(a, b) \mapsto (-a, b)$. Take f to be x^2, x^4, or $cos(x)$ as an example.
   b) if $f(-x) = -f(x)$ for all $x \in D_f$, then f is odd function and the curve is symmetric about the origin. That is, $(a, b) \mapsto (-a, -b)$. Take f to be x, x^3, or $sin(x)$ as an example.
4. Asymptotes: Horizontal asymptotes (H.A.) and Vertical asymptotes (V.A.).
5. Increasing/Decreasing Test: interval on which f is increasing or decreasing.
6. Local extrema: find all max. and min. local extrema.
7. Concavity and Inflection Points: find where the curve of f is CU or CD and find all of the inflection points.
8. Sketch: use all of the previous information to sketch the curve of f.

find the domain first

Graphing Rational Functions

A. Domain

A. Test to see if the graph has symmetry by plugging in (-x) in the function.

Options:
If the signs all stay the same or all change, f(-x) = f(x), then you have **even or y-axis symmetry**.

$$f(x) = \frac{x^2+3}{x^2-2}$$

$$f(-x) = \frac{(-x)^2+3}{(-x)^2-2}$$

$$f(-x) = \frac{x^2+3}{x^2-2}$$

If either the numerator or the denominator changes signs completely, f(-x)= -f(x) then you have **odd, or origin symmetry**.

$$f(x) = \frac{x^2}{x^3-x}$$

$$f(-x) = \frac{(-x)^2}{(-x)^3—(-x)}$$

$$= \frac{x^2}{-x^3+x}$$

$$= -\left(\frac{x^2}{x^3-x}\right)$$

If neither of the above, then there is no symmetry.

$$f(x) = \frac{x^3+3}{x^2-4x}$$

$$f(-x) = \frac{(-x)^3+3}{(-x)^2-4(-x)}$$

$$= \frac{-x^3+3}{x^2+4x}$$

B. Test to find y-intercepts by replacing x with 0.

$$f(x) = \frac{3x^2+8}{x^3-2}$$

$$f(0) = \frac{3(0)^2+8}{(0)^3-2}$$

$$= \frac{8}{-2} = -4$$

$$\text{y-int. } (0, -4)$$

C. Test to find x-intercepts by setting the numerator equal to 0.
Shortcut: Since multiplying by the denominator will eliminate it, you can just set the numerator equal to zero.

$$f(x) = \frac{2x}{x+1}$$

$$0 = 2x \rightarrow 0 = x$$

$$\text{x-int. } (0, 0)$$

D. Find the vertical asymptote by setting the denominator equal to zero.
The result is the equation of the vertical asymptote. Keep an eye out for holes. Holes occur when there is a factor that is the same both in the numerator and in the denominator. IT IS NOT A VERTICAL ASYMPTOTE because it simplifies away.

$$f(x) = \frac{x^2-9}{(x-3)(x+5)}$$

$$= \frac{(x-3)(x+3)}{(x-3)(x+5)}$$

$$x-3 = 0, \qquad x = 3$$

Therefore at x=3 there is a hole.

$$x + 5 = 0 \; x = -5$$

Therefore at x=-5 there is a vertical asymptote.

E. Find the horizontal asymptote.
To find the horizontal asymptote you compare the degrees of the numerator and the denominator. Note: horizontal asymptotes can be crossed while vertical asymptotes can never be crossed.

<u>Options:</u>
If the degrees are the same, you take the ratio of the leading coefficients and that is your asymptote.

$$f(x) = \frac{5x^2}{x^2 + 7}$$

$$\text{ratio} = \frac{5}{1} \text{ so the asymptote is at } y = 5$$

If the degree of the numerator is less than the degree of the denominator, y=0 is your asymptote.

$$f(x) = \frac{4x - 9}{x^2 + 3}$$

$$1 < 2 \text{ so the asymptote is at } y = 0$$

If the degree of the numerator is exactly one greater than the degree of the denominator, there is no horizontal asymptote but there is a slant asymptote.

$$f(x) = \frac{x^2 + 2}{x}$$

$$2 > 1 \text{ so there is no horizontal asymptote}$$

G. Find points.
Plot at least one point between and beyond each x-intercept and vertical asymptote. Plug a number in for x and solve for y. You want to do this so you can find which side of the asymptotes you are going to graph on.

H. Graph using the information you have found.

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

6. [$4 \times 10 = 40$ pts.] Let $f(x) = \dfrac{x^2 - x}{(x+1)^2}$.

(a) Find the vertical and horizontal asymptotes of the graph of f, if any.

(b) Given that $f'(x) = \dfrac{3x-1}{(x+1)^3}$.

  i. Find the intervals on which f is increasing and the intervals on which f is decreasing.

  ii. Find the local maximum and minimum values of f, if any.

(c) Given that $f''(x) = \dfrac{6(1-x)}{(x+1)^4}$.

  i. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

  ii. Find the points of inflection, if any.

(d) Sketch the graph of f.

5. [10 × 4 = 40 pts.] Let $f(x) = \ln(x^2 + 1)$.

(a) i) Study the symmetry of the graph of f.

ii) Find the horizontal asymptotes of the graph of f, if any.

(b) Given that $f'(x) = \dfrac{2x}{x^2 + 1}$.

i) Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

ii) Find the local maximum and minimum values of f, if any.

(c) Given that $f''(x) = \dfrac{2(1 - x^2)}{(x^2 + 1)^2}$.

i) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, if any.

ii) Find the points of inflection, if any.

(d) Sketch the graph of f.

7. [40 pts.] Let $f(x) = 1 + e^{-x^2}$.

(a) Find the horizontal asymptotes of the graph of f, if any.

(b) Given that $f'(x) = -2xe^{-x^2}$.

  i) Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

  ii) Find the local maximum and minimum values of f, if any.

(c) Given that $f''(x) = 2(2x^2 - 1)e^{-x^2}$.

  i) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, if any.

  ii) Find the points of inflection, if any.

(d) Sketch the graph of f.

6. [4 × 10 = 40 pts.] Let $f(x) = \dfrac{e^x}{1 - e^{x-1}}$.

(a) [5 + 5 = 10 pts.] Find the vertical and the horizontal asymptotes of the graph of f, if any.

(b) [5 + 2.5 + 2.5 = 10 pts.]

i. Show that $f'(x) = \dfrac{e^x}{(1 - e^{x-1})^2}$.

ii. Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

iii. Find the local maximum and minimum values of f, if any.

(c) [5 + 5 = 10 pts.] Given that $f''(x) = \dfrac{e^x + e^{2x-1}}{(1 - e^{x-1})^3}$.

i. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, if any.

ii. Find the points of inflection, if any.

(d) [10 pts.] Sketch the graph of f.

$$f(x) = \frac{x^2+1}{(x^2-4)^2}$$

$$f'(x) = \frac{-2x(x^2+6)}{(x^2-4)^3}$$

$$f''(x) = \frac{6(x^4+14x+8)}{(x^2-4)^4}$$

a) Find the vertical and horizontal asymptotes for the graph of f, if any.

b) Study the symmetry of the curve $y = f(x)$.

c) find x-intercept and y-intercept

i) Find the intervals on which f is increasing and the intervals on which f is decreasing.

ii) Find the local maximum and minimum values of f, if any.

i) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

ii) Find the points of inflection, if any.

e) Sketch the graph of f.

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim\limits_{x \to -\infty} f(x) = -2$, $\lim\limits_{x \to \infty} f(x) = 0$

4. [10 pts.] Sketch a possible graph of a function f that satisfies the following conditions:

(a) $f(0) = 0,\ f(4) = 3$ and $f(6) = 0$.

(b) $f'(x) < 0$ on $(-\infty, 0)$ and $(4, \infty)$, $f'(x) > 0$ on $(0, 4)$.

(c) $f''(x) < 0$ on $(-\infty, 0)$ and $(0, 6)$, $f''(x) > 0$ on $(6, \infty)$.

Calculus A

Chapter 4: Application of Differentiation

Sections: 4.7 Optimization Problems

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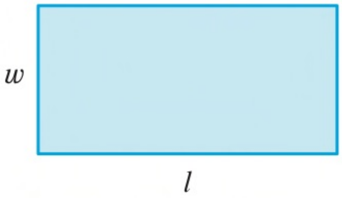
S=Surface Area = A = Area. P = Perimeter = Circumference = C. Volume = V

Rectangle

$A = lw$

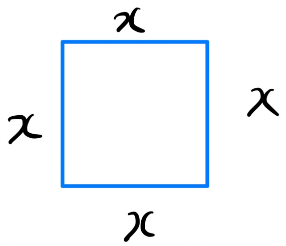
$P = 2l + 2w$



Square

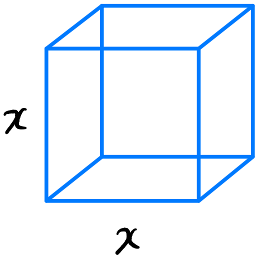
$A = x^2$

$P = 4x$



Cube

$V = x^3$

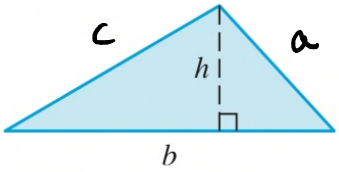


$S = 6x^2$

Triangle

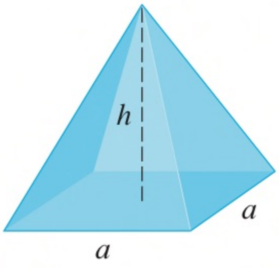
$A = \frac{1}{2}bh$

$P = a+b+c$



Pyramid

$V = \frac{1}{3}ha^2$



RECTANGULAR SOLID

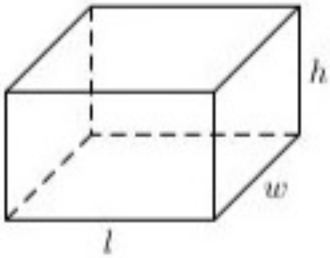
l = length, w = width,

h = height

Volume: $V = lwh$

Surface Area:

