Calculus A

Chapter 4: Application of Differentiation

Sections: 4.3 How Derivatives Affect the Shape of a Graph

A+

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* Increaing / Decreasing Test:

a) If $f'(x) > 0$ on an interval I, then f is an increasing ($\nearrow$) funtion on I.

b) If $f'(x) < 0$ on an interval I, then f is an decreasing ($\searrow$) funtion on I.

عشان تعرف إذ الدالة قاعدة تتزايد ولا تتناقص بفترة معينة

* The First Derivative Test:

عشان تعرف إذا الدالة لها local max و local min

Suppose that c is a critical number of a continuous function f.

a) If f' changes from $+$ to $-$ at c, then f has a local maximum at c.

b) If f' changes from $-$ to $+$ at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local extrema at c.

* Concavity Test:

عشان تعرف إذا الدالة مقعرة لأعلى أو لأسفل

a) If $f''(x) > 0$ for all x in interval I, then the graph of f is concave upward (CU) on I.

b) If $f''(x) < 0$ for all x in interval I, then the graph of f is concave downward (CD) on I.

* Curve Sketching Guidelines:

1. Domain: find the domain of the function f, denoted by D_f.

كيفك إلا إذا قالك ديبهم

2. intercepts: find y-intercept $(0, f(0))$ and x-intercept $(x, 0)$.
3. Symmetry:

   a) if $f(-x) = f(x)$ for all $x \in D_f$, then f is even function and the curve is symmetric about the y-axis. That is, $(a, b) \mapsto (-a, b)$. Take f to be x^2, x^4, or $cos(x)$ as an example.

   b) if $f(-x) = -f(x)$ for all $x \in D_f$, then f is odd function and the curve is symmetric about the origin. That is, $(a, b) \mapsto (-a, -b)$. Take f to be x, x^3, or $sin(x)$ as an example.

4. Asymptotes: Horizontal asymptotes (H.A.) and Vertical asymptotes (V.A.).
5. Increasing/Decreasing Test: interval on which f is increasing or decreasing.
6. Local extrema: find all max. and min. local extrema.
7. Concavity and Inflection Points: find where the curve of f is CU or CD and find all of the inflection points.
8. Sketch: use all of the previous information to sketch the curve of f.

Definition - Domain of a function:

The domain of a function is the set of all real numbers for which the function is well-defined.

Remark:

The domain may be stated explicitly by giving a specific interval that is a subset of the real domain of the function, for example, if we write

i. The domain of a polynomial is $\mathbb{R}$.

ii. The domain of a rational function $f(x) = \frac{P(x)}{Q(x)}$; where $p(x)$ and $Q(x)$ are polynomials, and $Q(x) \neq 0$; is the set of all real numbers except where $Q(x) = 0$. That is; the domain of a rational function is $\mathbb{R}/\{Q(x) = 0\}$.

iii. The domain of $\sqrt[n]{x}$, where n is an **even** integer, is $x \geq 0$.

iv. The domain of $\sqrt[n]{x}$, where n is an **odd** integer, is $\mathbb{R}$.

V) The domain of $\ln x$ $\log x$ is $(0, \infty)$

Vi) The domain e^x, $\sin x$, $\cos x$ is $\mathbb{R}$

$\mathbb{R} = (-\infty, \infty)$

find the domain

9. $f(x) = x^3 - 3x^2 - 9x + 4$ Domain : $\mathbb{R}$

$$f(x) = \frac{x^3}{x^2+1}.$$

Domain: $x^2 + 1 \neq 0$ impossible

$\therefore = \mathbb{R}$

$$y = \frac{2x^2}{x^2-1}.$$

$x^2 - 1 \neq 0 \Rightarrow x^2 \neq 1$

$x \neq 1 \ , \ x \neq -1$

$\therefore (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$= \mathbb{R} / \{-1, 1\}$

$y = 1/(1 + e^{-x})$ $\Rightarrow 1 + e^{-x} \neq 0 \Rightarrow e^{-x} \neq -1$

ما يصير داخل الln عدد سالب أو صفر لازم يكون
الرقم الي داخل الln أكبر من الصفر

$\ln e^{-x} \neq \ln(-1) \quad \therefore \mathbb{R}$

$y = (1 + e^x)^{-2} = \dfrac{1}{(1+e^x)^2}$

$\therefore 1 + e^x \neq 0$

$e^x \neq -1$

$\therefore \mathbb{R}$

find the domain

$f(x) = \dfrac{e^x}{1 - e^{x-1}}.$ $\Rightarrow 1 - e^{x-1} \neq 0$

$\Rightarrow e^{x-1} \neq 1$

$\ln e^{x-1} \neq \ln 1 \quad \Rightarrow x - 1 \neq 0 \quad \Rightarrow x \neq 1$

$\therefore (-\infty, 1) \cup (1, \infty) = \mathbb{R}/\{1\}$

$y = \ln(4 - x^2).$ $4 - x^2 > 0$

$4 > x^2 \quad \Rightarrow \quad 2 > |x|$

$\therefore 2 > x \qquad 2 > -x$

$(-\infty, 2) \qquad -2 < x$

$(-2, \infty)$

$\therefore$ Domain $= (-2, 2)$

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if $\ln(x^2 - 4)$ then $x^2 - 4 > 0 \quad x^2 > 4$

$|x| > 2 \qquad \therefore x > 2$ or $-x > 2$

$x < -2$

$(2, \infty) \qquad (-\infty, -2)$

$\therefore$ The domain $(-\infty, -2) \cup (2, \infty)$

find the domain

$$f(x) = \frac{x^2+2x+1}{x^2-4x+4} = \frac{(x+1)(x+1)}{(x-2)(x-2)}$$

$$= \frac{(x+1)^2}{(x-2)^2} = \left(\frac{x+1}{x-2}\right)^2$$

Domain :- $x-2 \neq 0 \quad \therefore x \neq 2$

$\therefore (-\infty, 2) \cup (2, \infty) = \mathbb{R} / \{2\}$

$f(x) = \ln(x^2+1).$

$$x^2+1>0$$

$$x^2 > -1 \quad \text{always}$$

$$\therefore \mathbb{R}$$

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

مو شرط كل دالة لها intercept مع axis ممكن تتقاطع وممكن لا

ex:

$$f(x) = \frac{x^2+2x+3}{x-2}$$

$$y = \frac{(x-1)(x+3)}{(x-2)}$$

for x-int "$y = 0$"

$$0 = \frac{(x-1)(x+3)}{(x-2)}$$

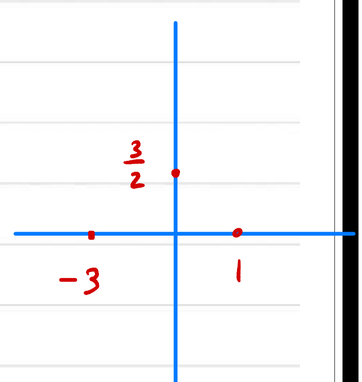
$\therefore (x-1)(x+3) = 0$

$\therefore x = 1 , x = -3$

$(1, 0) , (-3, 0)$

for y-int "$x = 0$"

$$y = \frac{(0-1)(0+3)}{0-2} = \frac{-3}{-2} = \frac{3}{2}$$



x-int $= -3 , 1$, y-int $= \frac{3}{2}$

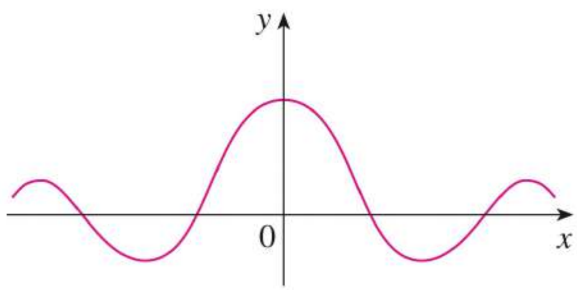
C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

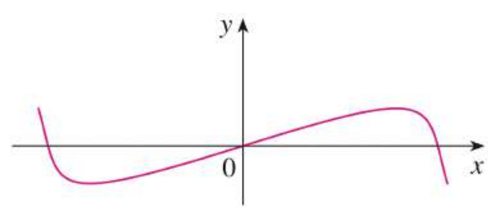
if $f(-x) = f(x)$

Then symmetric about y-axis



(a) Even function: reflectional symmetry

if $f(-x) = -f(x)$, Then it's symmetric about the origin



(b) Odd function: rotational symmetry

If neither of the above, then there is no symmetry.

$$f(x) = \frac{x^3+3}{x^2-4x}$$

$$f(-x) = \frac{(-x)^3+3}{(-x)^2-4(-x)}$$

$$= \frac{-x^3+3}{x^2+4x}$$

Examples (Even Functions):	**_Examples (Odd Functions):_**
$x^2 - 2$	$x^3 - x$
5	$\sqrt[5]{x}$
$x^2\lvert x\rvert$	$x^3\lvert x\rvert$
$\dfrac{x^4+1}{3x^8}$	$\dfrac{x^2+5}{x^3+2x}$
$\dfrac{x^3-2x}{x^5}$	$\dfrac{x^3-x^9}{x^4}$

Example: **Determine whether each function is Even, Odd or Neither.**

1. $f(x) = x^5 + x$

$(-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$ odd

2. $f(x) = 1 - x^4$

$1 - (-x)^4 = 1 - x^4 = f(x)$ even

3. $f(x) = 2x - x^2$

$2(-x) - (-x)^2 = -2x - x^2 = -(2x + x^2)$ Neither

4. $f(x) = |x| + 2$

$|-x| + 2 = |x| + 2 = f(x)$ even

5. $f(x) = 3$

$f(x) = 3 = f(-x)$ even

6. $f(x) = \frac{x}{x^2 + x^6}$

$$\frac{-x}{(-x)^2 + (-x)^6} = \frac{-x}{x^2 + x^6} = -f(x)$$ odd

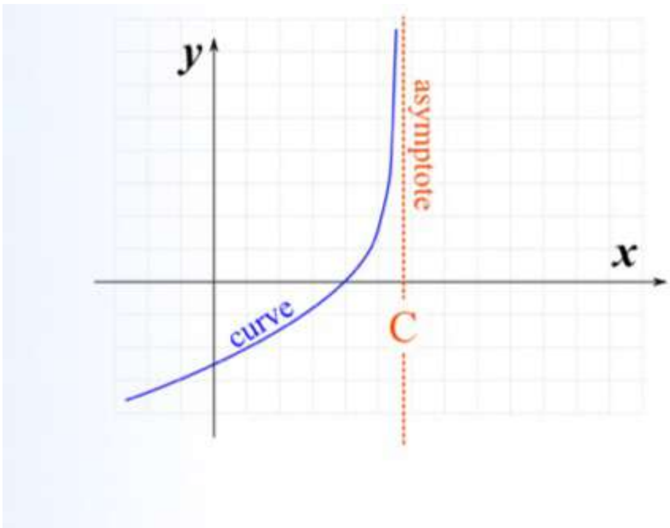
7. $f(x) = \frac{x^2}{x^4 + x}$

$$\frac{(-x^2)}{(-x)^4 + (-x)} = \frac{x^2}{x^4 - x}$$ Neither

D. Asymptotes

Vertical Asymptotes

شلون نطلع V.A



١) نطلع الارقام الي تخلي الدالة غير معرفة مثلا أصفار المقام

٢) كل رقم أطلعه ! لازم تدرس الlimit من اليمين أو اليسار

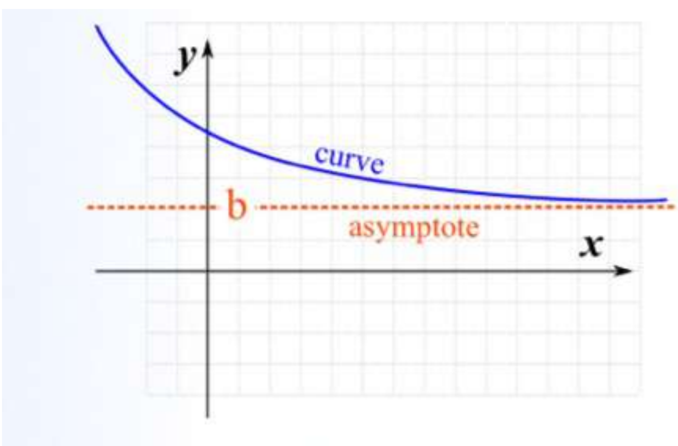
٣) إذا طلع لك الجواب $\pm\infty$ يعني العدد الي طلعته يعتبر V.A

It is a Vertical Asymptote when:

as x approaches some constant value **c** (from the left or right) then the curve goes towards infinity (or −infinity).

Horizontal Asymptotes

شلون نطلع H.A



١) نطلع $\lim_{x\to\infty} f(x)$

إذا طلع الlimit له قيمة ، إذا هذي القيمة أهيا H.A

٢) نطلع $\lim_{x\to-\infty} f(x)$

إذا طلع الlimit له قيمة ، إذا هذي القيمة أهيا H.A

It is a Horizontal Asymptote when:

as x goes to infinity (or −infinity) the curve approaches some constant value **b**

D. Asymptotes

طريقة الحل أن نقسم على أكبر أس بالمقام

find H.A $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

a) $f(x) = \dfrac{3x-1}{2x+5}$

درجة الأس البسط تساوي درجة
الأس في المقام

$$\lim_{x\to+\infty} \frac{3x-1}{2x+5} = \frac{\frac{3x}{x} - \frac{1}{x}}{\frac{2x}{x} + \frac{5}{x}}$$

$$\lim_{x\to+\infty} \frac{3 - \frac{1}{x}}{2 + \frac{5}{x}} = \frac{3 - \frac{1}{+\infty}}{2 + \frac{5}{+\infty}}$$

$$= \frac{3-0}{2-0} = \frac{3}{2}$$

$$\lim_{x\to-\infty} \frac{3-0}{2-0} = \frac{3}{2}$$

$y = \frac{3}{2}$ is H.A

b) $f(x)=\dfrac{3x^2+2x}{4x^3-5x+7}$

درجة المقام أكبر من البسط

$$\lim_{x\to\pm\infty}\frac{\frac{3x^2}{x^3}+\frac{2x}{x^3}}{\frac{4x^3}{x^3}-\frac{5x}{x^3}+\frac{7}{x^3}}$$

$$=\lim_{x\to\pm\infty}\frac{\frac{3}{x}+\frac{2}{x^2}}{4-\frac{5x}{x^3}+\frac{7}{x^3}}=\frac{0+0}{4-0+0}=0$$

$y=0$ is H.A

c) $f(x)=\dfrac{3x^2+4x}{x+2}$

درجة البسط أكبر من المقام

$$\lim_{x\to\pm\infty}f(x)=\frac{\frac{3x^2}{x}+\frac{4x}{x}}{\frac{x}{x}+\frac{2}{x}}=\lim_{x\to\pm\infty}\frac{3x+4}{1+\frac{2}{x}}$$

$$=\frac{\pm\infty+4}{1+\frac{2}{\pm\infty}}=\frac{\pm\infty+4}{1+0}=\pm\infty$$

No H.A

$$\lim_{x \to \infty} \frac{e^x + 300}{10^x + 3^x}$$

$$\lim_{x \to \infty} \frac{\frac{e^x}{10^x} + \frac{300}{10^x}}{\frac{10^x}{10^x} + \frac{3^x}{10^x}}$$

$$\lim_{x \to \infty} \frac{\left(\frac{e}{10}\right)^x + \frac{300}{10^x}}{1 + \left(\frac{3}{10}\right)^x} = \frac{\left(\frac{e}{10}\right)^\infty + \frac{300}{10^\infty}}{1 + \left(\frac{3}{10}\right)^\infty}$$

$$= \frac{0 + 0}{1 + 0} = 0$$

$y = 0$ is H.A

4) limit infinity for functions involving a radical

evalute $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$

$$f(x) = \frac{3x-2}{\sqrt{4x^2+5}}$$

$$\lim_{x \to \pm\infty} \frac{3x-2}{\sqrt{x^2\left(\frac{4x^2}{x^2} + \frac{5}{x^2}\right)}} = \frac{3x-2}{|x|\sqrt{4+\frac{5}{x^2}}}$$

$$\left.\begin{array}{l} \text{for } x \to \infty, \; |x| = x \\ \text{for } x \to -\infty, \; |x| = -x \end{array}\right\}$$

$$\lim_{x \to \pm\infty} \frac{x\left(\frac{3x}{x} - \frac{2}{x}\right)}{\pm x\sqrt{4+\frac{5}{x^2}}} = \lim_{x \to \pm\infty} \pm \frac{3 - \frac{2}{x}}{\sqrt{4 + \frac{5}{x^2}}}$$

$$= \pm \frac{3-0}{\sqrt{4+0}} = \pm \frac{3}{\sqrt{4}} = \pm \frac{3}{2}$$

$\therefore$ $y = \pm \frac{2}{3}$ are H.A

What Does f' Say About f?