$S = 2lw + 2lh + 2wh$

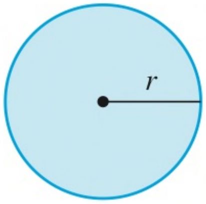


Circle

$A = \pi r^2$

$C = 2\pi r$

$D = 2r$

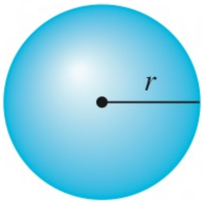


Sphere

$V = \frac{4}{3}\pi r^3$

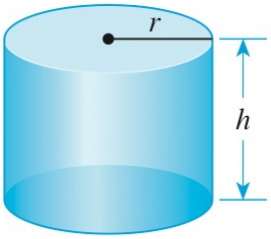
$A = 4\pi r^2$

$D = 2r$



Cylinder

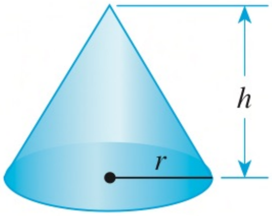
$V = \pi r^2 h$



$S = 2\pi rh + 2\pi r^2$

Cone

$V = \frac{1}{3}\pi r^2 h$



$S = \pi r\sqrt{r^2+h^2}$

Steps In Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols ($a, b, c, \ldots, x, y$) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.
6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

١ـ أفهم السؤال

٢ـ أرسم المسألة إذا احتجت

٣ـ أكتب المعادلة الي بدي نسويلها maximized or minimized

٤ـ خل المعادلة كمتغير واحد

٥ ـ طلع ال critical value و تأكد تنتمي لل Domain

٦ـ ال critical value رح تكون يا abs min أو abs max

٧ـ جاوب على السؤال

عندنا طريقتين للايجاد absolute max وال absoulte mini

١)

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

	$-\infty$		25		∞
f'		−		+	
↗↘		↘		↗	

Then the abs minimum is at $x = 25$

	$-\infty$		600		∞
f'		+		−	
↗↘		↗		↘	

Then the abs maximum is at $x = 600$

---

٢) أو عن طريق المشتقة الثانية

$f''(x) < 0$

إذا المشتقة الثانية دايما سالبة بكل الفترات يعني عندها local maximum

concave down ∩ abs max

$f''(x) > 0$

إذا المشتقة الثانية دايما موجبة بكل الفترات يعني عندها local minimum

concave up ∪ abs minimum

3. [10 pts.] Find the maximum product of two positive x and y such that $2x+y = 100$.

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

∴

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

21. Find the point on the line $y = 2x + 3$ that is closest to the origin.

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

4. [10 pts.] Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

3. [10 pts.] Find two nonnegative numbers x and y whose sum is 15 and $P = x^2 y^3$ is a maximum.

Q7. [10 pts.] Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(2, 0)$.

4. [10 pts.] Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Calculus A

Chapter 4: Application of Differentiation

Sections: 4.9 Antiderivatives

A+

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4.9 Antiderivatives

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

EXAMPLE 1 Find the most general antiderivative of each of the following functions.
(a) $f(x) = \sin x$ (b) $f(x) = 1/x$ (c) $f(x) = x^n, \quad n \neq -1$

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.
(b) Recall from Section 3.6 that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So on the interval $(0, \infty)$ the general antiderivative of $1/x$ is $\ln x + C$. We also learned that

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

for all $x \neq 0$. Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is $\ln |x| + C$ on any interval that doesn't contain 0. In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. So the general antiderivative of f is

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) We use the Power Rule to discover an antiderivative of x^n. In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

4.9 Antiderivatives

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$$

EXAMPLE 3 Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = 4x + 7$

3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$

5. $f(x) = x(12x + 8)$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$

5. $f(x) = x(12x + 8)$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

7. $f(x) = 7x^{2/5} + 8x^{-4/5}$

9. $f(x) = \sqrt{2}$

10. $f(x) = e^2$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

11. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$

12. $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

13. $f(x) = \frac{1}{5} - \frac{2}{x}$

15. $g(t) = \frac{1 + t + t^2}{\sqrt{t}}$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

14. $f(t) = \dfrac{3t^4 - t^3 + 6t^2}{t^4}$

22. $f(x) = \dfrac{2x^2 + 5}{x^2 + 1}$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

25–48 Find f.

27. $f''(x) = 2x + 3e^x$

28. $f''(x) = 1/x^2$

25–48 Find f.

34. $f'(t) = t + 1/t^3, \quad t > 0, \quad f(1) = 6$

25–48 Find f.

41. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

25–48 Find f.

45. $f''(x) = e^x - 2\sin x, \quad f(0) = 3, \quad f(\pi/2) = 0$

4. [10 pts.] Show that $F(x) = \frac{1}{2}\left(x^2 \tan^{-1} x - x + \tan^{-1} x\right)$ is an antiderivative of $f(x) = x \tan^{-1} x$.

5. [10 pts.] Let $f(x) = \sin(\ln x)$. Show that $F(x) = \frac{x}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big]$ is an antiderivative of $f(x)$.

Calculus A

Chapter 5: Integrals

Sections: 5.2 The Definite Integral

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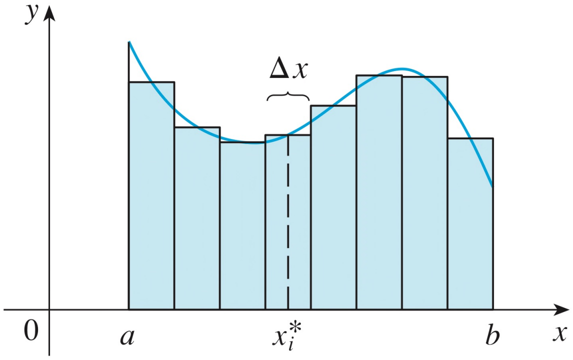
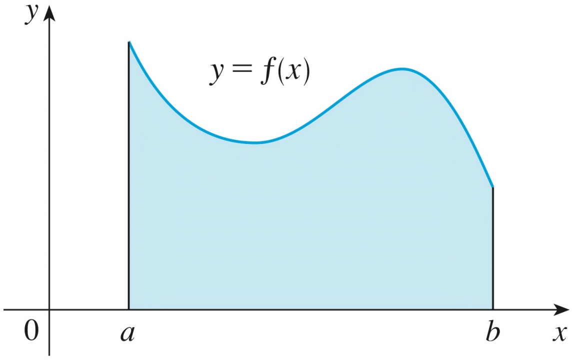
5.2 The Definite Integral

$\boxed{2}$ **Definition of a Definite Integral** If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.





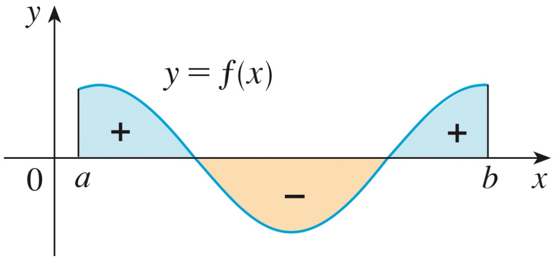
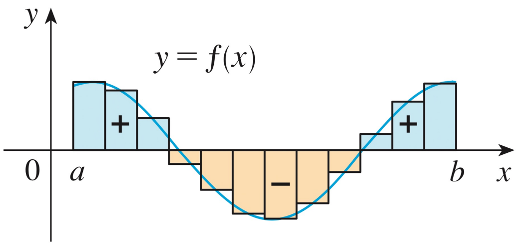


FIGURE 4

$\int_a^b f(x)\,dx$ is the net area.



التكامل هو المساحة تحت المنحى

$$\sum_{i=1}^{n} \underset{\text{الطول}}{f(x_i^*)} \quad \underset{\text{العرض}}{\Delta x}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

1) $\int$ is the integral sign

2) a and b the limits of the integral

3) f(x) is the integrand

4) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is called Riemann Sum

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Right end-points

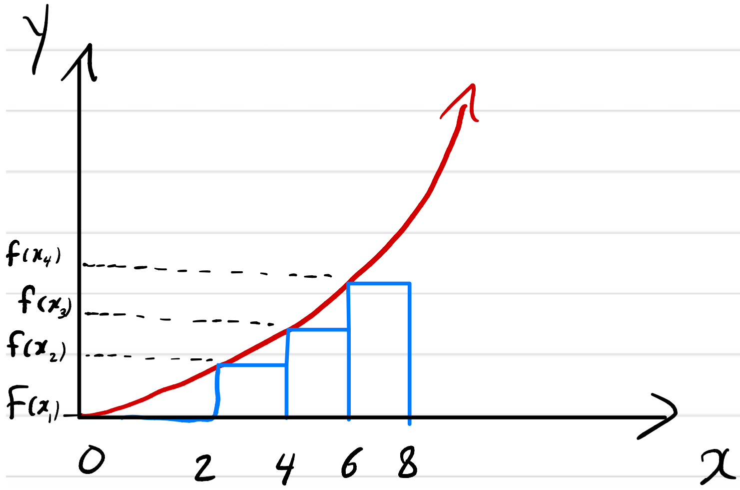
left $x_i = a + (i-1)\Delta x$

middle $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$$y = x^2 \qquad n = 4 \qquad [0, 8]$$

Taking sample points to be left

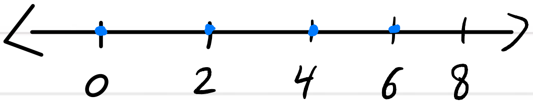
$$\text{Area} = \sum \underset{\text{عرض}}{\Delta x}\ \underset{\text{طول}}{f(x_i)}$$



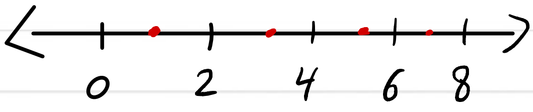
$$\Delta x = \frac{b-a}{n} = \frac{8-0}{4}$$

$$= 2$$

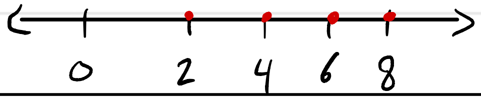
$$A_L = \Delta x [f(0) + f(2) + f(4) + f(6)]$$



$$A_m = \Delta x [f(1) + f(3) + f(5) + f(7)]$$



$$A_R = \Delta x [f(2) + f(4) + f(6) + f(8)]$$



خطوات حل Riemann sum

$f(x) = \sqrt{4-x^2}$, $-2 \leq x \leq 2$, $n=4$

by taking to the right

---

١) $f(x) = \sqrt{4-x^2}$

٢) أطلع حدود التكامل أو الفترة الي يبي مساحتها الي هي قيم a and b و عددها n

$\int_a^b$, $a \leq x \leq b$

$a = -2$

$b = 2$

$n=4$

٣) أشوف الsample points يبيها من أي طرف

Right

٤) أطلع المسافة الي بينهم

$$\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{4} = 1$$

٥) نسوي الجدول

x	-2	-1	0	1	2
$f(x)$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

٦) السؤال قايلي يبي
أربع قيم من اليمين

$$R_{4_{right}} = [\sqrt{3} + 2 + \sqrt{3} + 0](1) = 2 + 2\sqrt{3}$$

6. [10 pts.] Evaluate the Riemann sum for $f(x) = 3 - x^2$, taking the sample points to be right endpoints and $a = 1, b = 4$, and $n = 3$.

II. The Riemann sum of $f(x) = x^2$ on the interval $[1, 5]$ using $n = 4$ and taking the sample points to be the right endpoints is:

a) 25.
b) 54.
c) -30.
d) 0.
e) None of the above.

II. The Riemann sum of $f(x) = x^2$ on the interval $[1, 5]$ using $n = 4$ and taking the sample points to be the ~~right~~ endpoints is:

left

a) 25.

b) 54.

c) -30.

d) 0.

e) None of the above.

F) 30

I. The Riemann sum of $f(x) = \sin x,\ 0 \leqslant x \leqslant \pi$ with $n = 3$ taking the sample points to be the right endpoints is equal to

a) 0.
b) $\frac{\pi}{3}$.
c) $\frac{\pi}{\sqrt{3}}$.
d) $\frac{\pi(1+\sqrt{3})}{2}$.
e) None of the above.

نفس القيمة

$$\frac{\sqrt{3}\,\pi}{3} * \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\,\pi}{\cancel{3}\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

1. Evaluate the Riemann sum for $f(x) = x - 1$, $-6 \leqslant x \leqslant 4$, with five subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

or $\int_{-6}^{4} x - 1 \, dx$, $n = 5$ right end

3. If $f(x) = 16 - x$, $0 \leqslant x \leqslant 16$, find the Riemann sum with $n = 4$, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

The right end = 96 , The left end = 160

إجابة نفس السؤال لو طلب مني من جهة اليمين واليسار

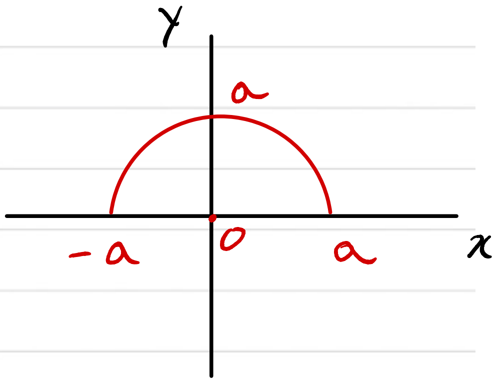
Integration in terms of areas

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$$

$$Y = \sqrt{a^2 - x^2}$$

$$x^2 + Y^2 = a^2$$

$$a^2 = r^2$$



$A = \pi r^2$ مساحة الدائرة

$A = \frac{1}{2} \pi r^2$ مساحة نصف دائرة

$A = \frac{1}{4} \pi r^2$ مساحة ربع الدائرة

شلون تعرف إذا نص دائرة أو ربع؟

تشوف حدود الكامل

Note: Not circle $\int_{-a}^{a} \sqrt{x^2 - a^2}$

EXAMPLE 4 Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1 - x^2}\, dx$

35–40 Evaluate the integral by interpreting it in terms of areas.

35. $\int_{-1}^{2} (1 - x)\, dx$

نقدر نرسم الدالة عن طريق التعويض بحدود التكامل داخل الدالة

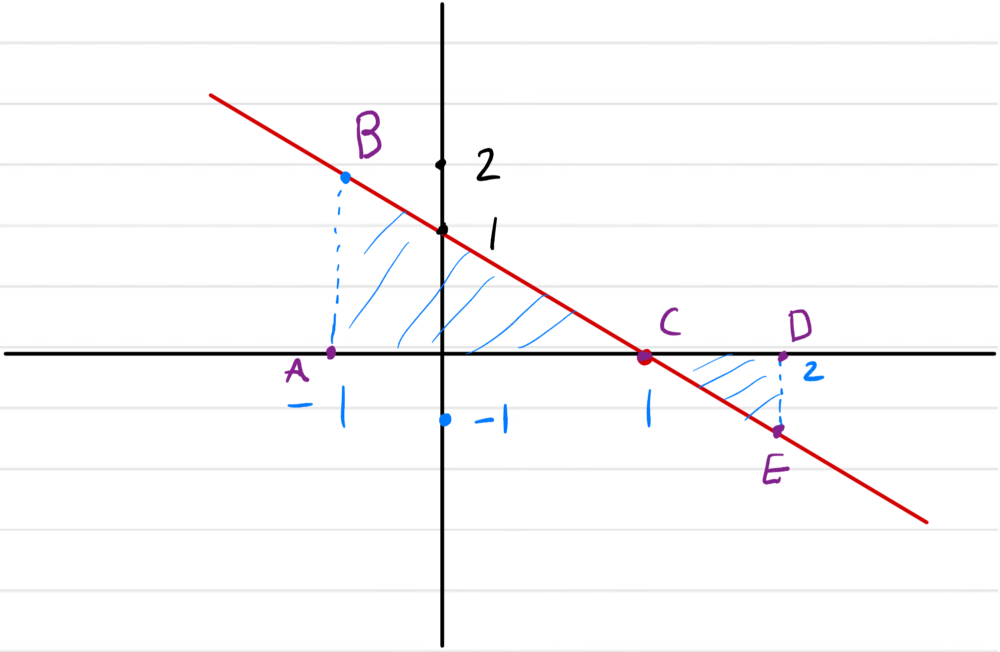
$f(x) = 1 - x$

$f(-1) = 2$

$f(0) = 1$

$f(1) = 0$

$f(2) = -1$



Area of Triangle $= \frac{1}{2}$ (Base) (height)

ملاحظة : أي مساحة تحت
الx-axis رح تكون سالبة

$\int_{-1}^{2} (1-x)\, dx$ = area of triangle ABC + area of triangle CDE

$= \frac{1}{2}(2)(2) + \left(\frac{1}{2}\right)(1)(-1)$

$= 2 - \frac{1}{2} = \frac{3}{2}$

35–40 Evaluate the integral by interpreting it in terms of areas.

37. $\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right) dx$

IV . $\displaystyle\int_{-1}^{1} \sqrt{1 - x^2}\, dx =$

a) 1.

b) $\pi/4$.

c) $\pi/2$.

d) 0.

e) None of the above.

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

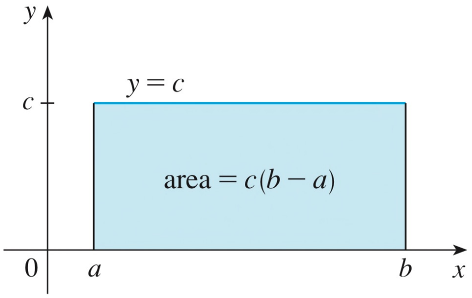


FIGURE 13

$\int_a^b c\,dx = c(b - a)$

EXAMPLE 6 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\, dx$.