E. Intervals of Increase or Decrease

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

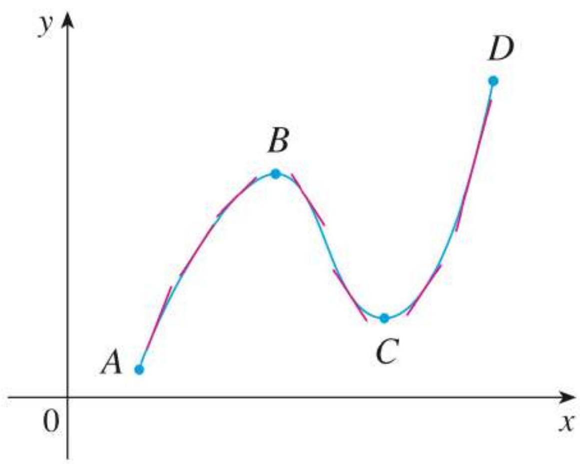


FIGURE 1

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

$c \in$ Domain

if not, we will not take it

Local Extreme Values

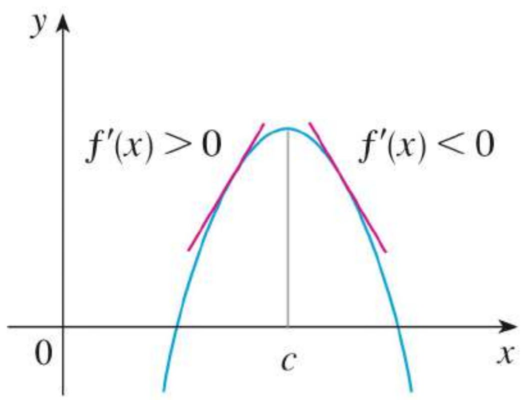
F. Local Maximum and Minimum Values

The First Derivative Test Suppose that c is a critical number of a continuous function f.

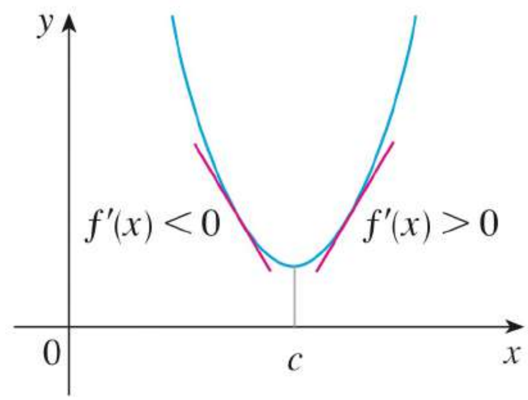
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

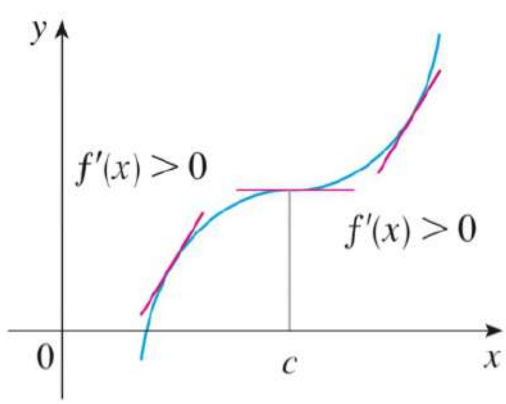
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



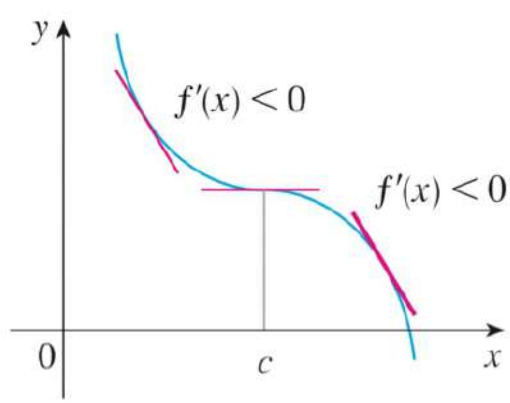
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum



(d) No maximum or minimum

تأكد أن قيمة الc تنتمي للdomain

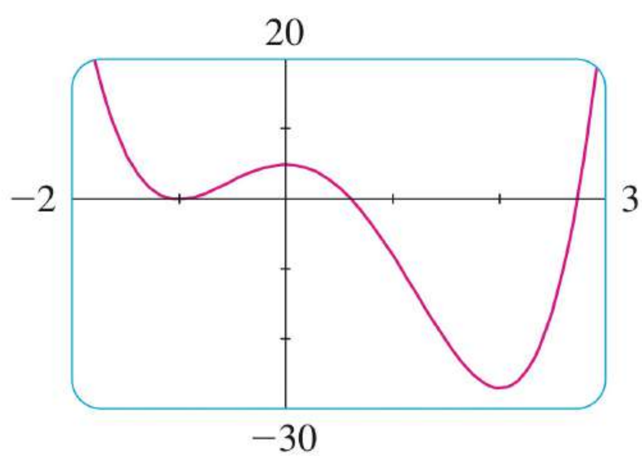
"C" should $\in$ Domain of the function $f(x)$

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

SOLUTION We start by differentiating f: Domain of f is $\mathbb{R}$

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$



$-\infty$ $\quad -2 \quad -1 \quad -1/2 \quad 0 \quad 1 \quad 2 \quad 100 \quad \infty$

f'	$-$	$+$	$-$	$+$
$\nearrow \searrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

f is increasing on $(-1, 0) \cup (2, \infty)$

f is decreasing $(-\infty, -1) \cup (0, 2)$

EXAMPLE 2 Find the local minimum and maximum values of the function f in Example 1.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

$-\infty$ $\quad -2 \quad -1 \quad -1/2 \quad 0 \quad 1 \quad 2 \quad 100 \quad \infty$

f'	$-$	$+$	$-$	$+$
↗ ↘	↘	↗	↘	↗

$x = -1 \in D_f$ "min", $x = 0 \in D_f$ "max" $x = 2 \in D_f$ "min"

$$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5$$

$$= 3 + 4 - 12 + 5 = 0$$

$$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 5$$

$$= 48 - 32 - 48 + 5 = -27$$

$$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 5 = 5$$

$\therefore$ f has Local minimum at $(-1, 0)$ and $(2, -27)$

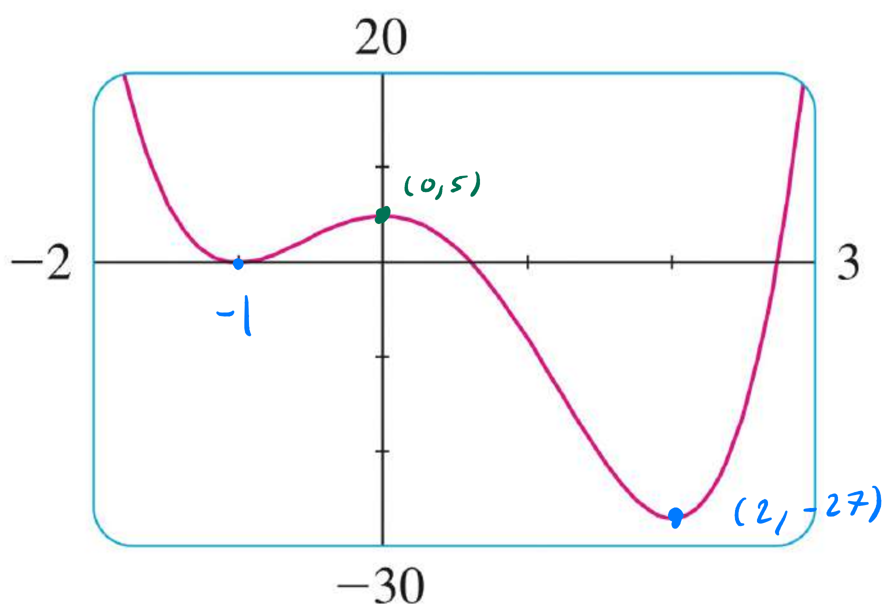
f has Local maximum at $(0, 5)$

f is increasing on $(-1,0) \cup (2,\infty)$

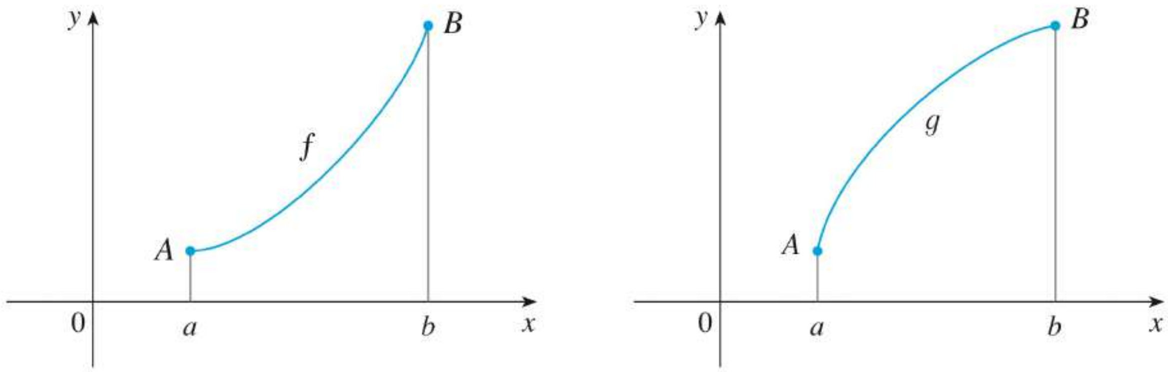
f is decreasing on $(-\infty,-1) \cup (0,2)$

$\therefore$ f has Local minimum at $(-1,0)$ and $(2,-27)$

$\therefore$ f has Local maximum at $(0,5)$



What Does f'' Say About f?



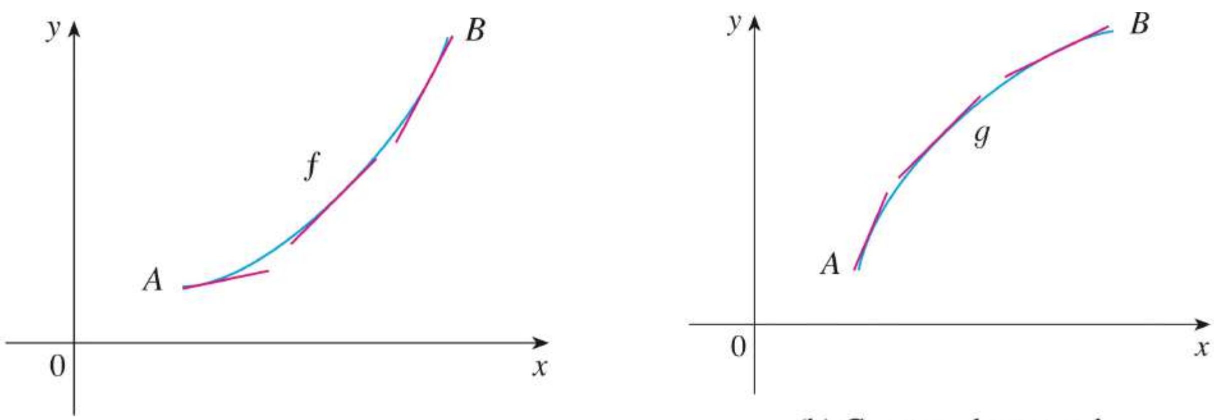
(a) (b)

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

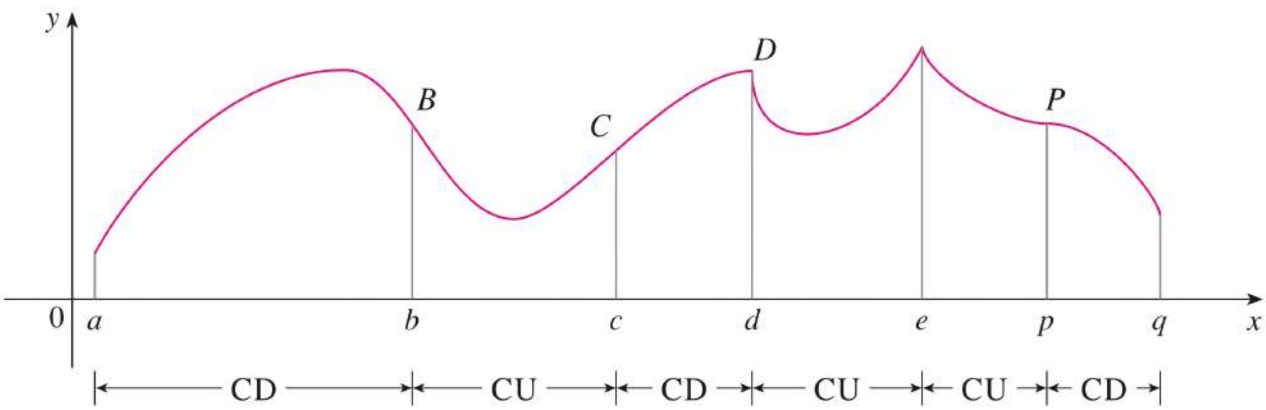
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.



(a) Concave upward (b) Concave downward

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).



(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

9. $f(x) = x^3 - 3x^2 - 9x + 4$

$$f(x) = x^3 - 3x^2 - 9x + 4$$

1) Domain of $f(x)$ is $\mathbb{R}$ (poly)

2) $f'(x) = 3x^2 - 6x - 9$

3) critical numbers $\to f'(x) = 0$; $\to f'(x) = DNE$

$f'(x) = 0 \Rightarrow 3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 3(x-3)(x+1) = 0$

$x = 3$, $x = -1$ are critical number

4) interval

$(-\infty, -1)$, $(-1, 3)$, $(3, \infty)$

4) interval f'

$(-\infty, -1)$, $(-1, 3)$, $(3, \infty)$

	$-\infty$ to -1	-1 to 3	3 to ∞
f'	$+$	$-$	$+$
↗ ↘	↗	↘	↗

f is increasing on the interval $(-\infty, -1) \cup (3, \infty)$

f is Decreasing on the interval $(-1, 3)$

for Local maximum:-

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 4$$

$$= -1 - 3 + 9 + 4 = 9$$

for local minimum

$$f(3) = (3)^3 - 3(3)^2 - 9(3) + 4$$

$$= 27 - 27 - 27 + 4 = -23$$

$\therefore$ f has local maximum on $(-1, 9)$

f has local minimum on $(3, -23)$

$f''(x) = 6x - 6$

$\longrightarrow f''(x) = DNE$

5) inflection point $\longrightarrow f''(x) = 0$

$\therefore 6x - 6 = 0 \Rightarrow 6x = 6$

$x = 1$

6) intervals for f''

$(-\infty, 1) \cup (1, \infty)$

	$-\infty$	1	∞
f''	$-$		$+$
$\cup \cap$	$\cap$		$\cup$

f has concave upward $(1, \infty)$

f has concave downward $(-\infty, 1)$

$f(1) = (1)^3 - 3(1)^2 - 9(1) + 4$

$= 1 - 3 - 9 + 4 = -7 \in$ Domain of f

The point $(1, -7)$ is inflection point

Calculus A

Chapter 4: Application of Differentiation

Sections: 4.5 Summary of Curve Sketching

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* Increaing / Decreasing Test:

a) If $f'(x) > 0$ on an interval I, then f is an increasing ($\nearrow$) funtion on I.

b) If $f'(x) < 0$ on an interval I, then f is an decreasing ($\searrow$) funtion on I.

* The First Derivative Test:

Suppose that c is a critical number of a continuous function f.

a) If f' changes from + to − at c, then f has a local maximum at c.

b) If f' changes from − to + at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local extrema at c.

* Concavity Test:

a) If $f''(x) > 0$ for all x in interval I, then the graph of f is concave upward (CU) on I.

b) If $f''(x) < 0$ for all x in interval I, then the graph of f is concave downward (CD) on I.

* The Second Derivative Test:

Suppose that f'' is continuous near a number c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

* Curve Sketching Guidelines:

1. Domain: find the domain of the function f, denoted by D_f.
2. intercepts: find y-intercept $(0, f(0))$ and x-intercept $(x, 0)$.
3. Symmetry:

   a) if $f(-x) = f(x)$ for all $x \in D_f$, then f is even function and the curve is symmetric about the y-axis. That is, $(a, b) \mapsto (-a, b)$. Take f to be x^2, x^4, or $cos(x)$ as an example.

   b) if $f(-x) = -f(x)$ for all $x \in D_f$, then f is odd function and the curve is symmetric about the origin. That is, $(a, b) \mapsto (-a, -b)$. Take f to be x, x^3, or $sin(x)$ as an example.