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

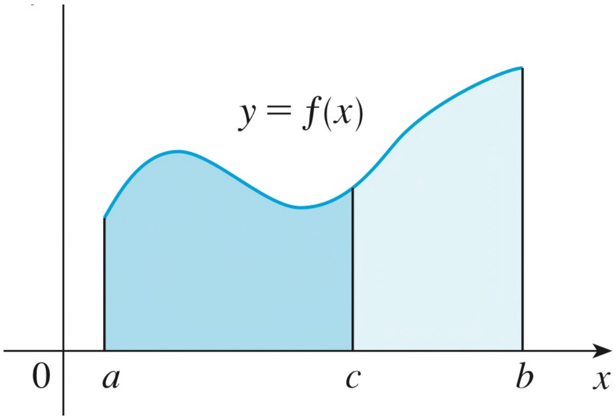


FIGURE 15

EXAMPLE 7 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$

50. Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

48. If $\int_2^8 f(x)\,dx = 7.3$ and $\int_2^4 f(x)\,dx = 5.9$, find $\int_4^8 f(x)\,dx$.

49. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find

$$\int_0^9 [2f(x) + 3g(x)]\,dx$$

II. If $\int\limits_{1}^{5} f(x)dx = 7$, then the value of $\int\limits_{1}^{5}(3f(x) + x)dx$ is equal to

a) 33.

b) 0.

c) 21.

d) 54.

e) None of the above.

6. [10 pts.] Let f and g be two continuous functions such that $\displaystyle\int_2^1 f(x)dx = -1$, $\displaystyle\int_2^3 f(x)dx = 3$, and $\displaystyle\int_1^3 g(x)dx = 5$. Find $\displaystyle\int_1^3 [f(x) - 2g(x)]dx$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

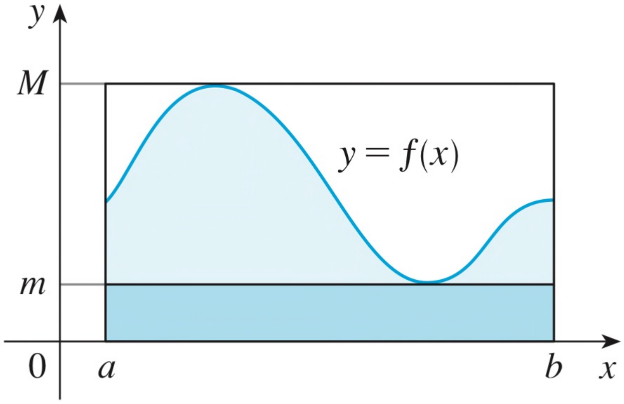


FIGURE 16

Show that (estimate) or use properties of integral

$$4 \leq \int_{1}^{3} \sqrt{x^2+3}\, dx \leq 4\sqrt{3}$$

55–58 Use the properties of integrals to verify the inequality without evaluating the integrals.

57. $2 \leqslant \int_{-1}^{1} \sqrt{1 + x^2}\, dx \leqslant 2\sqrt{2}$

55–58 Use the properties of integrals to verify the inequality without evaluating the integrals.

56. $\int_0^1 \sqrt{1 + x^2}\, dx \leqslant \int_0^1 \sqrt{1 + x}\, dx$

Show that (estimate) or

use properties of integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2 x + 3)\, dx \leq \frac{\pi}{3}$$

Calculus A

Chapter 5: Integrals

Sections: 5.3 The Fundamental Theorem of Calculus

A+

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5.3 The Fundamental Theorem of Calculus

نظرية مشتقة التكامل حفظ

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

مشتقة ال upper limit

$$\frac{d}{dx}\int_0^{\textcircled{x}} f(t)\,dt = f(x)\cdot(1)$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x)$$

حفظ رح نشرحها بسكشن الي بعده بالتفصيل

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

$$\int_{-2}^{1}\frac{1}{x}\,dx \neq \Big[\ln|x|\Big]_{-2}^{1}$$

not continous at $0 \in (-2,1)$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

---

find the derivative

$$f(x) = \int_0^x \sqrt{t^2+4}\,dt$$

---

where $f(t) = \sqrt{t^2+4}$

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt = \frac{d}{dx}\,\Big[F(t)\Big]_0^x\right.$$

$$\frac{d}{dx}\Big[F(x) - F(0)\Big] = f(x) - 0$$

$$\therefore f(x) = \sqrt{x^2+4}$$

find $\frac{d}{dx}\left[\int_{x}^{4} \sqrt{t^3+5}\ dt\right]$

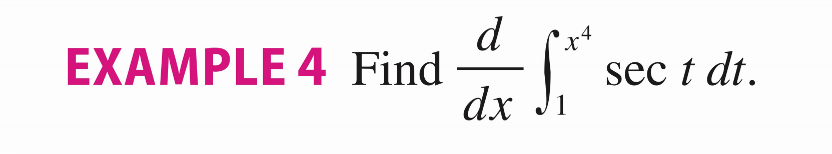
By the Fundamental Theorem of Calculus

$\because\ f(t) = \sqrt{t^3+5}$

$$\frac{d}{dx}\left[\int_{x}^{4} f(t)\,dt\right] = \frac{d}{dx}\left[F(t)\right]_{x}^{4}$$

$$= \frac{d}{dx}[F(4) - F(x] = 0 - f(x)$$

$$= -\sqrt{x^3-5}$$



EXAMPLE 4 Find $\frac{d}{dx}\int_{1}^{x^4} \sec t\,dt$.

By the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{1}^{x^4} \sec t\,dt = \sec(x)^4\left(\frac{dx^4}{dx}\right)$$

$$= 4x^3\sec(x)^4$$

7–18 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

7. $g(x) = \int_0^x \sqrt{t + t^3}\, dt$

9. $g(s) = \int_5^s (t - t^2)^8\, dt$

11. $F(x) = \int_x^0 \sqrt{1 + \sec t}\, dt$

$$\left[\textit{Hint: } \int_x^0 \sqrt{1 + \sec t}\, dt = -\int_0^x \sqrt{1 + \sec t}\, dt \right]$$

12. $R(y) = \int_y^2 t^3 \sin t\, dt$

13. $h(x) = \int_{1}^{e^x} \ln t \, dt$

17. $y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta \, d\theta$

18. $y = \int_{\sin x}^{1} \sqrt{1 + t^2}\, dt$

5. [10 pts.] Find the slope of the tangent line to $F(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \ln(1+2t)dt$ at $x = 1$.

7. [10 pts.] Find the inflection points, if any, of the curve

$$y = \int_{0}^{x} (1-t)e^{t}\,dt.$$

4. [10 pts.] Find an equation of the tangent line to the graph of the function

$$f(x) = \int_{-1}^{x^3} \frac{\ln(t+2)}{t^2+1}\,dt \quad \text{at } x = -1.$$

find f(x) if f is continous function such that:

$$\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt$$

5. [10 pts.] Let $f(x) = \displaystyle\int_1^{x^2+1} \frac{t^2}{\sqrt{t^2+1}}\, dt$. Find the absolute minimum value of f, if any.

Q9. [10 pts.] On what interval is the curve $y = \int_0^x \frac{1-t}{t^2+8}\,dt$ concave downward?

59–63 Find the derivative of the function

59. $g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\, du$

60. $g(x) = \int_{1-2x}^{1+2x} t \sin t\, dt$

59–63 Find the derivative of the function

61. $F(x) = \int_{x}^{x^2} e^{t^2}\, dt$

62. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t \, dt$

Calculus A

Chapter 5: Integrals

Sections: 5.4 Indefinite Integrals and the Net Change Theorem

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For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

حفظ

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

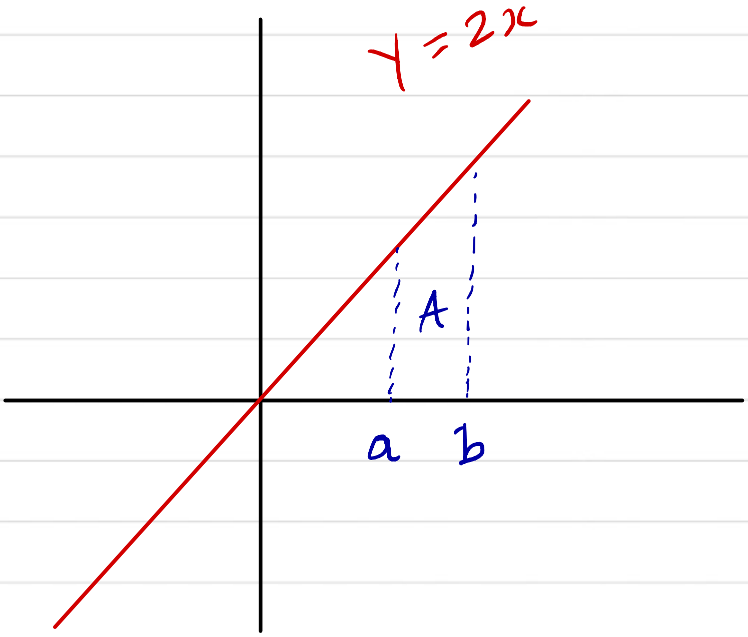
$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

شرح :-



$$\int_1^2 2x\,dx$$

we Know $\int 2x\,dx = x^2 + C$

at $x = 1$: $\int 2x\,dx = 1^2 + C$

at $x = 2$: $\int 2x\,dx = 2^2 + C$

* subtract $(2^2 + C) - (1^2 + C)$

$= 4 + C - 1 - C = 3$

حفظ

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

$$\int_1^2 2x\,dx = 2\left[\frac{x^2}{2}\right]_1^2 = \left[x^2\right]_1^2$$

$$= [2^2] - [1^2] = 3$$

ملاحظة :

التكامل يتوزع بالجمع والطرح

التكامل لا يتوزع بالضرب والقسمة

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

$$\int \frac{1}{x^2+1}\,dx =$$

$$\int \sqrt{x}\,dx =$$

$$\int 10^x\,dx =$$

$$\int x^{10}\,dx =$$

$$\int e^x\,dx =$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\, dx$$

EXAMPLE 4 Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2 + 1}\right) dx$ and interpret the result in terms of areas.

1–4 Verify by differentiation that the formula is correct.

1. $\int \frac{1}{x^2\sqrt{1 + x^2}}\,dx = -\frac{\sqrt{1 + x^2}}{x} + C$

1–4 Verify by differentiation that the formula is correct.

3. $\int \tan^2 x \, dx = \tan x - x + C$

EXAMPLE 5 Evaluate the integral $\int_1^3 e^x\,dx$.

SOLUTION The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

SOLUTION An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} \qquad ■$$

EXAMPLE 7 Evaluate $\int_3^6 \frac{dx}{x}$.

SOLUTION The given integral is an abbreviation for

$$\int_3^6 \frac{1}{x}\,dx$$

An antiderivative of $f(x) = 1/x$ is $F(x) = \ln|x|$ and, because $3 \leqslant x \leqslant 6$, we can write $F(x) = \ln x$. So

$$\int_3^6 \frac{1}{x}\,dx = \ln x\Big]_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2 \qquad ■$$

EXAMPLE 8 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

5–18 Find the general indefinite integral.

7. $\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right) dx$

5–18 Find the general indefinite integral.

9. $\int (u + 4)(2u + 1)\, du$

19–44 Evaluate the integral.

19. $\int_{1}^{3} (x^2 + 2x - 4)\, dx$

14. $\int \left(\frac{1 + r}{r} \right)^2 dr$

21. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt$

25. $\int_{\pi/6}^{\pi} \sin\theta \, d\theta$

26. $\int_{-5}^{5} e \, dx$

23. $\int_1^9 \sqrt{x}\, dx$

29. $\int_{1}^{4} \frac{2 + x^2}{\sqrt{x}}\,dx$

35. $\displaystyle\int_1^2 \frac{v^3 + 3v^6}{v^4}\,dv$

36. $\int_{1}^{18} \sqrt{\frac{3}{z}}\, dz$

21. $\int_{-2}^{3} (x^2 - 3)\, dx$

27. $\int_0^{\pi} (5e^x + 3 \sin x)\, dx$

37. $\int_0^1 (x^e + e^x)\,dx$

41. $\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1 - r^2}}$

30. $\int_0^1 \frac{4}{1 + p^2}\, dp$

34. $\int_0^1 (5x - 5^x)\, dx$

35. $\int_0^1 (x^{10} + 10^x)\, dx$

Integrating an Absolute Value

مافي تكامل حق المطلق

ولكن كل الي نقدر نسويه أن نعيد تعريفه

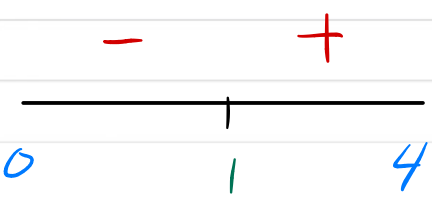
نطلع أصفار المطلق ولازم نتأكد من أصفار المطلق تنتمي للفترة الي أبي أكاملها

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_0^4 |x-1|\,dx$$

$x - 1 = 0$

$x = 1 \in (0,4)$



$$\int_0^4 |x-1|\,dx = \int_0^1 -(x-1)\,dx + \int_1^4 +(x-1)\,dx$$

$$\int_0^3 |x^2+2x-3| \, dx$$

44. $\int_0^2 |2x - 1|\, dx$

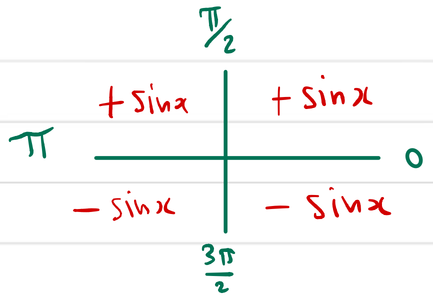
$2x - 1 = 0$

$x = \frac{1}{2}$

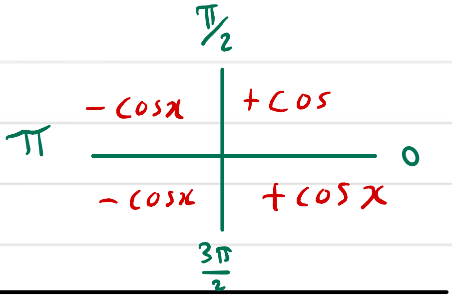
$-$ $+$

0 $\frac{1}{2}$ 2

46. $\int_0^{3\pi/2} |\sin x|\, dx$



$$\int_0^{\pi} |\cos x| \, dx$$



Calculus A

Chapter 5: Integrals

Sections: 5.5 The Substitution Rule

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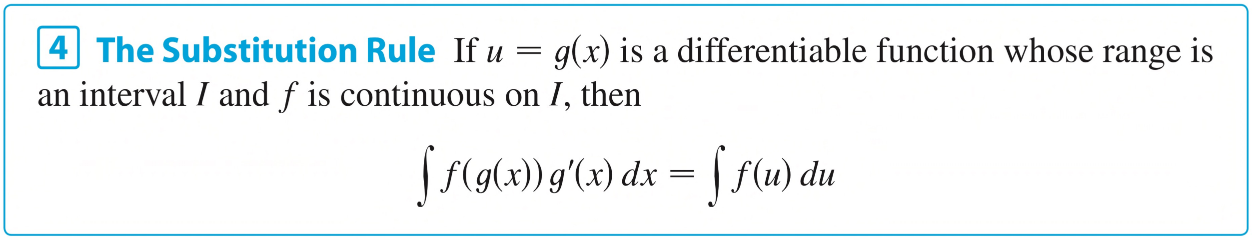

4 **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

شلون أختار u :

١) أختار الu للدالة الي مشتقتها موجودة معاها

٢) جرب داخل الدوال

٣) جرب الدالة الي مرفوعة أكبر أس

٤) جرب الدالة ال موجودة بالمقام

خطوات الحل

١- أختار الu
٢- اشتقها بالنسبة للمتغير
٣- لازم التكامل يصير بمتغير واحد الحين بس
٤- ييب التكامل الغير محدد
٥- شيل u وحط مكانها الدالة الاصلية (مع تكاملها إذا كان محدد)

---

* special case **46.** $\int x^2\sqrt{2+x}\,dx$ $= \int (u-2)^2\sqrt{u}\,du$

$u = 2 + x \Rightarrow (u-2) = x \Rightarrow (u-2)^2 = x^2$

$du = dx$

1–6 Evaluate the integral by making the given substitution.

1. $\int \cos 2x \, dx, \quad u = 2x$

2. $\int xe^{-x^2}\,dx, \quad u = -x^2$

$u = -x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

$\int xe^{-x^2}\,dx = \int e^u \frac{-du}{2}$

$= \frac{-1}{2}\int e^u\,du = \frac{-1}{2}e^u + C$

$= -\frac{1}{2}e^{-x^2} + C$

1–6 Evaluate the integral by making the given substitution.

6. $\int \sqrt{2t + 1}\, dt, \quad u = 2t + 1$

3. $\int x^2\sqrt{x^3+1}\,dx, \quad u = x^3+1$

4. $\int \sin^2\theta \cos\theta \, d\theta, \quad u = \sin\theta$

15. $\int \cos^3\theta \sin\theta \, d\theta$

$$\int \cos^3(5x)\,\sin(5x)\,dx$$

5. $\int \frac{x^3}{x^4 - 5}\,dx, \quad u = x^4 - 5$

9. $\int (1 - 2x)^9 \, dx$

16. $\int e^{-5r}\,dr$

17. $\int \frac{e^{u}}{(1-e^{u})^{2}}\,du$

19. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

21. $\int \frac{(\ln x)^2}{x}\,dx$

23. $\int \sec^2\theta \ \tan^3\theta \ d\theta$

25. $\int e^x \sqrt{1 + e^x}\, dx$

28. $\int e^{\cos t} \sin t \, dt$

30. $\int \frac{\sec^2 x}{\tan^2 x}\, dx$

44. $\displaystyle\int \frac{x}{1 + x^4}\,dx$

46. $\int x^2\sqrt{2 + x}\, dx$

47. $\int x(2x + 5)^8 \, dx$

55. $\int_0^1 \sqrt[3]{1 + 7x}\, dx$

59. $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

60. $\int_0^1 xe^{-x^2}\,dx$

62. $\int_0^{\pi/2} \cos x \ \sin(\sin x)\, dx$

64. $\int_0^a x\sqrt{a^2 - x^2}\,dx$

71. $\int_0^1 \frac{e^z + 1}{e^z + z}\, dz$

Examples (Even Functions):	***Examples (Odd Functions):***
$x^2 - 2$	$x^3 - x$
5	$\sqrt[5]{x}$
$x^2\lvert x\rvert$	$x^3\lvert x\rvert$
$\dfrac{x^4+1}{3x^8}$	$\dfrac{x^2+5}{x^3+2x}$
$\dfrac{x^3-2x}{x^5}$	$\dfrac{x^3-x^9}{x^4}$