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4. Asymptotes: Horizontal asymptotes (H.A.) and Vertical asymptotes (V.A.).
5. Increasing/Decreasing Test: interval on which f is increasing or decreasing.
6. Local extrema: find all max. and min. local extrema.
7. Concavity and Inflection Points: find where the curve of f is CU or CD and find all of the inflection points.
8. Sketch: use all of the previous information to sketch the curve of f.

find the domain first

Graphing Rational Functions

A. Domain

A. Test to see if the graph has symmetry by plugging in (-x) in the function.

Options:
If the signs all stay the same or all change, f(-x) = f(x), then you have **even or y-axis symmetry**.

$$f(x) = \frac{x^2+3}{x^2-2}$$

$$f(-x) = \frac{(-x)^2+3}{(-x)^2-2}$$

$$f(-x) = \frac{x^2+3}{x^2-2}$$

If either the numerator or the denominator changes signs completely, f(-x)= -f(x) then you have **odd, or origin symmetry**.

$$f(x) = \frac{x^2}{x^3-x}$$

$$f(-x) = \frac{(-x)^2}{(-x)^3-(-x)}$$

$$= \frac{x^2}{-x^3+x}$$

$$= -\left(\frac{x^2}{x^3-x}\right)$$

If neither of the above, then there is no symmetry.

$$f(x) = \frac{x^3+3}{x^2-4x}$$

$$f(-x) = \frac{(-x)^3+3}{(-x)^2-4(-x)}$$

$$= \frac{-x^3+3}{x^2+4x}$$

B. Test to find y-intercepts by replacing x with 0.

$$f(x) = \frac{3x^2+8}{x^3-2}$$

$$f(0) = \frac{3(0)^2+8}{(0)^3-2}$$

$$= \frac{8}{-2} = -4$$

y-int. $(0, -4)$

C. Test to find x-intercepts by setting the numerator equal to 0.
Shortcut: Since multiplying by the denominator will eliminate it, you can just set the numerator equal to zero.

$$f(x) = \frac{2x}{x+1}$$

$$0 = 2x \rightarrow 0 = x$$

x-int. $(0, 0)$

D. Find the vertical asymptote by setting the denominator equal to zero.
The result is the equation of the vertical asymptote. Keep an eye out for holes. Holes occur when there is a factor that is the same both in the numerator and in the denominator. IT IS NOT A VERTICAL ASYMPTOTE because it simplifies away.

$$f(x) = \frac{x^2-9}{(x-3)(x+5)}$$

$$= \frac{(x-3)(x+3)}{(x-3)(x+5)}$$

$$x - 3 = 0, \qquad x = 3$$

Therefore at x=3 there is a hole.

$$x + 5 = 0 \; x = -5$$

Therefore at x=-5 there is a vertical asymptote.

E. Find the horizontal asymptote.
To find the horizontal asymptote you compare the degrees of the numerator and the denominator. Note: horizontal asymptotes can be crossed while vertical asymptotes can never be crossed.

Options:
If the degrees are the same, you take the ratio of the leading coefficients and that is your asymptote.

$$f(x) = \frac{5x^2}{x^2 + 7}$$

$$\text{ratio} = \frac{5}{1} \text{ so the asymptote is at } y = 5$$

If the degree of the numerator is less than the degree of the denominator, y=0 is your asymptote.

$$f(x) = \frac{4x - 9}{x^2 + 3}$$

$$1 < 2 \text{ so the asymptote is at } y = 0$$

If the degree of the numerator is exactly one greater than the degree of the denominator, there is no horizontal asymptote but there is a slant asymptote.

$$f(x) = \frac{x^2 + 2}{x}$$

$$2 > 1 \text{ so there is no horizontal asymptote}$$

G. Find points.
Plot at least one point between and beyond each x-intercept and vertical asymptote. Plug a number in for x and solve for y. You want to do this so you can find which side of the asymptotes you are going to graph on.

H. Graph using the information you have found.

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} = f(x)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.
$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

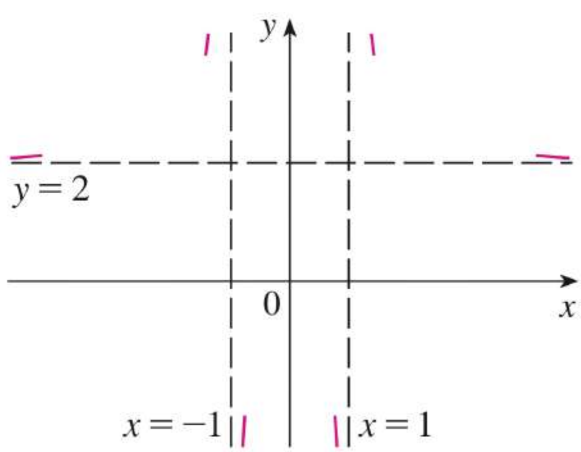
Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

E.
$$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$



$$f(x) = \frac{2x^2}{x^2-1} \quad , \quad f'(x) = \frac{-4x}{(x^2-1)^2}$$

* critical number $f'(x) \begin{cases} \to = 0 \\ \to = DNE \end{cases}$

$$f'(x) = 0 \Rightarrow -4x = 0$$

$$x = 0 \in \text{Domain}$$

$$f'(x) = DNE \Rightarrow x^2 - 1 = 0 \Rightarrow x = -1 , x = 1$$

But $x = 1, x = -1 \notin D_f$

	$-\infty$		-1		0		1		∞
f'		+		+		−		−	
↗ ↘		↗		↗	•	↘		↘	

f is increasing on $(-\infty, -1)$ and $(-1, 0)$

f is decreasing on $(0, 1)$ and $(1, \infty)$

$$f(0) = \frac{2(0)^2}{0^2 + 1} = 0$$

$\therefore$ local max at $(0, 0)$

* concavity & inf points

$$f''(x) = \frac{12x^2+4}{(x^2-1)^3}$$

$$f''(x) \begin{cases} \to = 0 \\ \to = DNE \end{cases}$$

$f''(x) = 0 \Rightarrow 12x^2 + 4 = 0$ impossible

$f''(x) = DNE \rightarrow x^2 - 1 = 0$

$x = \pm 1 \notin D_f$

	$-\infty$ to -1	-1 to 1	1 to ∞
f''	+	−	+
$\cup \cap$	$\cup$	$\cap$	$\cup$

Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

Therefore the line $y = 2$ is a horizontal asymptote.

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2$$

$x = 1$ and $x = -1$ are vertical asymptotes.

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty \qquad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty \qquad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = \infty$$

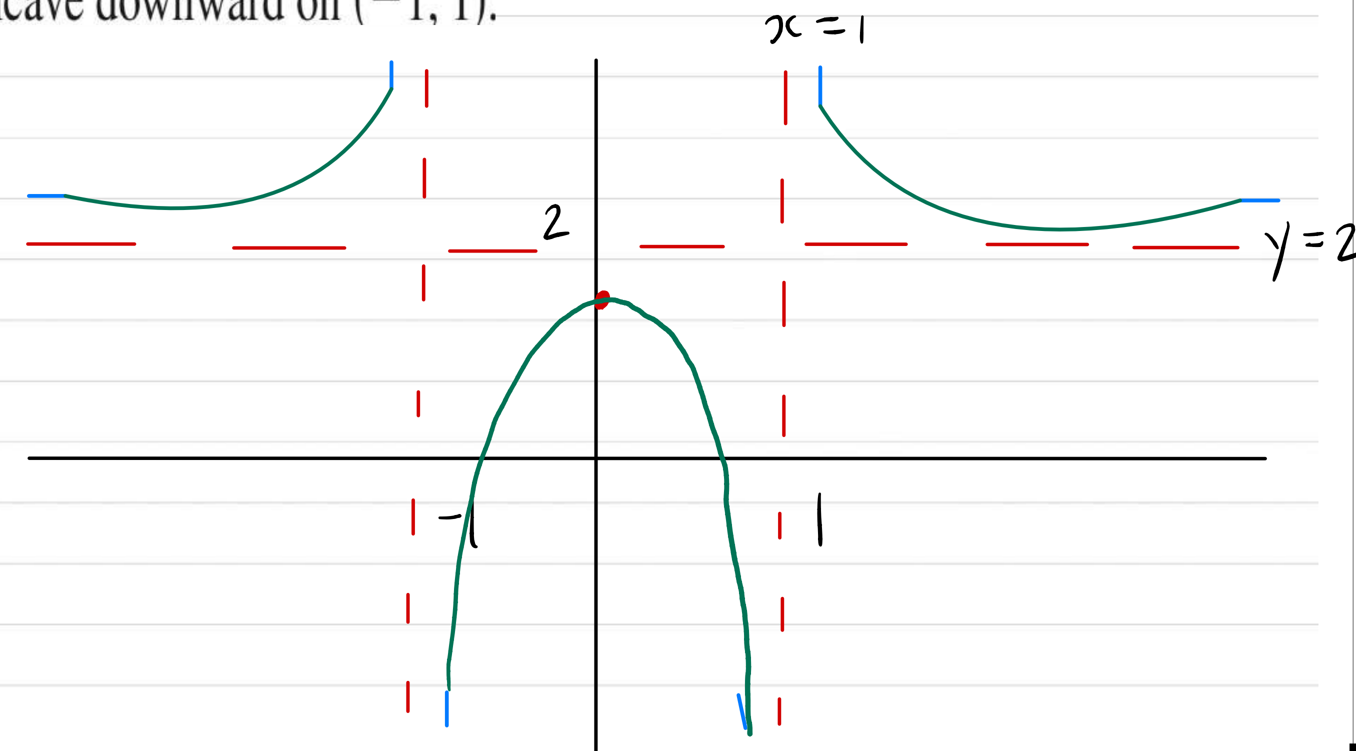
f is increasing on $(-\infty,-1)$ and $(-1,0)$

f is decreasing on $(0,1)$ and $(1,\infty)$

$\therefore$ local max at $(0,0)$

concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$

concave downward on $(-1, 1)$.



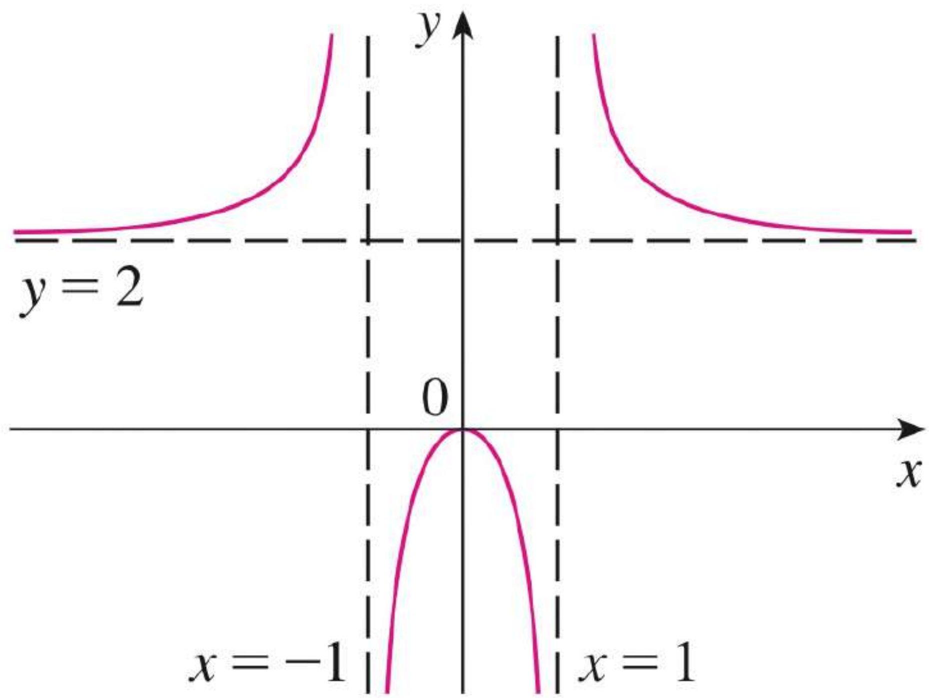


FIGURE 6
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

6. [$4 \times 10 = 40$ pts.] Let $f(x) = \dfrac{x^2 - x}{(x+1)^2}$.

(a) Find the vertical and horizontal asymptotes of the graph of f, if any.

(b) Given that $f'(x) = \dfrac{3x-1}{(x+1)^3}$.

i. Find the intervals on which f is increasing and the intervals on which f is decreasing.

ii. Find the local maximum and minimum values of f, if any.

(c) Given that $f''(x) = \dfrac{6(1-x)}{(x+1)^4}$.

i. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

ii. Find the points of inflection, if any.

(d) Sketch the graph of f.

---

6. [$4 \times 10 = 40$ pts.] Let $f(x) = \dfrac{x^2 - x}{(x+1)^2}$. The domain of f is: $\boxed{D_f = (-\infty, -1) \cup (-1, \infty)}$.

(a) We have: $\boxed{\lim_{x \to \pm\infty} \dfrac{x^2 - x}{(x+1)^2} = 1}$. Therefore, $y = 1$ is a horizontal asymptote of the graph of f.

Furthermore, $\boxed{\lim_{x \to -1} \dfrac{x^2 - x}{(x+1)^2} = \infty}$ and hence $x = -1$ is a vertical asymptote of the graph of f.

$f'(x) = \begin{cases} \to 0 \\ \to \text{DNE} \end{cases}$ عشان أطلع الفترات

(b) Since $f'(x) = \dfrac{3x-1}{(x+1)^3}$, we have: $f'(x) = 0$ iff $x = 1/3$.

i. Moreover,

x	$-\infty$		-1		$1/3$		∞
f'		$+$	$\|$	$-$	0	$+$	
f		$\nearrow$	$\|$	$\searrow$	$-1/8$	$\nearrow$	

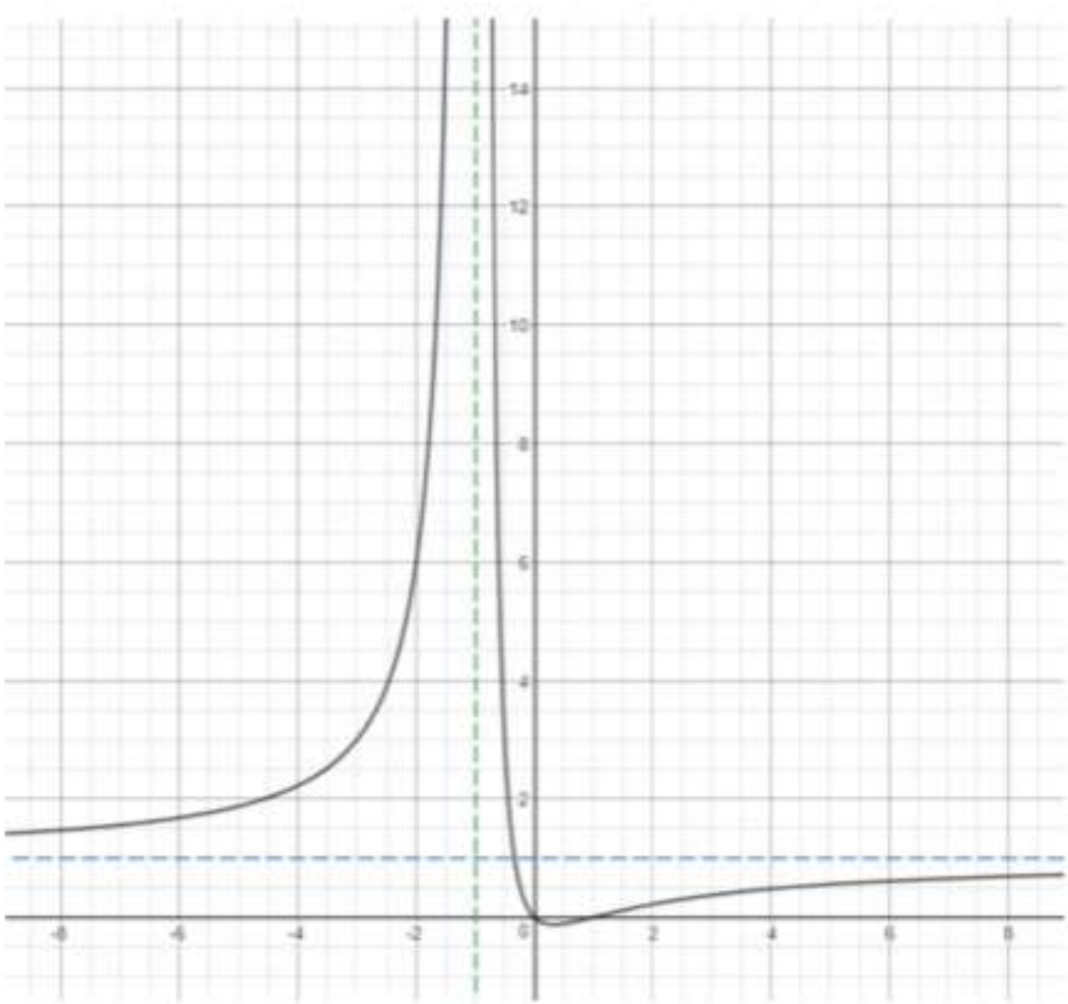
f is increasing in the interval $(-\infty, -1) \cup (1/3, \infty)$.
f is decreasing in the interval $(-1, 1/3)$.

ii. The only critical number of f is $\boxed{x = 1/3}$. Since f' changes sign from negative to positive at $1/3$, we conclude that $\boxed{f(1/3) = -1/8 \text{ is a local minimum value by the FDT}}$. In fact, $f(1/3) = -1/8$ is the absolute minimum value of f.

(c) Since $f''(x) = \dfrac{6(1-x)}{(x+1)^4}$, we have: $f''(x) = 0$ iff $x = 1$.

i. Moreover, $\boxed{f''(x) > 0 \text{ iff } x \in (-\infty, -1) \cup (-1, 1), \text{ where the concavity is upward}}$. In turn, $\boxed{f''(x) < 0 \text{ iff } x \in (1, \infty), \text{ where the concavity is downward}}$.

ii. As the concavity changes at $x = 1$ and f is continuous on its domain, we conclude that $\boxed{(1, 0) \text{ is an inflection point}}$.