Example: **Determine whether each function is Even, Odd or Neither.**

1. $f(x) = x^5 + x$

$$(-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$$

odd

2. $f(x) = 1 - x^4$

$$1 - (-x)^4 = 1 - x^4 = f(x)$$

even

3. $f(x) = 2x - x^2$

$$2(-x) - (-x)^2 = -2x - x^2 = -(2x + x^2)$$

Neither

4. $f(x) = |x| + 2$

$$|-x| + 2 = |x| + 2 = f(x)$$

even

5. $f(x) = 3$

$$f(x) = 3 = f(-x)$$

even

6. $f(x) = \dfrac{x}{x^2 + x^6}$

$$\frac{-x}{(-x)^2 + (-x)^6} = \frac{-x}{x^2 + x^6} = -f(x)$$

odd

7. $f(x) = \dfrac{x^2}{x^4 + x}$

$$\frac{(-x^2)}{(-x)^4 + (-x)} = \frac{x^2}{x^4 - x}$$

Neither

Symmetry

The next theorem uses the Substitution Rule for Definite Integrals (6) to simplify the calculation of integrals of functions that possess symmetry properties.

7 **Integrals of Symmetric Functions** Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Even-Odd Identities:

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

EXAMPLE 11 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$ ■

(b) $\displaystyle\int_{-1}^{1} (x^2 + \cos x) \tan x \, dx.$

38. $\int_{-5}^{5} \left(x - \sqrt{25 - x^2}\right) dx$

9. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

(a) $\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx.$

IV. $\displaystyle\int_{-1}^{1} x^2 \tan x \, dx =$

a) 0.

b) 1.

c) −1.

d) 2.

e) None of the above.

7. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

I. $\displaystyle\int \sin^3(x)\cos(x)\,dx.$

II. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx.$

7. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

(i) $\displaystyle\int \frac{\cos(\pi x)}{(1+\sin(\pi x))^3}\,dx.$

(ii) $\displaystyle\int_1^e \frac{\ln(x^2)}{x}\,dx.$

7. [10 + 10 pts.] Evaluate the integral

(a) $\displaystyle\int \frac{dx}{x(\ln x)^5}$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2\sec^2 x}{1+\tan x}dx$

EXAMPLE 5 Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

Q8. [5+5=10 pts.] Evaluate each of the following integrals.

(a) $\displaystyle\int (\cos x)\,(\sin(\sin x))\;\, dx$

(b) $\displaystyle\int_0^{\pi/4} 4(1+\tan x)^2 \sec^2 x \; dx$

EXAMPLE 6 Calculate $\int \tan x \, dx$.

Calculus A

Chapter 6: Application of Integration

Sections: 6.1 Area Between Curves

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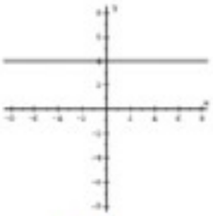
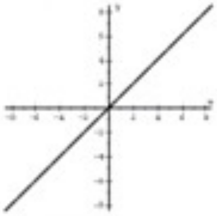
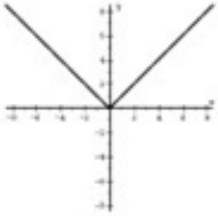
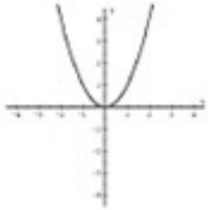
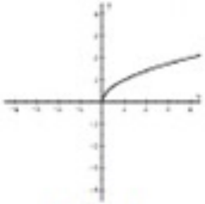
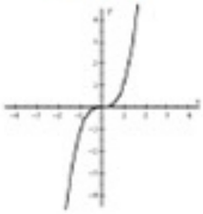
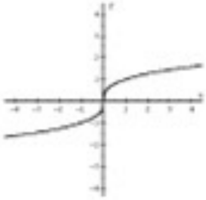
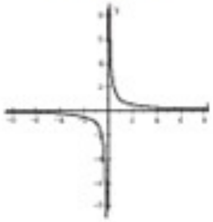
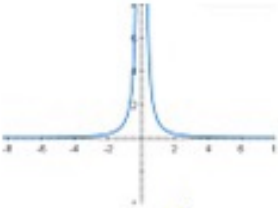
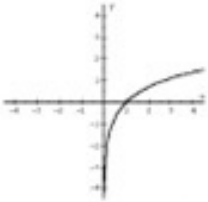
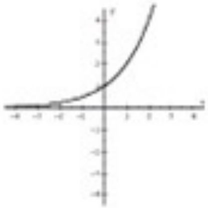
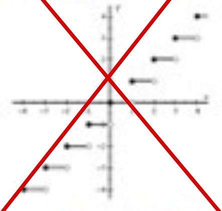
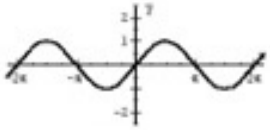
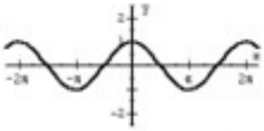
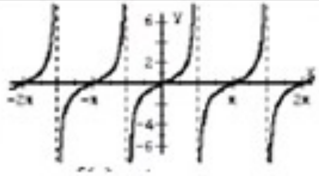
$\boxed{2}$ The area A of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

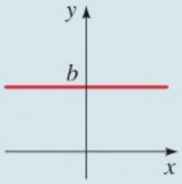
$$A = \int_a^b [f(x) - g(x)]\,dx$$

Constant  ✓ f(x) = c	Linear  ✓ f(x) = x	Absolute Value  ✓ f(x) = |x|	Quadratic  ✓ f(x) = x^2
Square Root  ✓ f(x)= $\sqrt{x}$	Cubic  ✓ f(x) = x^3	Cube Root  ✓ f(x) = $\sqrt[3]{x}$	Reciprocal/Inverse/ Rational  f(x) = $\frac{1}{x}$
Rational  f(x) = $\frac{1}{x^2}$	Logarithmic  ✓ f(x) = ln (x)	Exponential  ✓ f(x) = e^x	~~Greatest Integer (Step Function)~~  ~~f(x) = [[x]]~~
Trigonometric Functions →	 ✓ f(x) = sin (x)	 ✓ f(x) = cos (x)	 f(x) = tan (x)

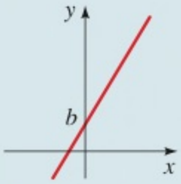
SOME FUNCTIONS AND THEIR GRAPHS

Linear functions

$f(x) = mx + b$



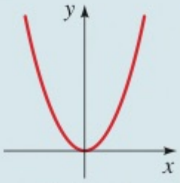
$f(x) = b$



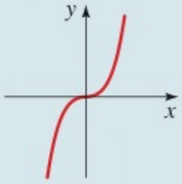
$f(x) = mx + b$

Power functions

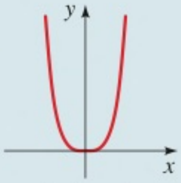
$f(x) = x^n$



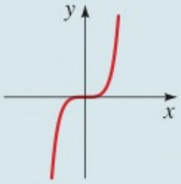
$f(x) = x^2$



$f(x) = x^3$



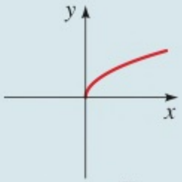
$f(x) = x^4$



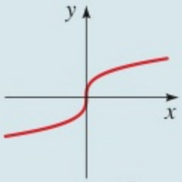
$f(x) = x^5$

Root functions

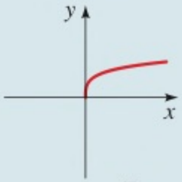
$f(x) = \sqrt[n]{x}$



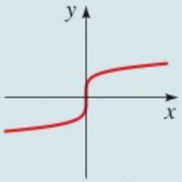
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



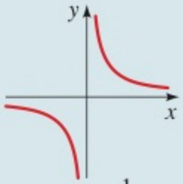
$f(x) = \sqrt[4]{x}$



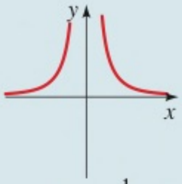
$f(x) = \sqrt[5]{x}$

Reciprocal functions

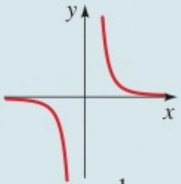
$f(x) = \frac{1}{x^n}$



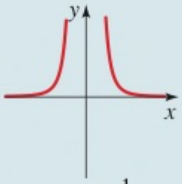
$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$