5. [10 × 4 = 40 pts.] Let $f(x) = \ln(x^2+1)$.

(a) i) Study the symmetry of the graph of f.

ii) Find the horizontal asymptotes of the graph of f, if any.

(b) Given that $f'(x) = \dfrac{2x}{x^2+1}$.

i) Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

ii) Find the local maximum and minimum values of f, if any.

(c) Given that $f''(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

i) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, if any.

ii) Find the points of inflection, if any.

(d) Sketch the graph of f.

---

5. [10 × 4 = 40 pts.] Let $f(x) = \ln(x^2+1)$.

$\boxed{D_f = (-\infty, \infty)}$.

(a) i) Since $\boxed{f(-x) = \ln((-x)^2+1) = f(x)}$ for all $x \in D_f$, the graph of f is symmetric about the $y-$axis.

ii) Since $\boxed{\lim\limits_{x\to\pm\infty} \ln(x^2+1) = \infty}$, $\boxed{\text{The graph of } f \text{ has no horizontal asymptotes}}$.

(a) Given that $f'(x) = \dfrac{2x}{x^2+1}$.

i) We have $\boxed{f'(x) = 0 \text{ when } x = 0}$. Since $\boxed{f'(x) < 0 \text{ when } x < 0 \text{ and } f'(x) > 0 \text{ when } x > 0}$, therefore, f is decreasing on $\boxed{(-\infty, 0)}$ and increasing on $\boxed{(0, \infty)}$.

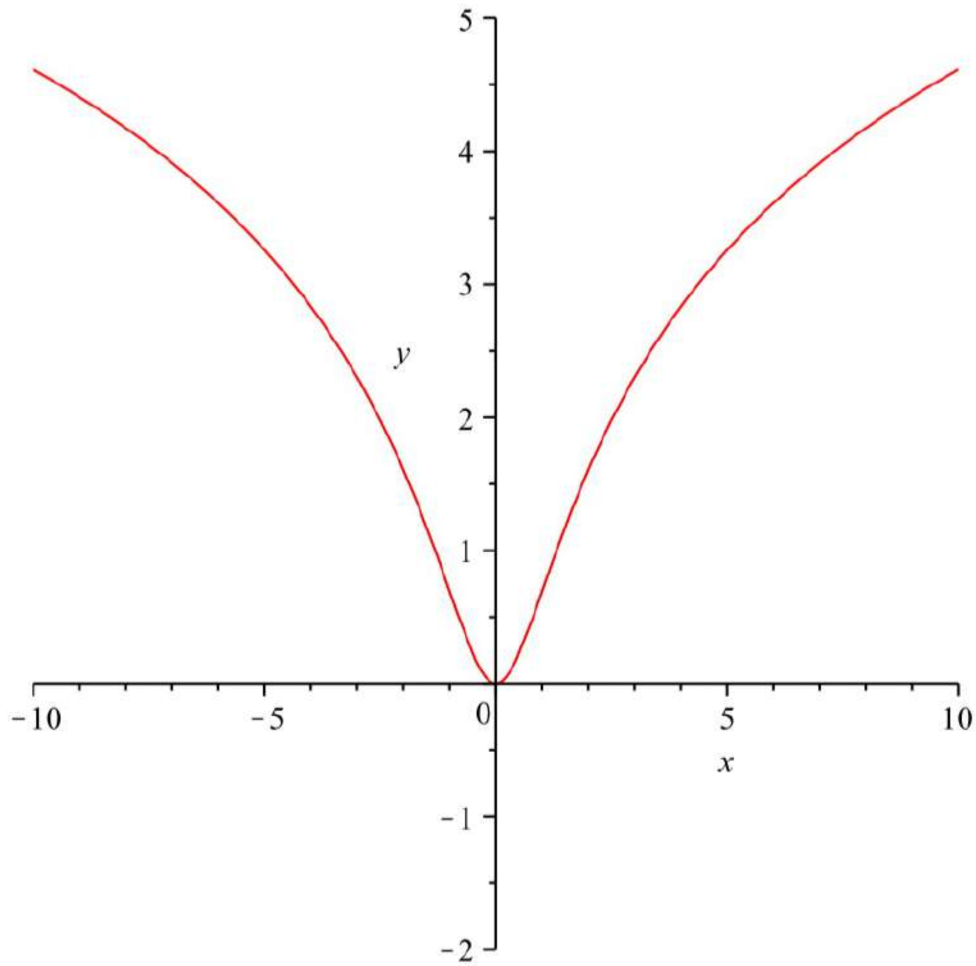
ii) The only critical number of f is $\boxed{x = 0}$.
Since f' changes sign from negative to positive at 0, $\boxed{f(0) = 0 \text{ is a local minimum}}$ by the First Derivative Test. Furthermore, this local minimum is an absolute minimum value of f.

(b) Given that $f''(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

i) $\boxed{f''(x) = \dfrac{2-2x^2}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}}$. We have $\boxed{f''(x) = 0 \text{ when } x = \pm 1}$.
Since $f''(x) < 0$ on $\boxed{(-\infty, -1) \text{ and } (1, \infty)}$, and $f''(x) > 0$ on $\boxed{(-1, 1)}$, therefore, the graph of f is concave downward on $\boxed{(-\infty, -1) \text{ and } (1, \infty)}$, and concave upward on $\boxed{(-1, 1)}$.

ii) Since f is continuous, the points $\boxed{(-1, \ln 2)}$ and $\boxed{(1, \ln 2)}$ are points of inflection.

(d) The graph of f



7. [40 pts.] Let $f(x) = 1 + e^{-x^2}$.

(a) Find the horizontal asymptotes of the graph of f, if any.

(b) Given that $f'(x) = -2xe^{-x^2}$.

i) Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

ii) Find the local maximum and minimum values of f, if any.

(c) Given that $f''(x) = 2(2x^2 - 1)e^{-x^2}$.

i) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, if any.

ii) Find the points of inflection, if any.

(d) Sketch the graph of f.

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7. [40 pts] For the function $f(x) = 1 + e^{-x^2}$, we have $\boxed{D_f = (-\infty, \infty)}$.

(a) [10 pts] $\boxed{\lim_{x \to \pm\infty} 1 + e^{-x^2} = 1}$. Therefore, $y = 1$ is the only horizontal asymptote.

(b) [10 pts] We have $\boxed{f'(x) = 0 \text{ only at } x = 0 \text{ and } f'(x) \text{ exists for all } x}$.

Therefore, $\boxed{\text{the only critical number is } x = 0}$.

Using the Increasing/Decreasing Test, we conclude that the function

$\boxed{f \text{ is increasing on the interval } (-\infty, 0)}$ and $\boxed{\text{it is decreasing on the interval } (0, \infty)}$.

By the First Derivative Test, this implies that $\boxed{f \text{ has a local maximum at } x = 0,\ f(0) = 2}$.

Moreover, $f(0) = 2$ is the absolute maximum value of f.

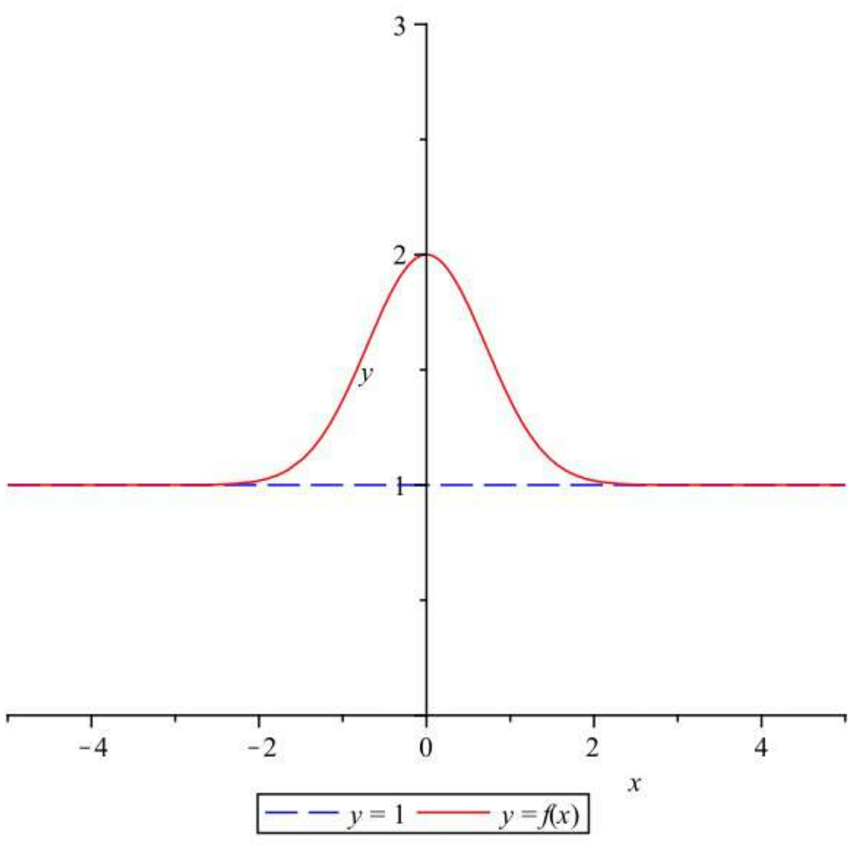
(c) [10 pts] We have $\boxed{f''(x) = 0 \text{ only at } x = \frac{\pm 1}{\sqrt{2}} \text{ and } f''(x) \text{ is defined for all } x}$.

Using the Concavity Test, we conclude that

$\boxed{\text{the graph of } f \text{ is concave upward on } (-\infty, \frac{-1}{\sqrt{2}}) \text{ and } (\frac{1}{\sqrt{2}}, \infty) \text{ and concave downward on } (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$.

Since f is continuous, $\boxed{\text{the points } \left(\frac{\pm 1}{\sqrt{2}}, 1 + e^{\frac{-1}{2}}\right)}$ are points of inflection.

(d) [10 pts] The graph of the function f:



6. [$4 \times 10 = 40$ pts.] Let $f(x) = \dfrac{e^x}{1 - e^{x-1}}$.

(a) [$5 + 5 = 10$ pts.] Find the vertical and the horizontal asymptotes of the graph of f, if any.

(b) [$5 + 2.5 + 2.5 = 10$ pts.]

i. Show that $f'(x) = \dfrac{e^x}{(1 - e^{x-1})^2}$.

ii. Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

iii. Find the local maximum and minimum values of f, if any.

(c) [$5 + 5 = 10$ pts.] Given that $f''(x) = \dfrac{e^x + e^{2x-1}}{(1 - e^{x-1})^3}$.

i. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward, if any.

ii. Find the points of inflection, if any.

(d) [10 pts.] Sketch the graph of f.

الحل في الصفحة التالية

1) Domain of $f(x)$:-

$1 - e^{x-1} \neq 0 \Rightarrow e^{x-1} \neq 1$

$\ln e^{x-1} \neq \ln 1 \Rightarrow x - 1 \neq 0$

$x \neq 1$ $\therefore$ Domain of $f(x) = \mathbb{R} / \{1\}$

a) horizontal asymptotes

$$\lim_{x \to \infty} \frac{e^x}{1 - e^{x-1}} = \frac{e^x}{1 - e^x e^{-1}}$$

$$\lim_{x \to \infty} \frac{\frac{e^x}{e^x}}{\frac{1}{e^x} - \frac{e^x e^{-1}}{e^x}} = \frac{1}{\frac{1}{e^\infty} - e^{-1}} = \frac{1}{-e^{-1}} = -e$$

$$\lim_{x \to -\infty} \frac{\frac{e^x}{e^x}}{\frac{1}{e^x} - \frac{e^x e^{1}}{e^x}} = \frac{1}{\frac{1}{e^{-\infty}} - e^{1}} = \frac{1}{\infty - e} = 0$$

$\therefore$ $y = 0$, $y = e^{-}$ are H.A

b) Vertical asymptotes

$$\lim_{x \to 1^{\pm}} \frac{e^x}{1 - e^{x-1}} = \frac{e^1}{1 - e^0} = \frac{e}{1-1} = \frac{e}{0} = \pm\infty$$

$x = 1$ is V.A

b) i) $f'(x) = \dfrac{e^x(1-e^{x-1}) - e^x(-e^{x-1})}{(1-e^{x-1})^2}$

$$= \frac{e^x(1-e^{x-1}+e^{x-1})}{(1-e^{x-1})^2} = \frac{e^x}{(1-e^{x-1})^2}$$

ii) critical number $\begin{cases} f'(x) = 0 \\ f'(x) = DNE \end{cases}$

1) $f'(x) = 0 \Rightarrow e^x = 0$ impossible

2) $f'(x) = DNE \Rightarrow (1-e^{x-1})^2 = 0$

$1 - e^{x-1} = 0 \Rightarrow 1 = e^{x-1} \Rightarrow \ln 1 = \ln e^{x-1}$

$x - 1 = 0 \Rightarrow x = 1 \notin \mathbb{R}/\{1\}$

No critical number ∴ No local extreme value

3) interval f' $(-\infty, 1) \cup (1, \infty)$

	$-\infty$ 0	1	∞
f'	$+$	$+$	
$\nearrow \searrow$	$\nearrow$	$\nearrow$	

f is increasing on it's Domain

d) given $f''(x) = \dfrac{e^x + e^{2x-1}}{(1-e^{x-1})^3}$

for concavity:-

$$f''(x) = \begin{cases} f''(x) = 0 \\ f''(x) = DNE \end{cases}$$

$f''(x) = 0 \Rightarrow e^x + e^{2x-1} = 0$ impossible

$f''(x) = DNE \Rightarrow (1-e^{x-1})^3 = 0$

$1 - e^{x-1} = 0 \Rightarrow 1 = e^{x-1} \Rightarrow \ln 1 = \ln e^{x-1}$

$x - 1 = 0 \Rightarrow x = 1 \notin \mathbb{R}/\{1\}$

3) interval f'' $(-\infty, 1) \cup (1, \infty)$

	$-\infty$ 0	1 2 ∞
f''	+	−
∪ ∩	∪	∩

There is no inflection point since $1 \notin$ Domain

f is concave upward on $(-\infty, 1)$

f is concave downward on $(1, \infty)$

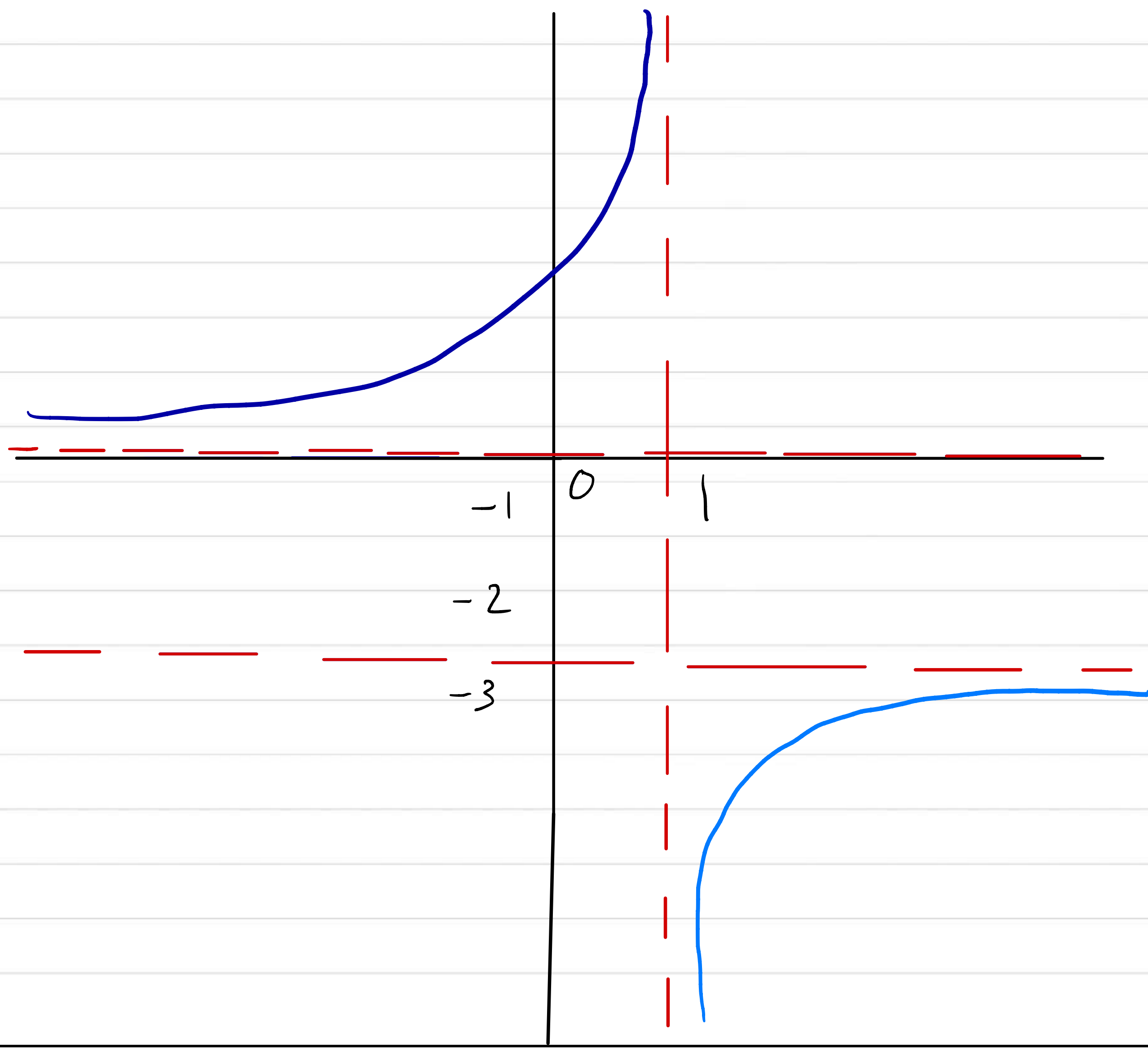
$$f(x) = \frac{e^x}{1 - e^{x-1}}.$$

1) Asymptotes $y = -e$, $y = 0$

$x = 1$

	$-\infty$	0	1	∞
f'		$+$	$+$	
$\nearrow \searrow$		$\nearrow$	$\nearrow$	

	$-\infty$	0	1	2	∞
f''		$+$	$-$		
$\cup \cap$		$\cup$	$\cap$		



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6. [$4 \times 10 = 40$ pts.] Let $f(x) = \dfrac{e^x}{1-e^{x-1}}$.

$$D_f = (-\infty, 1) \cup (1, \infty).$$

(a) We have: $\boxed{\displaystyle\lim_{x\to\infty} \frac{e^x}{1-e^{x-1}} = -e}$ and $\boxed{\displaystyle\lim_{x\to-\infty} \frac{e^x}{1-e^{x-1}} = 0}$.