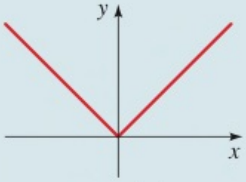
$f(x) = \frac{1}{x^3}$



$f(x) = \frac{1}{x^4}$

Absolute value function

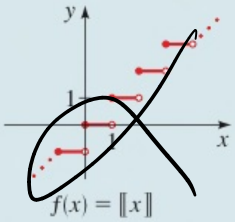
$f(x) = |x|$



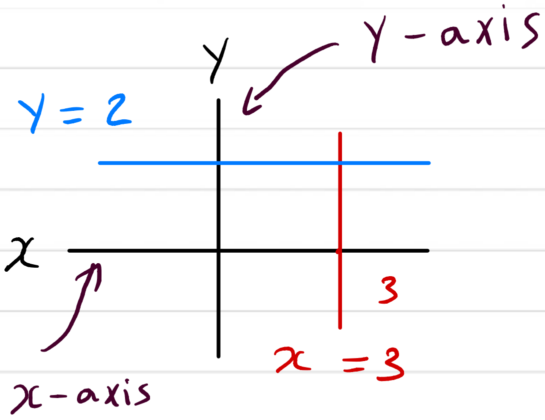
$f(x) = |x|$

Greatest integer function

$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$

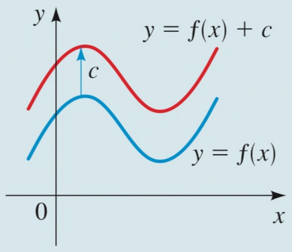


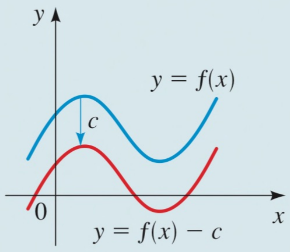
VERTICAL SHIFTS OF GRAPHS

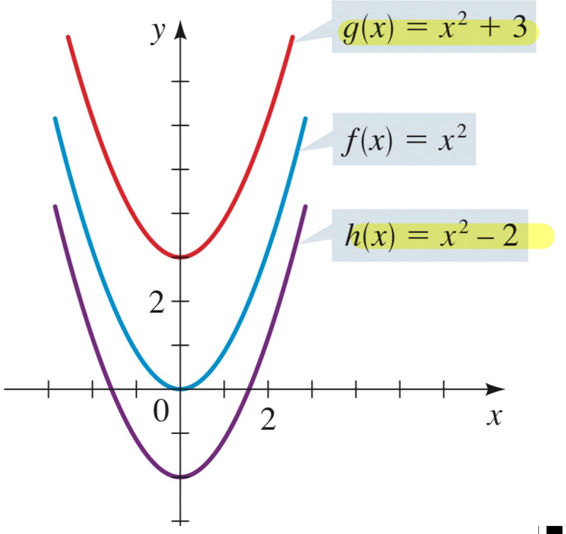
Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.





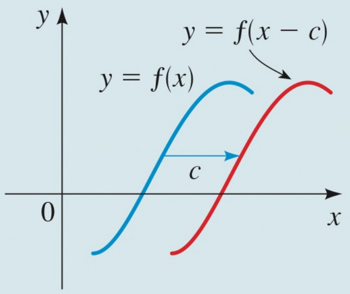


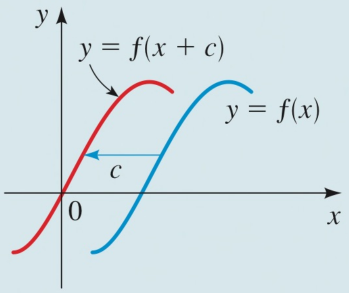
HORIZONTAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.





The graphs of g and h are sketched in Figure 2.

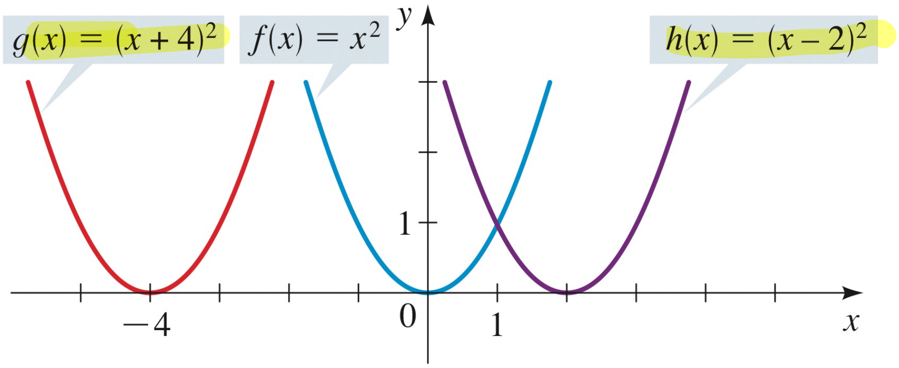
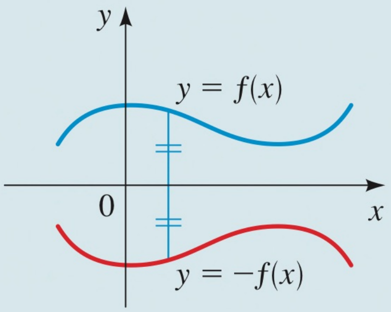
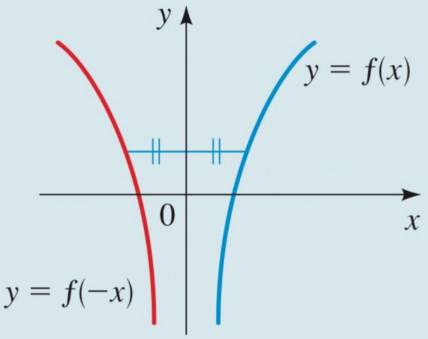


FIGURE 2

REFLECTING GRAPHS

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.





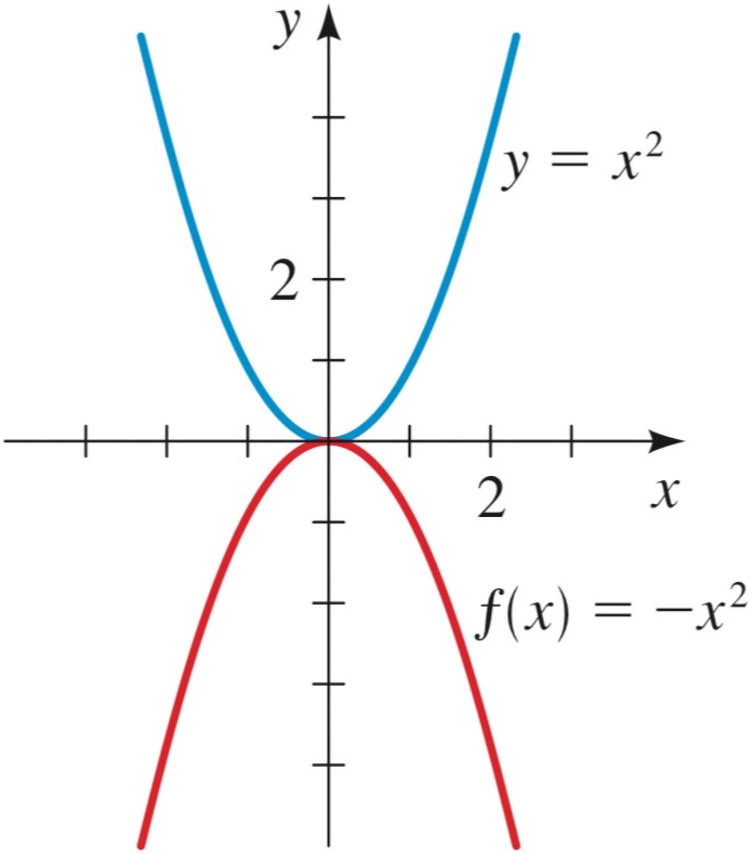


FIGURE 4

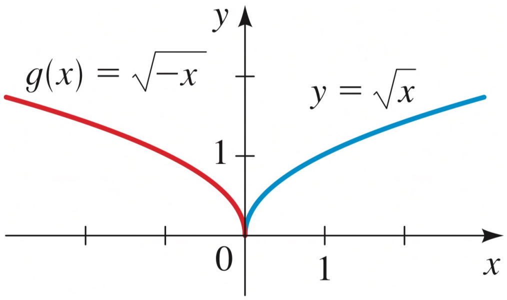


FIGURE 5

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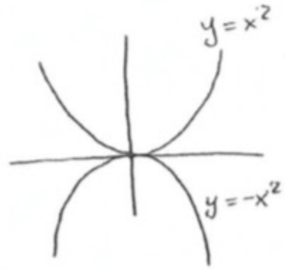
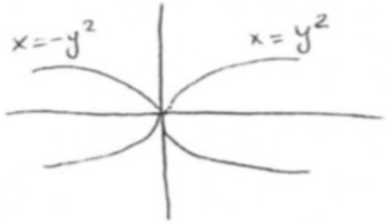
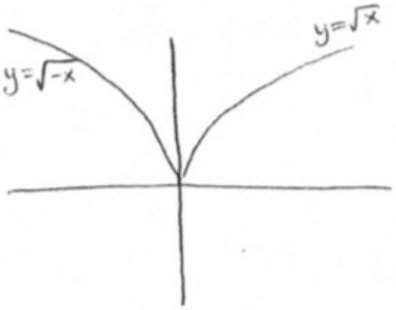
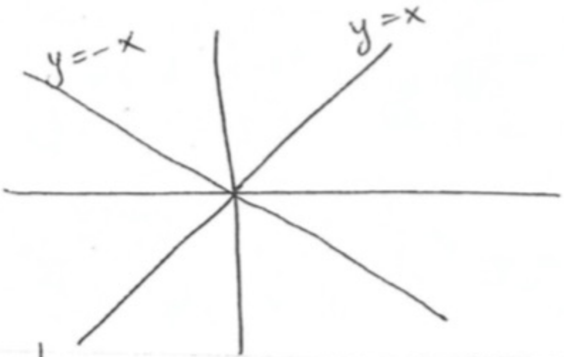
Basic Curves