Therefore, the lines $y = -e$ and $y = 0$ are horizontal asymptotes of the graph of f.

Furthermore, $\boxed{\displaystyle\lim_{x\to 1^{\pm}} \frac{e^x}{1-e^{x-1}} = \mp\infty}$ and hence $x = 1$ is a vertical asymptote of the graph of f.

(b) $\boxed{f'(x) = \dfrac{(e^x)(1-e^{x-1}) - (e^x)(-e^{x-1})}{(1-e^{x-1})^2} = \dfrac{e^x}{(1-e^{x-1})^2}}$.

Since $\boxed{f'(x) = \dfrac{e^x}{(1-e^{x-1})^2},\ f'(x) > 0 \text{ for all } x \neq 1}$.

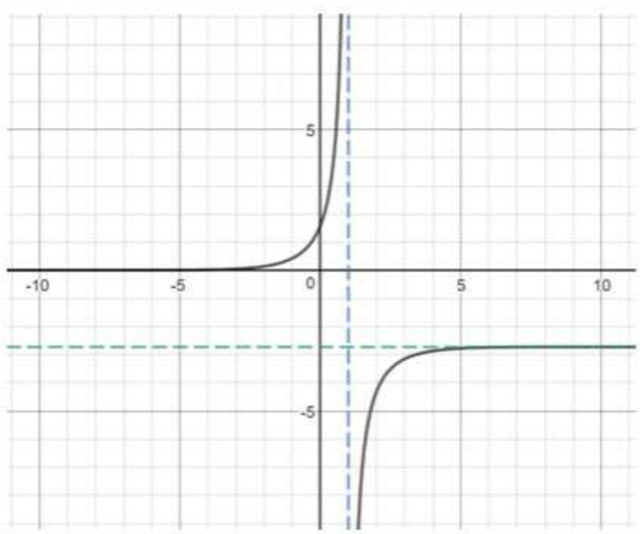
Thus $\boxed{f \text{ is increasing on its domain}}$. Furthermore, $\boxed{f \text{ has no critical numbers}}$. Hence $\boxed{f \text{ has no local extreme values}}$.

(c) Since $\boxed{f''(x) = \dfrac{e^x + e^{2x-1}}{(1-e^{x-1})^3},\ f''(x) > 0 \text{ on } (-\infty, 1) \text{ and } f''(x) < 0 \text{ on } (1, \infty)}$.

Therefore, $\boxed{\text{the graph of } f \text{ is concave upward on } (-\infty, 1) \text{ and it is concave downward on } (1, \infty)}$.

$\boxed{\text{There are no inflection points}}$ (f is discontinuous at $x = 1$).

(d) Sketch



$$f(x) = \frac{x^2+1}{(x^2-4)^2}$$

$$f'(x) = \frac{-2x(x^2+6)}{(x^2-4)^3}$$

$$f''(x) = \frac{6(x^4+14x+8)}{(x^2-4)^4}$$

a) Find the vertical and horizontal asymptotes for the graph of f, if any.

b) Study the symmetry of the curve $y = f(x)$.

c) find x-intercept and y-intercept

i) Find the intervals on which f is increasing and the intervals on which f is decreasing.

ii) Find the local maximum and minimum values of f, if any.

i) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

ii) Find the points of inflection, if any.

e) Sketch the graph of f.

---

1) Domain of $f(x)$

$x^2 - 4 \neq 0 \Rightarrow x^2 \neq 4$

$\therefore x = \pm 2$

Domain of f is $\mathbb{R} / \{\pm 2\}$

a) for Vertical asymptotes

$$\lim_{x \to 2^+} \frac{x^2+1}{(x^2-4)^2} = \infty$$

$$\lim_{x \to 2^-} \frac{x^2+1}{(x^2-4)^2} = \infty$$

$$\lim_{x \to -2^+} \frac{x^2+1}{(x^2-4)^2} = \infty$$

$$\lim_{x \to -2^-} \frac{x^2+1}{(x^2-4)^2} = \infty$$

$\therefore x = -2 \ , \ x = 2$ are V.A

b) for Horizontal asymptotes

$$\lim_{x \to \infty} \frac{x^2+1}{(x^2-4)^2} = 0$$

$y = 0$ is H.A

$$\lim_{x \to -\infty} \frac{x^2+1}{(x^2-4)^2} = 0$$

b) symmetry

$$f(-x) = \frac{(-x)^2+1}{\left((-x)^2-4\right)^2} = \frac{x^2+1}{(x^2-4)^2}$$

$= f(x)$ $\quad\therefore$ symmetric about y-axis

---

c) intercept

for x-int: set $y=0$

$$\frac{x^2+1}{(x^2-4)^2} = 0$$

impossible

No x-int

for y-int: set $x=0$

$$f(0) = \frac{0+1}{(0-4)^2} = \frac{1}{16}$$

$\therefore \left(0, \frac{1}{16}\right)$ is the y-int

$$f'(x) = \frac{-2x(x^2+6)}{(x^2-4)^3}$$

ii) critical number $\begin{cases} f'(x) = 0 \\ f'(x) = DNE \end{cases}$

1) $f'(x) = 0 \Rightarrow -2x(x^2+6) = 0$

$-2x = 0 \Rightarrow x = 0 \in D$

$x^2 + 6 \neq 0$

2) $f'(x) = DNE \Rightarrow x^2 - 4 = 0$

$x^2 = 4 \qquad x = -2,\ x = 2$

But $x = 2$ and $x = -2 \notin \mathbb{R} / \{\pm 2\}$

3) interval $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$

$-\infty$	-3	-2 -1	0 1	2 ∞
f'	$+$	$-$	$+$	$-$
↗ ↘	↗	↘	↗	↘

$f(0) = \frac{0+1}{(0-4)^2} = \frac{1}{16}$

$(0, \frac{1}{16})$ is the local minimum

increasing interval $(-\infty, -2) \cup (0, 2)$

Decreasing interval $(-2, 0) \cup (2, \infty)$

$$f''(x) = \frac{6(x^4+14x+8)}{(x^2-4)^4}$$

$$f''(x) = \begin{cases} \to 0 \\ \to DNE \end{cases}$$

$f''(x) = 0 \Rightarrow 6(x^4+14x+8) = 0$

impossible

$f''(x) = DNE \Rightarrow x^2-4 = 0$

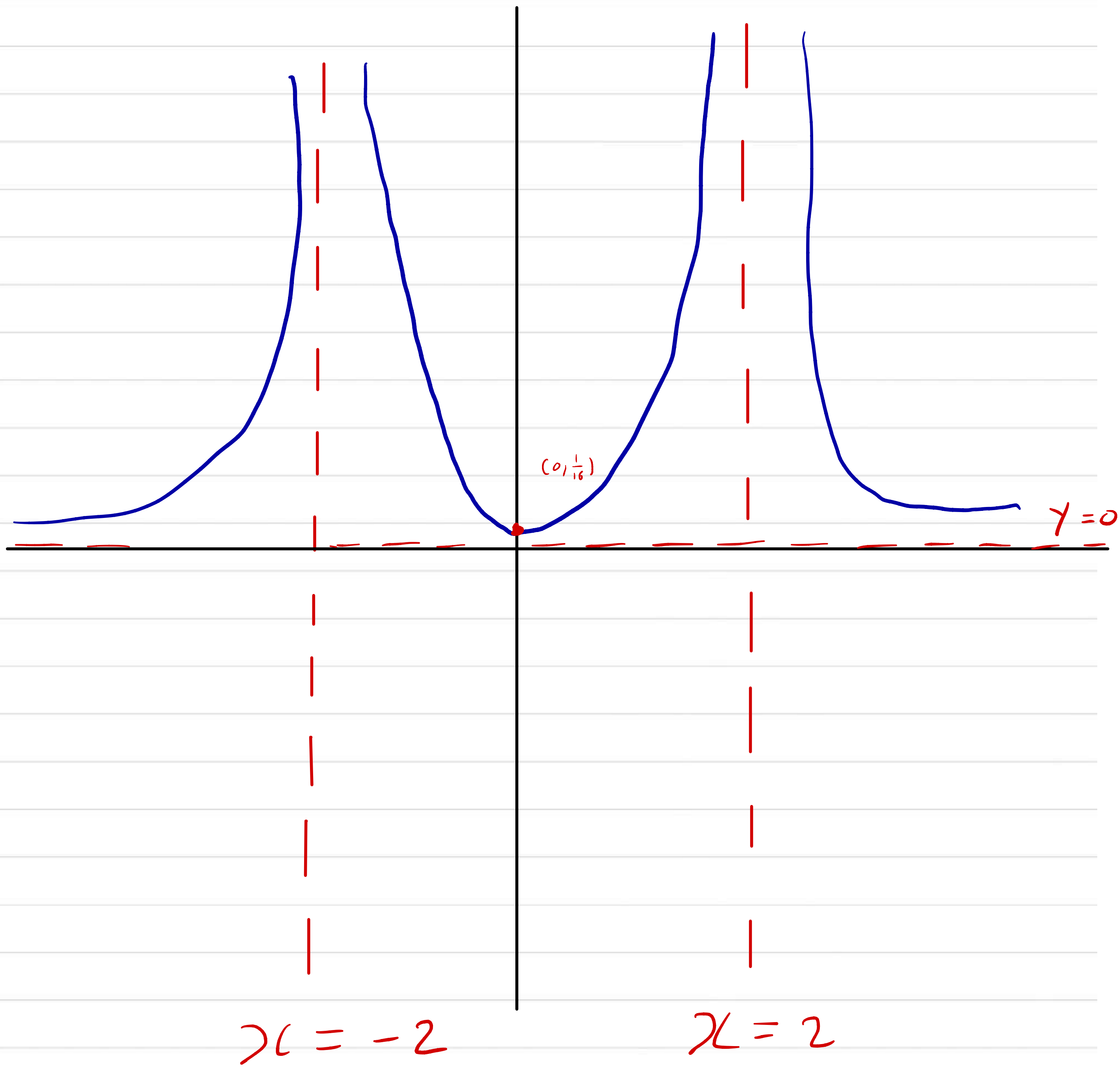
$x = 2$, $x = -2$

interval for f'' $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$

	$-\infty$ to -2	-2 to 2	2 to ∞
f''	+	+	+
U ∩	U	U	U

f is concaving up on its Domain

No inflection Point



(0, 1/16)
y = 0
x = -2
x = 2

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

SOLUTION Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$. From condition (iii) we know that the graph of f has two horizontal asymptotes: $y = -2$ (to the left) and $y = 0$ (to the right).

We first draw the horizontal asymptote $y = -2$ as a dashed line (see Figure 9). We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at $x = 1$, and decreasing toward the x-axis as at the far right. We also make sure that the graph has inflection points when $x = -2$ and 2. Notice that we made the curve bend upward for $x < -2$ and $x > 2$, and bend downward when x is between -2 and 2. ■

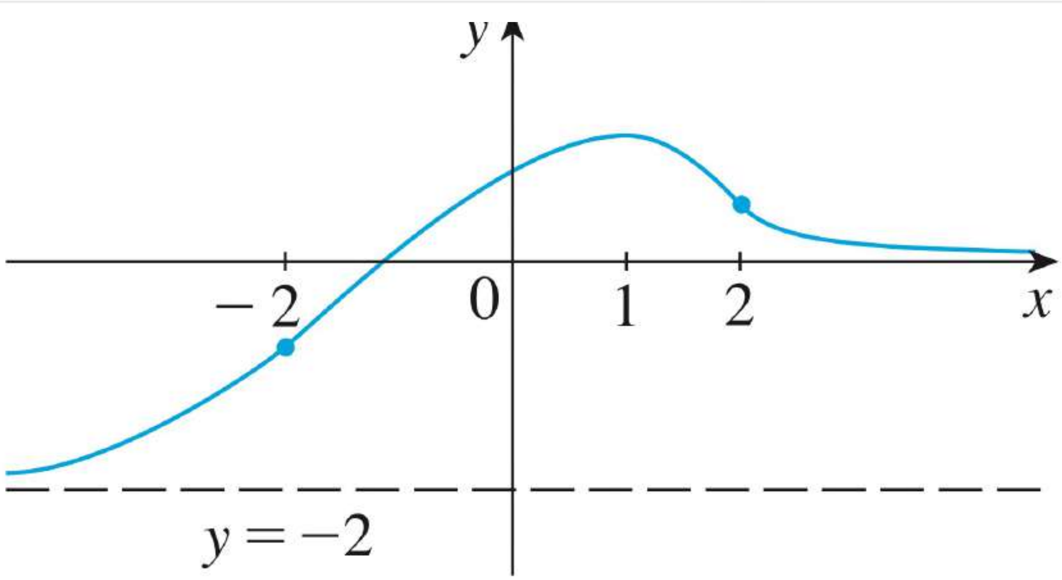


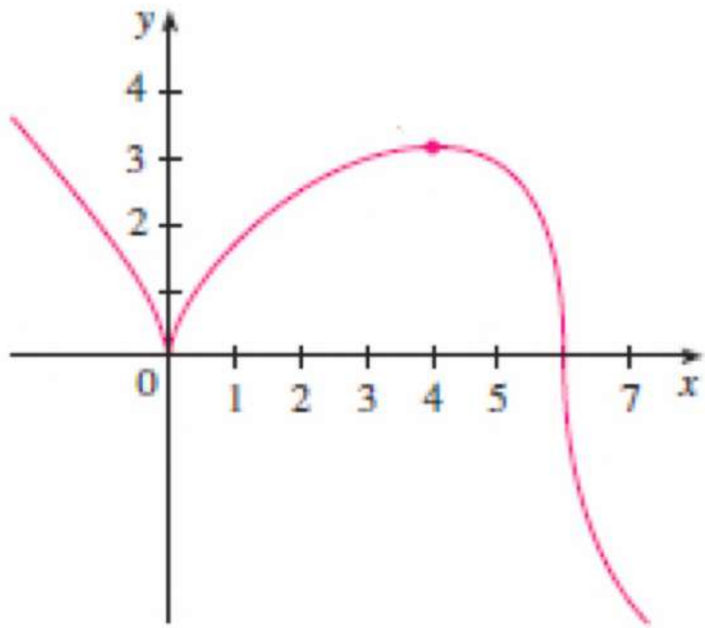
FIGURE 9

4. [10 pts.] Sketch a possible graph of a function f that satisfies the following conditions:

(a) $f(0) = 0,\ f(4) = 3$ and $f(6) = 0$.

(b) $f'(x) < 0$ on $(-\infty, 0)$ and $(4, \infty)$, $f'(x) > 0$ on $(0, 4)$.

(c) $f''(x) < 0$ on $(-\infty, 0)$ and $(0, 6)$, $f''(x) > 0$ on $(6, \infty)$.



Calculus A

Chapter 4: Application of Differentiation

Sections: 4.7 Optimization Problems

A+

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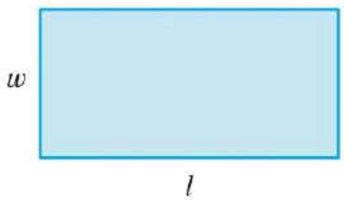
S=Surface Area = A = Area. P = Perimeter = Circumference = C. Volume = V

Rectangle

$A = lw$

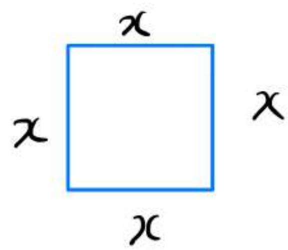
$P = 2l + 2w$



Square

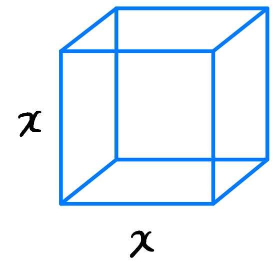
$A = x^2$

$P = 4x$



Cube

$V = x^3$

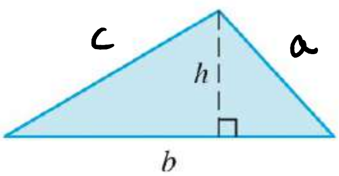


$S = 6x^2$

Triangle

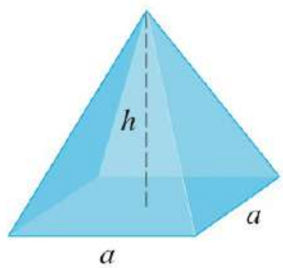
$A = \frac{1}{2}bh$

$P = a+b+c$



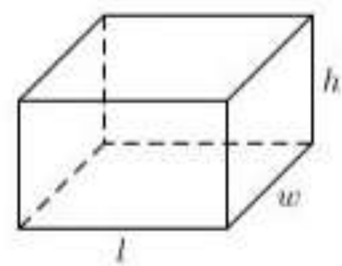
Pyramid

$V = \frac{1}{3}ha^2$



RECTANGULAR SOLID

l = length, w = width,
h = height
Volume: $V = lwh$
Surface Area:
$S = 2lw + 2lh + 2wh$

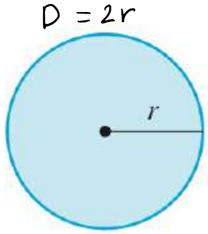


Circle

$A = \pi r^2$

$C = 2\pi r$

$D = 2r$

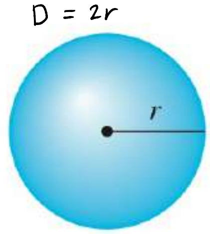


Sphere

$V = \frac{4}{3}\pi r^3$

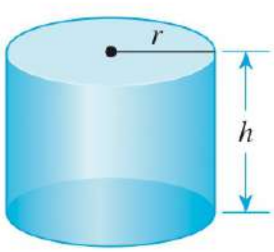
$A = 4\pi r^2$

$D = 2r$



Cylinder

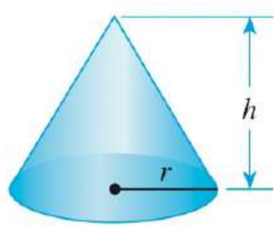
$V = \pi r^2 h$



$S = 2\pi rh + 2\pi r^2$

Cone

$V = \frac{1}{3}\pi r^2 h$



$S = \pi r\sqrt{r^2+h^2}$

Steps In Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?

2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.

6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

١ـ أفهم السؤال

٢ـ أرسم المسألة إذا احتجت

٣ـ أكتب المعادلة الي بدي نسويلها maximized or minimized

٤ـ حل المعادلة كمتغير واحد

٥ ـ طلع ال critical value و تأكد تنتمي لل Domain

٦ـ ال critical value رح تكون يا abs min أو abs max

٧ ـ جاوب على السؤال

عندنا طريقتين للايجاد absolute max والabsoulte mini

١)

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

	$-\infty$	25	∞
f'	$-$		$+$
↗ ↘	↘		↗

Then the abs minimum is at $x = 25$

	$-\infty$	600	∞
f'	$+$		$-$
↗ ↘	↗		↘

Then the abs maximum is at $x = 600$

٢) أو عن طريق المشتقة الثانية

$f''(x) < 0$

إذا المشتقة الثانية دايما سالبة بكل الفترات يعني عندها
local maximum

concave down $\cap$ abs max

$f''(x) > 0$

إذا المشتقة الثانية دايما موجبة بكل الفترات يعني عندها local minimum

concave up $\cup$ abs minimum

3. [10 pts.] Find the maximum product of two positive x and y such that $2x+y = 100$.

We want to maximize $xy = x(100-2x) = 100x - 2x^2$. For this purpose, we let $f(x) = 100x - 2x^2$. We have $f'(x) = 100 - 4x$. The only critical number of f is $x = 25$. Now $f''(x) = -4$, using the second derivative test $f''(25) = -4 < 0$, so f has absolute maximum value $f(25) = 1250$.

$2x + y = 100 \Rightarrow y = 100 - 2x$

$xy = ??$ Domain: $x > 0$, $y > 0$

$f(x) = x(100 - 2x) = 100x - 2x^2$

for critical value:- $f'(x) = 0$

$f'(x) = 100 - 4x = 0$

$4x = 100 \Rightarrow x = 25$

	$-\infty$	25	∞
f'	$+$		$-$
↗ ↘	↗		↘

Then the abs maximum is at $x = 25$

$\therefore y = 100 - 2(25)$

$y = 50$

$\therefore xy = 25 * 50 = 1250$

2. Find two numbers whose difference is 100 and whose product is a minimum.

1) $x - y = 100$

$\therefore \quad x = 100 + y$ تقدر تحلها بالنسبة ل x أو y كيفك

$xy = ??$ Domain: $\mathbb{R}$

$\therefore f(y) = (100 + y)\,y$

$= f(y) = 100y + y^2$

$f'(y) = 100 + 2y$

for critical number $f'(y) = 0$

$\therefore \; 100 + 2y = 0 \Rightarrow y = -50$

	$-\infty$	-50	∞
f'	$-$		$+$
$\nearrow$ $\searrow$	$\searrow$		$\nearrow$

Then the abs minimum is at $y = -50$

$\therefore \; x = 100 - 50 \;,\; x = 50$

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

$xy = 100$, $y = \dfrac{100}{x}$

$x + y = ??$

$f(x) = x + \dfrac{100}{x}$ Domain : $x > 0$, $y > 0$

$f'(x) = 1 - \dfrac{100}{x^2}$

for critical number

$f'(x) = 0 \Rightarrow f'(x) = 1 - \dfrac{100}{x^2} = 0$

$\Rightarrow 1 = \dfrac{100}{x^2} \Rightarrow x^2 = 100$

$\therefore x = 10$ ✓ and $x = -10$ X

السؤال طالب
منّي أنّا تكون
الأعداد موجبة

	$-\infty$	10	∞
f'	$-$		$+$
↗↘	↘		↗

Then the abs minimum is at $x = 10$

$\therefore y = \frac{100}{10} \Rightarrow y = 10$

$\boxed{x = 10 \quad y = 10}$

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

$x + y = 16 \quad , \quad y = 16 - x$

$x^2 + y^2 = ??$

Domain: $x > 0$
$y > 0$

$f(x)^2 = x^2 + (16 - x)^2$

تقدر تشتق

$f'(x) = 2x + 2(16 - x)(-1)$

$= 2x - 32 + 2x$

$f'(x) = 4x - 32$

$f(x) = x^2 + (256 - 32x + x^2)$

$f(x) = 2x^2 - 32x + 256$

$f'(x) = 4x - 32$

$f'(x) = 4x - 32 = 0 \Rightarrow 4x = 32$

$x = \dfrac{32}{4} = 8$

لا تنسى القيم الي
اختبرها الي في
الجدول نعوضها
بالمشتقة ونشوف
إشارتها

	$-\infty$	8	∞
f'	$-$	$+$	
$\nearrow \searrow$	$\searrow$	$\nearrow$	

$y = 8$

$x = 8$

Then the abs minimum is at $x = 8$

$\therefore y = 16 - 8 = 8 \quad , \quad \therefore x^2 + y^2 = 8^2 + 8^2 = 64 + 64 = 128$

21. Find the point on the line $y = 2x + 3$ that is closest to the origin.

$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$\therefore$ The distance between $(\overset{x_0}{0}, \overset{y_0}{0})$ and $(\overset{x}{x}, \overset{y}{2x+3})$

$$D = \sqrt{(x - 0)^2 + (2x + 3 - 0)^2}$$

$$D^2 = (x)^2 + (2x+3)^2$$

$$f(x) = (x)^2 + (2x+3)^2$$

$$f'(x) = 2x + 2(2x+3)(2)$$

$$f'(x) = 2x + 8x + 12 = 10x + 12$$

$$\therefore x = \frac{-12}{10} = \frac{-6}{5}$$

	$-\infty$		$\frac{-6}{5}$		∞
f'		$-$		$+$	
$\nearrow \searrow$		$\searrow$		$\nearrow$	

Then the abs minimum is at $x = -\frac{6}{5}$

$$\therefore y = 2\left(\frac{-6}{5}\right) + 3 = \frac{-12}{5} + 3 = \frac{-12}{5} + \frac{15}{5} = \frac{3}{5}$$

$\therefore \left(-\frac{6}{5}, \frac{3}{5}\right)$ is the closest point to the origin

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$\because\ y^2 = 2x \Rightarrow x = \frac{1}{2} y^2$$

$\therefore$ The distance between $(\overset{x_0}{1}, \overset{y_0}{4})$ and $(\overset{x}{\tfrac{1}{2}y^2}, \overset{y}{y})$

$$D = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y - 4)^2}$$

$$D^2 = \left(\frac{1}{2}y^2 - 1\right)^2 + (y - 4)^2$$

$$f(y) = \left(\frac{1}{2}y^2 - 1\right)^2 + (y - 4)^2$$

$$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right)y + 2(y - 4)$$

ليش ما حطينا y' ؟
لأن قاعدين نشتق
بالنسبة ل y مو
بالنسبة ل x

$$f'(y) = y^3 - 2y + 2y - 8 = y^3 - 8$$

$$f'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y^3 = 8 \Rightarrow y = 2$$

	$-\infty$	2	∞
f'	$-$		$+$
↗ ↘	↘		↗

Then the abs minimum is at $y = 2$

$\therefore x = \frac{1}{2}(2)^2 = 2$

$\therefore (2, 2)$ is the closet point to $(1, 4)$

4. [10 pts.] Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

If the rectangle has dimensions x and y then its area $\boxed{A = xy = 100 \text{ m}^2, \text{ so } y = 100/x}$. The perimeter $\boxed{P = 2x + 2y = 2x + 200/x}$. We wish to minimize the function $\boxed{P(x) = 2x + 200/x}$ for $x > 0$. Since $\boxed{P'(x) = 2 - 200/x^2 = (2/x^2)(x^2 - 100)}$, the only critical number of P is $\boxed{x = \sqrt{100} = 10}$. Now $\boxed{P''(x) = 400/x^3 > 0}$, so the graph of P is concave upward on $(0, \infty)$ and $\boxed{P(10) = 40 \text{ is an absolute minimum value}}$. The dimensions of the rectangle with minimal perimeter are $\boxed{x = y = 10 \text{ m}}$. (The rectangle is a square).

1

3. [10 pts.] Find two nonnegative numbers x and y whose sum is 15 and $P = x^2y^3$ is a maximum.

Answer. We have $x+y = 15$ and hence $y = 15-x$ with $x \in [0, 15]$. The objective is to maximize $P(x) = x^2y^3 = x^2(15-x)^3$ on $[0, 15]$. The derivative is $P'(x) = 2x(15-x)^3 - 3x^2(15-x)^2 = x(15-x)^2[2(15-x) - 3x] = x(15-x)^2(30-5x)$. Therefore, P has one critical number in $(0, 15)$, $x = 6$. Since $P(0) = P(15) = 0$, and $P(6) = 6^29^3 > 0$, the Closed Interval Method gives the maximum value as $P(6) = 6^29^3$. Therefore, the two numbers are $x = 6$ and $y = 9$.

Q7. [10 pts.] Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(2, 0)$.

The distance d from the point $(2, 0)$ to a point (x, y) on the curve $y = \sqrt{x}$ is: $\boxed{d = \sqrt{(x-2)^2 + (\sqrt{x} - 0)^2}}$ and the square of the distance is $\boxed{D = f(x) = (x-2)^2 + x}$.

Differentiating, we obtain $\boxed{f'(x) = 2(x-2) + 1 = 2x - 3}$.

$\boxed{f'(x) = 0 \text{ when } x = 3/2}$. Observe that $\boxed{f'(x) < 0 \text{ when } x < 3/2 \text{ and } f'(x) > 0 \text{ when } x > 3/2}$, so by the FDT for Absolute Extreme values, $\boxed{\text{the absolute minimum occurs when } x = 3/2}$. The corresponding value for y is $\boxed{y = \sqrt{3/2}}$. Thus, the point on the curve $y = \sqrt{x}$ closest to the point $(2, 0)$ is $\boxed{(3/2, \sqrt{3/2})}$.

4. [10 pts.] Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

$$A = xy = 100$$

$$\therefore y = \frac{100}{x}$$

The perimeter $= p = 2x + 2y$

$$\therefore p(x) = 2x + 2\,\frac{100}{x} = 2x + \frac{200}{x}$$

$$\therefore p'(x) = 2 - \frac{200}{x^2}$$

for critical value :- $p'(x) = 0$

$$2 - \frac{200}{x^2} = 0 \Rightarrow 2 = \frac{200}{x^2}$$

$$\therefore x^2 = 100 \Rightarrow x = 10 \checkmark, \quad x = -10 \times$$

$x > 0$ because its length

	$-\infty$	10	∞
f'	$-$		$+$
$\nearrow \searrow$	$\searrow$		$\nearrow$

Then the abs minimum is at $x = 10$

$$\therefore y = \frac{100}{10}$$

$$y = 10$$

$\therefore$ Dimensions

$x = 10$

$y = 10$

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

$$V = 1 \text{ Litre} = 1000 \text{ cm}^3$$

we want to minimize the surface area

Domain

$r > 0$

$h > 0$

$$\because SA = 2\pi r^2 + 2\pi r h$$

$$\because V = \pi r^2 h \implies h = \frac{1000}{\pi r^2}$$

$$\therefore SA = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

$$\therefore SA' = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}$$

find the critical values:-

$$\frac{4\pi r^3 - 2000}{r^2} = 0 \implies 4\pi r^3 - 2000 = 0$$

$$4\pi r^3 = 2000 \implies r^3 = \frac{500}{\pi} \implies r = \sqrt[3]{\frac{500}{\pi}}$$

	$-\infty$		$\sqrt[3]{\frac{500}{\pi}}$		∞
f'		$-$		$+$	
		$\searrow$		$\nearrow$	

$$\therefore h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2}$$

Then the abs minimum is at $r = \sqrt[3]{\frac{500}{\pi}}$

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

$$x + x + y = 2400$$

$x > 0$

$y > 0$

$$2x + y = 2400$$

$$\therefore \; y = 2400 - 2x \qquad 0 \leq x \leq 1200$$

$$\therefore \; A = xy$$

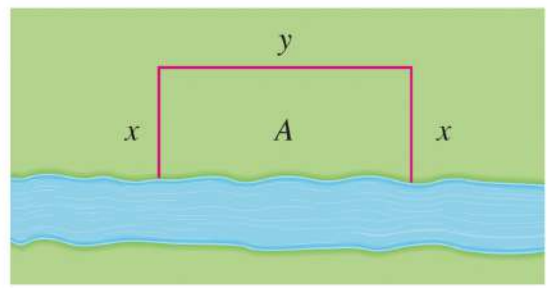


FIGURE 2

$$\therefore \; A(x) = x(2400 - 2x)$$

$$A(x) = 2400x - 2x^2$$

$$A'(x) = 2400 - 4x$$

for critical value :- $A'(x) = 0$

$$\Rightarrow A'(x) = 2400 - 4x = 0$$

$$\Rightarrow 4x = 2400 \Rightarrow x = 600$$

	$-\infty$	600	∞
f'	$+$	$-$	
$\nearrow \searrow$	$\nearrow$	$\searrow$	

$$\therefore \; y = 2400 - 2(600)$$
$$= 2400 - 1200$$
$$= 1200$$

Then the abs maximum is at $x = 600$

$\therefore$ Dimensions

$x = 600$

$y = 1200$

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

SOLUTION 1 Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

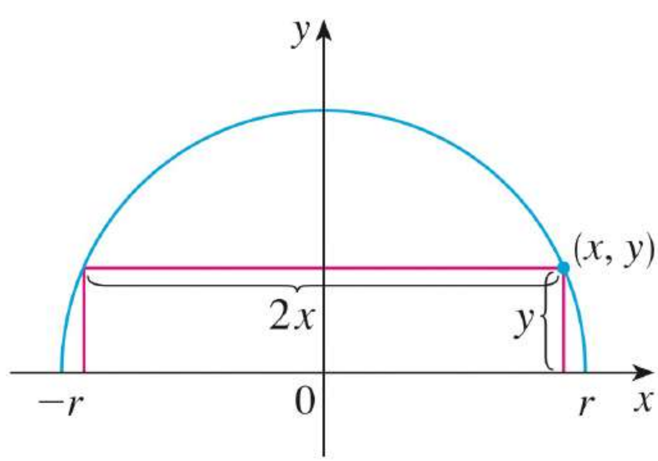


FIGURE 9

Calculus A

Chapter 4: Application of Differentiation

Sections: 4.9 Antiderivatives

A+

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4.9 Antiderivatives

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

EXAMPLE 1 Find the most general antiderivative of each of the following functions.
(a) $f(x) = \sin x$ (b) $f(x) = 1/x$ (c) $f(x) = x^n, \quad n \neq -1$

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.
(b) Recall from Section 3.6 that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So on the interval $(0, \infty)$ the general antiderivative of $1/x$ is $\ln x + C$. We also learned that