75

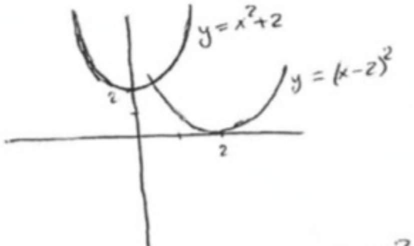


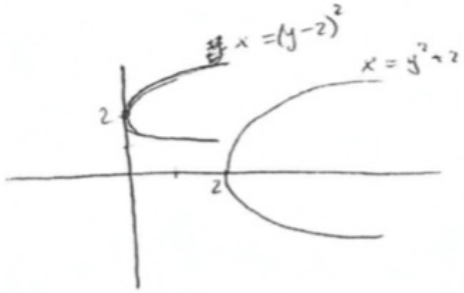


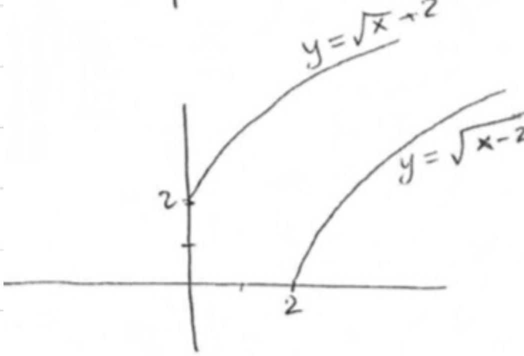


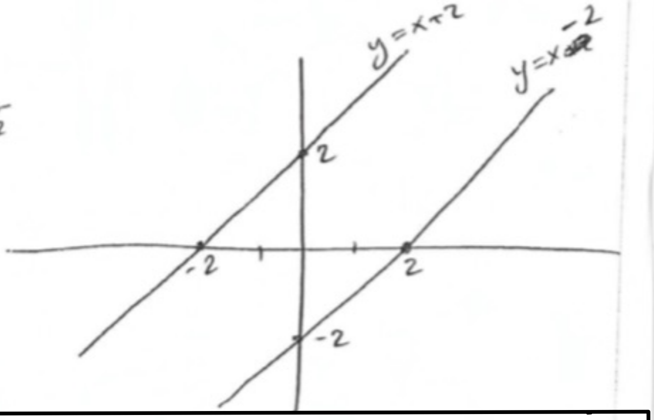


Shift

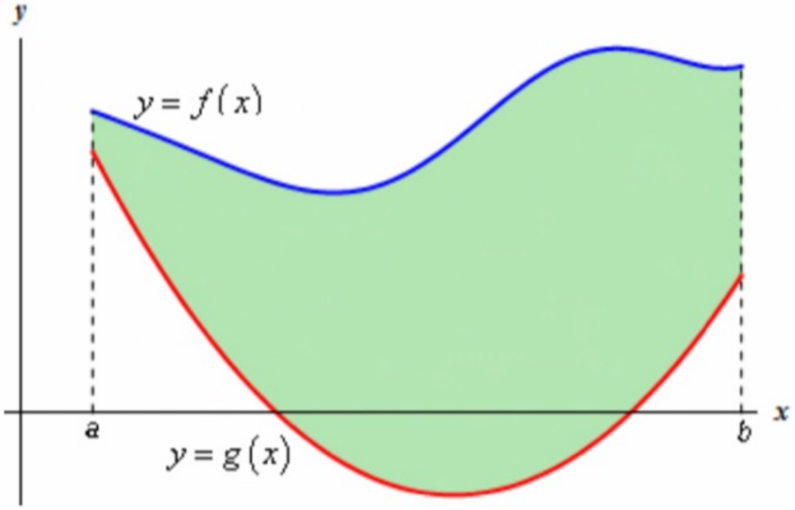








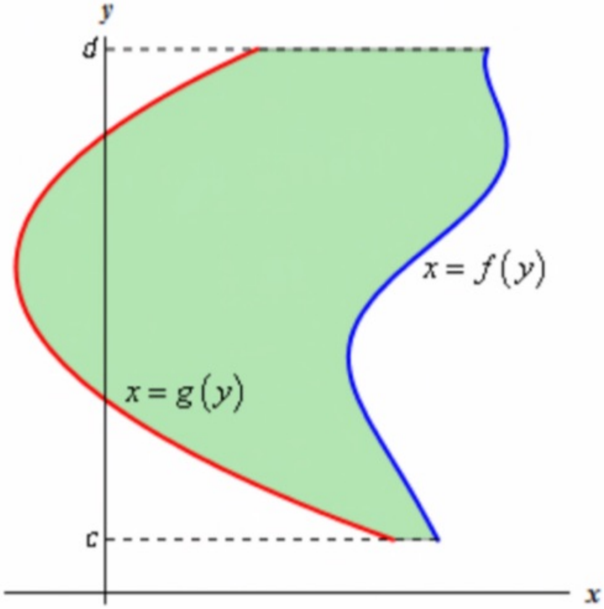
Vertical Shift	$f(x) \pm c$	$+c$ فوق $-c$ تحت
Horizontal Shift	$f(x \pm c)$	$+c$ يسار $-c$ يمين
Reflect x-axis	$-f(x)$	
Reflect y-axis	$f(-x)$	



In the Area and Volume Formulas section of the Extras chapter we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\,dx \qquad (1)$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\,dy \qquad (2)$$

In the first case we will use,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx, \qquad a \leq x \leq b \qquad (3)$$

In the second case we will use,

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \leq y \leq d \qquad (4)$$

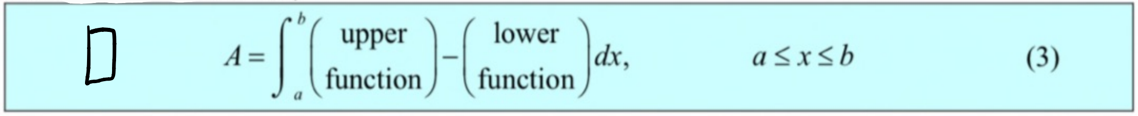
خطوات الحل :

١) رح نطلع نقطة تقاطع الدالتين

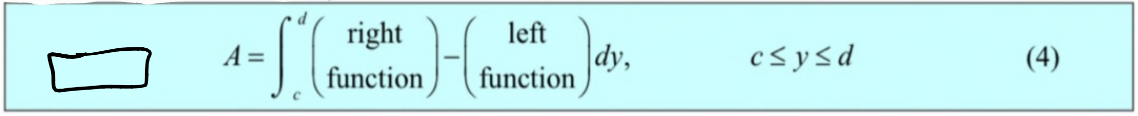
٢) نرسم الدالة عن طريق تعويض(نعوض بقيم التقاطع ونقطة بينهم)

٣) نختار الشريحة الاسهل للتكامل

٤) نطبق قانون مساحة المحصورة



$$A=\int_a^b\left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right)-\left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right)dx, \qquad a\le x\le b \qquad (3)$$



$$A=\int_c^d\left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right)-\left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right)dy, \qquad c\le y\le d \qquad (4)$$

ملاحظات :

الرسم مهم لانه يخليك تعرف حدود التكامل مالتك + تعرف منو الدالة الأكبر

المساحة المحصورة أهيا المساحات الي رح نطلعها في هذا الدرس ، دائما موجبة و دائما رح يكون الشكل مسكر.. يعني بإختصار إذا طلعت المساحة المحصورة سالبة يعني عندك غلط

إذا طلب منك بالامتحان set up the integral يعني ما يبيبك تكامل بس أكتبله التكامل الي تبي تسويه بدون لا تحله

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

6. [10 pts.] Sketch the region enclosed by the curves $f(x) =$ x and $g(x) =$ x^2 and find its area.

Q10. [10 pts.] Find the area of the region enclosed by the curves $y = 4 - x^2$ and $y = 3$.

6. [10 pts.] Let R be the region between the curves $y = x^2$ and $y = \sqrt{x}$, from $x = 0$ to $x = 4$. Sketch the region R, and find its area.

5. [$2 + 8 = 10$ pts.] Let R be the region between the curves $y = x$ and $y = x^2$, from $x = 0$ to $x = 2$. Sketch the region R and find its area.

6. [10 pts.] Sketch the region enclosed by the curves $f(x) = \sqrt{x-1}$ and $g(x) = x - 1$ and find its area.

8. [2 + 8 = 10 pts.] Let R be the region enclosed by the curves $y = x - 2$ and $y = x^2 - 4$.

(a) Sketch the region R.

(b) Find the area of the region R.

6. [10 pts.] Sketch the region enclosed by the curves $f(x) = \sqrt{x-1}$ and $g(x) = x - 1$ and find its area.

EXAMPLE 6 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

EXAMPLE 1 Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.