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

for all $x \neq 0$. Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is $\ln|x| + C$ on any interval that doesn't contain 0. In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. So the general antiderivative of f is

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) We use the Power Rule to discover an antiderivative of x^n. In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

4.9 Antiderivatives

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

SOLUTION We first rewrite the given function as follows:

$$g'(x) = 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4\sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$g(x) = 4(-\cos x) + 2\,\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4\cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C$$

EXAMPLE 3 Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

SOLUTION The general antiderivative of

$$f'(x) = e^x + \frac{20}{1 + x^2}$$

is

$$f(x) = e^x + 20\tan^{-1}x + C$$

To determine C we use the fact that $f(0) = -2$:

$$f(0) = e^0 + 20\tan^{-1}0 + C = -2$$

Thus we have $C = -2 - 1 = -3$, so the particular solution is

$$f(x) = e^x + 20\tan^{-1}x - 3$$ ■

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

SOLUTION The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

■

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = 4x + 7$

$$F(x) = 4\left(\frac{x^2}{2}\right) + 7x + C$$

$$F(x) = 2x^2 + 7x + C$$

3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$

$$F(x) = 2\frac{x^4}{4} - \frac{2}{3}\frac{x^3}{3} + 5\frac{x^2}{2} + C$$

$$F(x) = \frac{x^4}{2} - \frac{2}{9}x^3 + \frac{5x^2}{2} + C$$

5. $f(x) = x(12x + 8)$

$$f(x) = 12x^2 + 8x$$

$$F(x) = 12\frac{x^3}{3} + 8\frac{x^2}{2} + C = 4x^3 + 4x^2 + C$$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$

$$F(x) = 2\frac{x^4}{4} - \frac{2}{3}\frac{x^3}{3} + 5\frac{x^2}{2} + C$$

$$F(x) = \frac{x^4}{2} - \frac{2}{9}x^3 + \frac{5x^2}{2} + C$$

5. $f(x) = x(12x + 8)$

$$f(x) = 12x^2 + 8x$$

$$F(x) = 12\frac{x^3}{3} + 8\frac{x^2}{2} + C = 4x^3 + 4x^2 + C$$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

7. $f(x) = 7x^{2/5} + 8x^{-4/5}$

$$F(x) = 7\left(\frac{x^{7/5}}{\frac{7}{5}}\right) + 8\left(\frac{x^{1/5}}{\frac{1}{5}}\right) + C$$

$$F(x) = 7\left(\frac{5}{7}x^{7/5}\right) + 8(5x^{1/5}) + C$$

$$= 5x^{7/5} + 40x^{1/5} + C$$

9. $f(x) = \sqrt{2}$

$$F(x) = \sqrt{2}\,x + C$$

10. $f(x) = e^2$

$$F(x) = e^2 x + C$$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

11. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$

$f(x) = 3x^{1/2} - 2x^{1/3}$

$F(x) = 3\left(\frac{2}{3}x^{3/2}\right) - 2\left(\frac{3}{4}x^{4/3}\right) + C$

$= 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

12. $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$

$f(x) = x^{\frac{2}{3}} + x\,x^{1/2} = x^{2/3} + x^{3/2}$

$F(x) = \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

13. $f(x) = \dfrac{1}{5} - \dfrac{2}{x}$

$$f(x) = \frac{1}{5} - 2\left(\frac{1}{x}\right)$$

$$F(x) = \frac{1}{5}x - 2\ln|x| + c$$

15. $g(t) = \dfrac{1 + t + t^2}{\sqrt{t}}$

$$g(t) = \frac{1}{\sqrt{t}} + \frac{t}{\sqrt{t}} + \frac{t^2}{\sqrt{t}}$$

$$g(t) = t^{-1/2} + t^{1/2} + t^{3/2}$$

$$G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

14. $f(t) = \dfrac{3t^4 - t^3 + 6t^2}{t^4}$

$$f(t) = \frac{3t^4}{t^4} - \frac{t^3}{t^4} + \frac{6t^2}{t^4}$$

$$f(t) = 3 - t^{-1} + 6t^{-2}$$

$$\boxed{t^{-1} = \frac{1}{t}}$$

$$F(t) = 3t - \ln|t| + 6\frac{t^{-1}}{-1} + c$$

$$F(t) = 3t - \ln|t| - 6t^{-1} + c$$

$$F(t) = 3t - \ln|t| - \frac{6}{t} + c$$

22. $f(x) = \dfrac{2x^2 + 5}{x^2 + 1}$

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)

$$\begin{array}{r|l} & 2 \\ \hline x^2+1 & 2x^2 + 5 \\ & -(2x^2 + 2) \\ \hline & 0 + 3 \end{array}$$

$$\therefore f(x) = 2 + \frac{3}{x^2+1}$$

$$\therefore F(x) = 2x + 3\tan^{-1}x + C$$

25–48 Find f.

27. $f''(x) = 2x + 3e^x$

$$f'(x) = x^2 + 3e^x + c_1$$

$$f(x) = \frac{x^3}{3} + 3e^x + c_1 x + c_2$$

28. $f''(x) = 1/x^2$

$$f''(x) = x^{-2}$$

$$f'(x) = \frac{x^{-1}}{-1} + c_1 = -x^{-1} + c_1 = -\frac{1}{x} + c_1$$

$$f(x) = -\ln|x| + c_1 x + c_2$$

25–48 Find f.

34. $f'(t) = t + 1/t^3, \quad t > 0, \quad f(1) = 6$

$$f'(t) = t + t^{-3}$$

$$f(t) = \frac{t^2}{2} - \frac{t^{-2}}{-2} + C = \frac{t^2}{2} - \frac{1}{2t^2} + C$$

قيمة المتغير　　قيمة ال f(t)

We have $f(1) = 6$

$$\therefore 6 = \frac{1^2}{2} - \frac{1}{2(1)^2} + C \qquad \therefore C = 6$$

$$\therefore f(t) = \frac{t^2}{2} - \frac{1}{2t^2} + 6$$

25–48 Find f.

41. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

$f'(\theta) = -\cos\theta + \sin\theta + c_1$

$\because f'(0) = 4$

$\therefore 4 = -\cos 0 + \sin 0 + c_1$

$\therefore 4 = -1 + c_1 \Rightarrow c_1 = 5$

$\therefore f'(\theta) = -\cos\theta + \sin\theta + 5$

$\therefore f(\theta) = -\sin\theta - \cos\theta + 5\theta + c_2$

$\because f(0) = 3$

$\therefore 3 = -\sin 0 - \cos 0 + 5(0) + c_2$

$\therefore 3 = -1 + c_2 \Rightarrow c_2 = 4$

$\therefore f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$

25–48 Find f.

45. $f''(x) = e^x - 2\sin x$, $\quad f(0) = 3$, $\quad f(\pi/2) = 0$

$f'(x) = e^x + 2\cos x + c_1$

$f(x) = e^x + 2\sin x + c_1 x + c_2$

We have $f(0) = 3$

$f(0) = e^0 + 2\sin 0 + c_1 0 + c_2$

$3 = 1 + c_2 \Rightarrow c_2 = 2$

$\therefore f(x) = e^x + 2\sin x + c_1 x + 2$

We have $f\left(\frac{\pi}{2}\right) = 0$

$f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} + 2\sin\frac{\pi}{2} + c_1\frac{\pi}{2} + 2$

$0 = e^{\frac{\pi}{2}} + 2 + \frac{\pi}{2} c_1 + 2$

$\therefore -\frac{\pi}{2} c_1 = e^{\frac{\pi}{2}} + 4$

$\Rightarrow c_1 = -\frac{e^{\frac{\pi}{2}}}{\frac{\pi}{2}} - \frac{4}{\frac{\pi}{2}} = -\frac{2e^{\frac{\pi}{2}}}{\pi} - \frac{8}{\pi}$

$f(x) = e^x + 2\sin x - \left(\frac{2e^{\frac{\pi}{2}} + 8}{\pi}\right)x + 2$

4. [10 pts.] Show that $F(x) = \frac{1}{2}\left(x^2 \tan^{-1} x - x + \tan^{-1} x\right)$ is an antiderivative of $f(x) = x \tan^{-1} x$.

$$F'(x) = \frac{1}{2}\left[\left(2x \tan^{-1} x + x^2\left(\frac{1}{1+x^2}\right) - 1 + \left(\frac{1}{1+x^2}\right)\right]$$

$$\frac{1}{2}\left[\left(2x \tan^{-1} x + \frac{x^2}{1+x^2} - 1 + \frac{1}{1+x^2}\right]$$

$$\frac{1}{2}\left[\left(2x \tan^{-1} x + \frac{x^2+1}{1+x^2} - 1\right]$$

$$\frac{1}{2}\left[\left(2x \tan^{-1} x + 1 - 1\right]$$

$$\frac{1}{2}\left[\left(2x \tan^{-1} x\right] = x \tan^{-1} x$$

$\therefore$ F is an antiderivative of f

We have $F'(x) = \frac{1}{2}\left(\frac{x^2}{1+x^2} + 2x \tan^{-1} x - 1 + \frac{1}{1+x^2}\right) = x \tan^{-1} x$, so F is an antiderivative of f.

5. [10 pts.] Let $f(x) = \sin(\ln x)$. Show that $F(x) = \frac{x}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big]$ is an antiderivative of $f(x)$.

$$F'(x) = \frac{1}{2}(\sin(\ln x) - \cos(\ln x)] + \frac{x}{2}\left[\cos(\ln x)\frac{1}{x} - \left(-\sin(\ln x)\frac{1}{x}\right)\right.$$

$$= \frac{1}{2}\sin(\ln x) - \frac{1}{2}\cos(\ln x) + \frac{x}{2}\cdot\frac{1}{x}\cos(\ln x) + \frac{x}{2}\cdot\frac{1}{x}\sin(\ln x)$$

$$= \frac{1}{2}\sin(\ln x) - \frac{1}{2}\cos(\ln x) + \frac{1}{2}\cos(\ln x) + \frac{1}{2}\sin(\ln x)$$

$$= \frac{1}{2}\sin(\ln x) + \frac{1}{2}\sin(\ln x)$$

$$= \sin(\ln x)$$

$\therefore$ F is an antiderivative of f

Since $\boxed{F'(x) = \frac{1}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big] + \frac{x}{2}\Big[\frac{1}{x}\cos(\ln x) + \frac{1}{x}\sin(\ln x)\Big] = \sin(\ln x) = f(x)}$, we conclude that $F(x) = \frac{x}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big]$ is an antiderivative of $f(x) = \sin(\ln x)$.

Calculus A

Chapter 5: Integrals

Sections: 5.2 The Definite Integral

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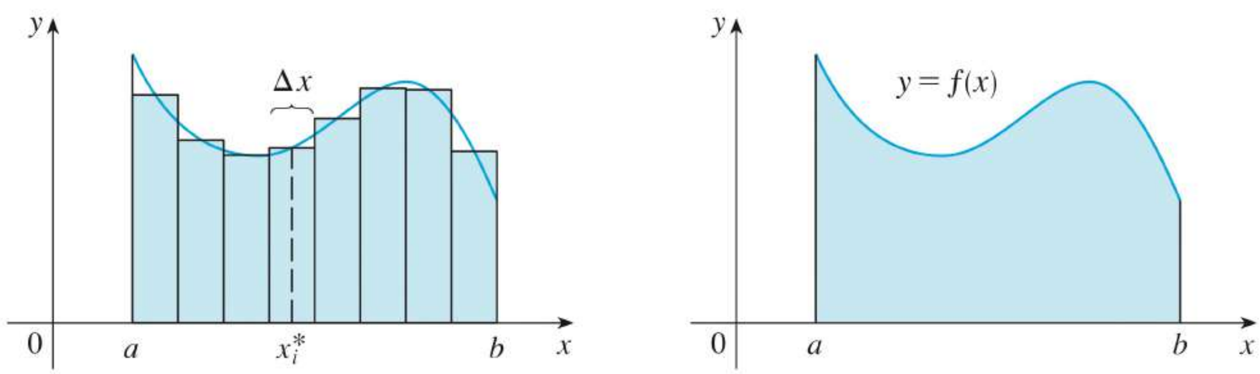
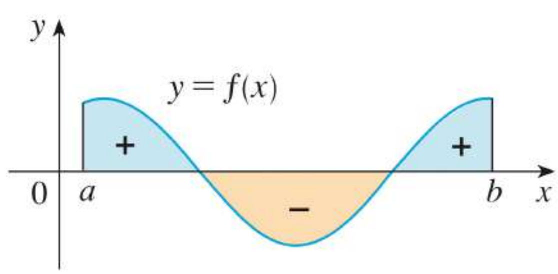
5.2 The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.





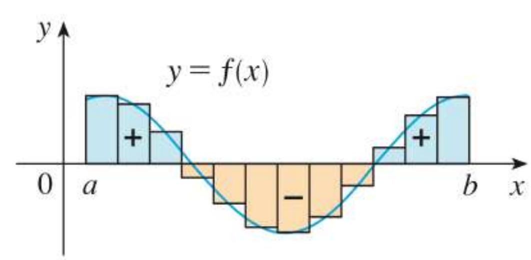


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

التكامل هو المساحة تحت المنحنى

$$\sum_{i=1}^{n} f(x_i^*)\ \Delta x$$

الطول العرض

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

1) $\int$ is the integral sign

2) a and b the limits of the integral

3) $f(x)$ is the integrand

4) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is called Riemann Sum

$\boxed{4}$ **Theorem** If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

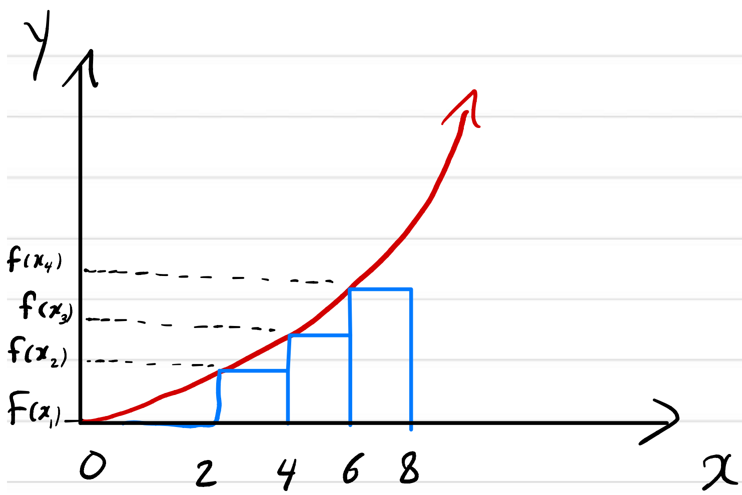
where $\Delta x = \dfrac{b-a}{n}$ and $\underset{\text{Right end-points}}{\underline{x_i = a + i\,\Delta x}}$

left $x_i = a + (i-1)\Delta x$

middle $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$$y = x^2 \qquad n = 4 \qquad [0, 8]$$

Taking sample points to be left



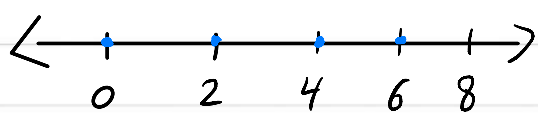
$$\text{Area} = \sum \Delta x \, f(x_i)$$

عرض طول

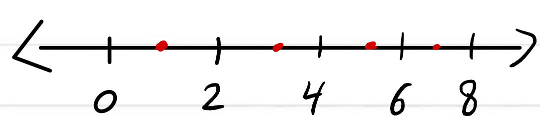
$$\Delta x = \frac{b-a}{n} = \frac{8-0}{4}$$

$$= 2$$

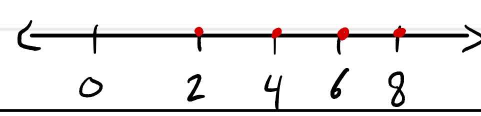
$$A_L = \Delta x [f(0) + f(2) + f(4) + f(6)]$$



$$A_m = \Delta x [f(1) + f(3) + f(5) + f(7)]$$



$$A_R = \Delta x [f(2) + f(4) + f(6) + f(8)]$$



خطوات حل Riemann sum

$f(x) = \sqrt{4 - x^2}$, $-2 \leq x \leq 2$, $n = 4$

by taking to the right

$f(x) = \sqrt{4 - x^2}$ (1

٢) أطلع حدود التكامل أو الفترة الي يبي مساحتها الي هي قيم a and b و عددها n

$\int_a^b$, $a \leq x \leq b$ $a = -2$ $n = 4$

$b = 2$

٣) أشوف الsample points يبيها من أي طرف

Right

٤) أطلع المسافة الي بينهم

$\Delta x = \dfrac{b - a}{n} = \dfrac{2 - (-2)}{4} = 1$

٥) نسوي الجدول

x	-2	-1	0	1	2
$f(x)$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

٦) السؤال قايلي يبي
أربع قيم من اليمين

$R_{4\,right} = [\sqrt{3} + 2 + \sqrt{3} + 0](1) = 2 + 2\sqrt{3}$

6. [10 pts.] Evaluate the Riemann sum for $f(x) = 3 - x^2$, taking the sample points to be right endpoints and $a = 1, b = 4$, and $n = 3$.

We have $\boxed{\Delta x = \frac{4-1}{3} = 1 \text{ and } x_i = a + i\Delta x = 1 + i, \text{ for } i = 1, 2, 3}$.

Therefore, $\boxed{R_3 = [f(2) + f(3) + f(4)] \times 1 = [(-1) + (-6) + (-13)] = -20}$.

$f(x) = 3 - x^2$

$a = 1$, $b = 4$, $n = 3$

* sample points to be right

$\Delta x = \frac{b-a}{n} = \frac{4-1}{3} = 1$

x	2	3	4
$f(x)$	-1	-6	-13

$f(2) = -1$
$f(3) = -6$
$f(4) = -13$

$\therefore R_3 = [f(x_1)^* + f(x_2)^* + f(x_3)^*] \Delta x$

$= [-1 + -6 + -13](1) = -20$

II. The Riemann sum of $f(x) = x^2$ on the interval $[1, 5]$ using $n = 4$ and taking the sample points to be the right endpoints is:

a) 25.

✓ b) 54.

c) -30.

d) 0.

e) None of the above.

$f(x) = x^2$

$a = 1$, $b = 5$, $n = 4$

* Sample points to be right

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$$

x	2	3	4	5
$f(x)$	4	9	16	25

$f(2) = 4$
$f(3) = 9$
$f(4) = 16$
$f(5) = 25$

$$\therefore R_4 = [f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)]\,\Delta x$$

$$= [4 + 9 + 16 + 25](1) = 54$$

II. The Riemann sum of $f(x) = x^2$ on the interval $[1, 5]$ using $n = 4$ and taking the sample points to be the ~~right~~ left endpoints is:

a) 25.

b) 54.

c) -30.

d) 0.

e) None of the above.

F) 30 ✓

$$f(x) = x^2$$

$a = 1$, $b = 5$, $n = 4$

* Sample points to be left

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$$

x	1	2	3	4
$f(x)$	1	4	9	16

$f(1) = 1$
$f(2) = 4$
$f(3) = 9$
$f(4) = 16$

$$\therefore R_4 = [f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)]\,\Delta x$$

$$= [1 + 4 + 9 + 16](1) = 30$$

I. The Riemann sum of $f(x) = \sin x$, $0 \leqslant x \leqslant \pi$ with $n = 3$ taking the sample points to be the right endpoints is equal to

a) 0.
b) $\frac{\pi}{3}$.
✓ c) $\frac{\pi}{\sqrt{3}}$.
d) $\frac{\pi(1+\sqrt{3})}{2}$.
e) None of the above.

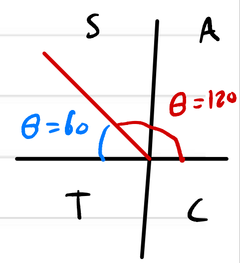
نفس القيمة

$$\frac{\sqrt{3}\pi}{3} * \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\pi}{3\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

$f(x) = \sin x$

$\therefore \sin\frac{2\pi}{3} = +\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$a = 0$, $b = \pi$, $n = 3$



* Sample points to be right

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{3} = \frac{\pi}{3}$$

x	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{3} = \pi$
$f(x)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0

$f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$

$f(\pi) = 0$

$$\therefore R_3 = [f(x_1)^* + f(x_2)^* + f(x_3)^*]\,\Delta x$$

$$= [\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0]\frac{\pi}{3} = (\frac{2\sqrt{3}}{2})(\frac{\pi}{3}) = \frac{\sqrt{3}\pi}{3}$$

1. Evaluate the Riemann sum for $f(x) = x - 1$, $-6 \leqslant x \leqslant 4$, with five subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

or $\int_{-6}^{4} x - 1 \, dx$, $n = 5$ right end

$f(x) = x - 1$

$a = -6$, $b = 4$, $n = 5$

* Sample points to be right

$$\Delta x = \frac{b-a}{n} = \frac{4-(-6)}{5} = 2$$

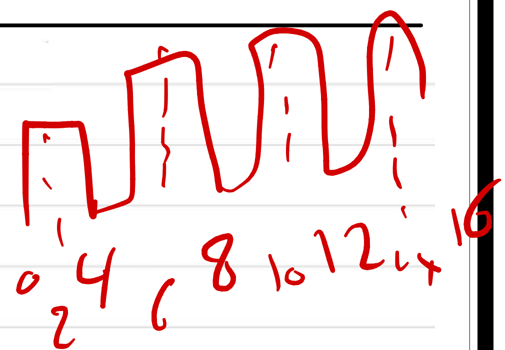
x	-4	-2	0	2	4
$f(x)$	-5	-3	-1	1	3

$$\therefore R_5 = [f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + f(x_5^*)] \, \Delta x$$

$$= [\underbrace{-5 + -3 + -1}_{(-9} + \underbrace{1 + 3}_{+\ 4\ )\ 2}](2) = -10$$

3. If $f(x) = 16 - x$, $0 \leqslant x \leqslant 16$, find the Riemann sum with $n = 4$, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

$f(x) = 16 - x$



$a = 0$, $b = 16$, $n = 4$

* Sample points to be midpoints

$$\Delta x = \frac{b-a}{n} = \frac{16-0}{4} = 4$$

x	2	6	10	14
$f(x)$	14	10	6	2

$$\therefore R_4 = [f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)]\, \Delta x$$

$$= [14 + 10 + 6 + 2](4) = 128$$

The right end $= 96$, The left end $= 160$

إجابة نفس السؤال لو طلب مني من جهة اليمين واليسار

Integration in terms of areas

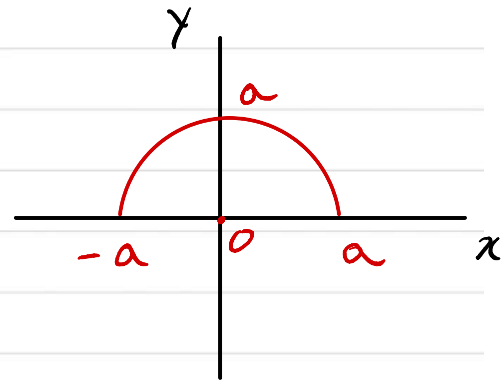
$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$$

$$Y = \sqrt{a^2 - x^2}$$

$$x^2 + Y^2 = a^2$$

$$a^2 = r^2$$

$A = \pi r^2$ مساحة الدائرة



شلون تعرف إذا نص دائرة أو ربع؟

تشوف حدود الكامل

$A = \frac{1}{2} \pi r^2$ مساحة نصف دائرة

$A = \frac{1}{4} \pi r^2$ مساحة ربع الدائرة

Note: Not circle $\int_{-a}^{a} \sqrt{x^2 - a^2}$

EXAMPLE 4 Evaluate the following integrals by interpreting each in terms of areas.

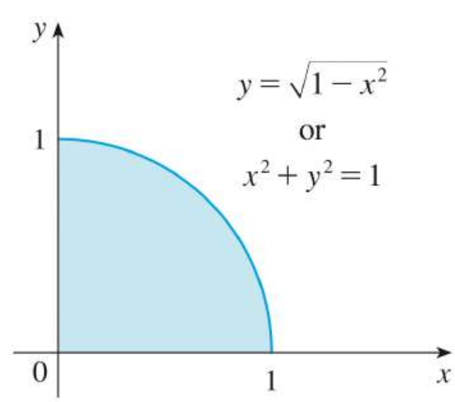
(a) $\int_0^1 \sqrt{1 - x^2}\, dx$

$Y = \sqrt{1 - x^2} \Rightarrow Y^2 = 1 - x^2$

$Y^2 + x^2 = 1$, Area of quarter circle

$A = \frac{1}{4} * \pi * r^2$

$$\int_0^1 \sqrt{1 - x^2}\, dx = \tfrac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$



35–40 Evaluate the integral by interpreting it in terms of areas.

35. $\int_{-1}^{2} (1 - x)\, dx$

نقدر نرسم الدالة عن طريق التعويض بحدود التكامل داخل الدالة

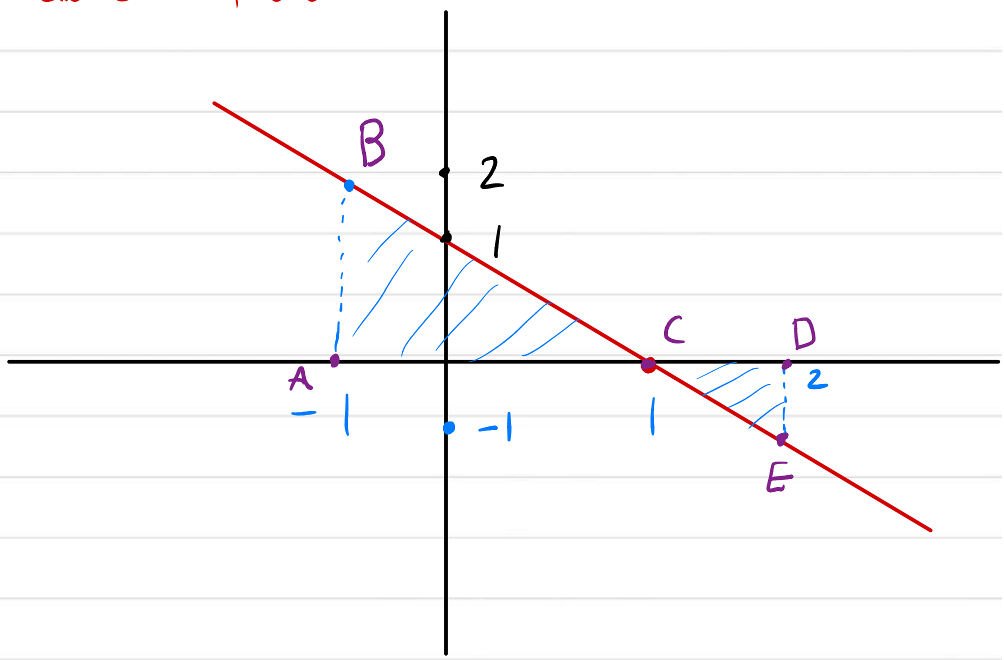
$f(x) = 1 - x$

$f(-1) = 2$

$f(0) = 1$

$f(1) = 0$

$f(2) = -1$



$$\text{Area of Triangle} = \frac{1}{2}(\text{Base})(\text{height})$$

ملاحظة : أي مساحة تحت

الـx-axis رح تكون سالبة

$$\int_{-1}^{2} (1-x)\, dx = \text{area of triangle } ABC + \text{area of triangle } CDE$$

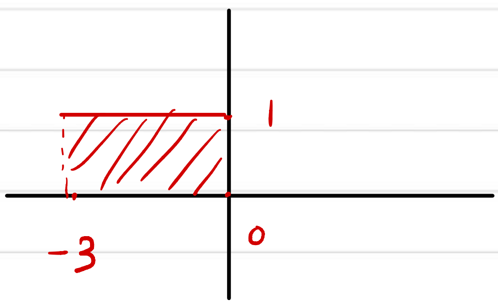
$$= \frac{1}{2}(2)(2) + \left(\frac{1}{2}\right)(1)(-1)$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

35–40 Evaluate the integral by interpreting it in terms of areas.

37. $\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right) dx$

$$\int_{-3}^{0} 1 \, dx + \int_{-3}^{0} \sqrt{9-x^2} \, dx$$



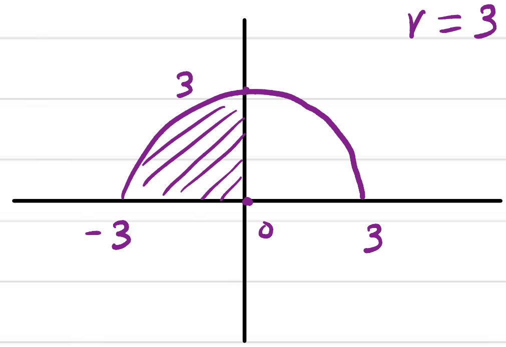
Area of rectangle = L * W

= 1 * 3 = 3

$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$ → upper part of quarter circle with radius $r = 3$



Area of the circle $= \pi r^2$

But we have quarter circle

$\therefore A = \frac{1}{4}\pi(r)^2$

$= \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi$

$\therefore \int_{-3}^{0} 1 \, dx + \int_{-3}^{0} \sqrt{9-x^2} \, dx$

$= 3 + \frac{9\pi}{4}$

IV . $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx =$

a) 1.

b) $\pi/4$.

✓ c) $\pi/2$.

d) 0.

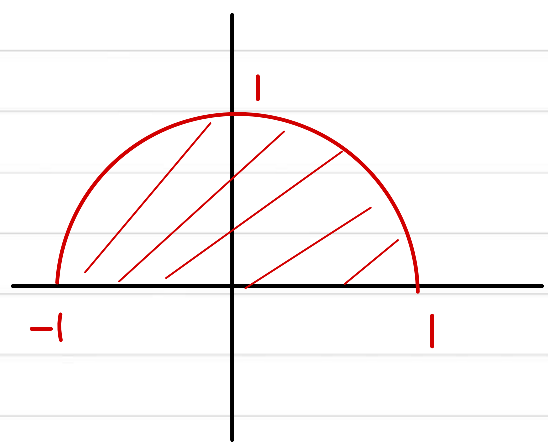
e) None of the above.

$y = \sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2$

$\Rightarrow x^2 + y^2 = 1$ → upper part of semi-circle with $r=1$

Area of the circle $= \pi r^2$

But we have semi-circle



$\therefore A = \frac{1}{2}\pi r^2$

$\therefore \displaystyle\int_{-1}^{1} \sqrt{1-x^2} = \frac{1}{2}\pi(r)^2 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

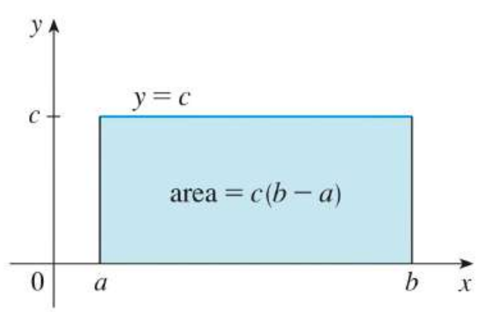


FIGURE 13

$\int_a^b c\,dx = c(b - a)$

EXAMPLE 6 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0) = 4$$

and we found in Example 5.1.2 that $\int_0^1 x^2\,dx = \frac{1}{3}$. So

$$\begin{aligned}\int_0^1 (4 + 3x^2)\,dx &= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx \\ &= 4 + 3 \cdot \tfrac{1}{3} = 5\end{aligned}$$

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

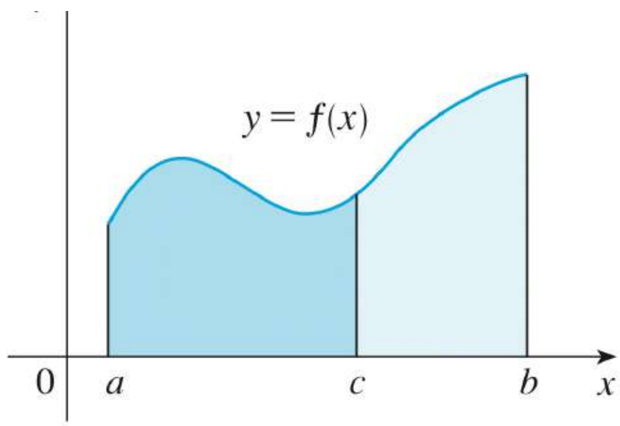


FIGURE 15

EXAMPLE 7 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$

SOLUTION By Property 5, we have

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

so $$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$$ ■

50. Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= \int_0^3 3\,dx + \int_3^5 x\,dx$$

$$= \left[3x\right]_0^3 + \left[\frac{x^2}{2}\right]_3^5$$

$$= 3(3-0) + \frac{1}{2}(5^2-3^2)$$

$$9 + \frac{1}{2}(16)$$

$$9 + 8 = 17$$

48. If $\int_2^8 f(x)\,dx = 7.3$ and $\int_2^4 f(x)\,dx = 5.9$, find $\int_4^8 f(x)\,dx$.

$$\because \int_2^8 f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^8 f(x)\,dx$$

$$\therefore 7.3 = 5.9 + \int_4^8 f(x)\,dx$$

$$\therefore \int_4^8 f(x)\,dx = 1.4$$

49. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find

$$\int_0^9 [2f(x) + 3g(x)]\,dx$$

$$2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx$$

$$2(37) + 3(16) = 74 + 48 = 122$$

II. If $\int_1^5 f(x)dx = 7$, then the value of $\int_1^5 (3f(x)+x)dx$ is equal to

a) 33.

b) 0.

c) 21.

d) 54.

e) None of the above.

$$\int_1^5 3f(x)\,dx + \int_1^5 x\,dx$$

$$3\int_1^5 f(x)\,dx + \left[\frac{x^2}{2}\right]_1^5$$

$$3(7) + \frac{1}{2}(25-1)$$

$$21 + \frac{1}{2}(24) = 21 + 12 = 33$$

6. [10 pts.] Let f and g be two continuous functions such that $\int_2^1 f(x)dx = -1$, $\int_2^3 f(x)dx = 3$, and $\int_1^3 g(x)dx = 5$. Find $\int_1^3 [f(x) - 2g(x)]dx$.

$$\int_1^2 f(x)\,dx = -\int_2^1 f(x)\,dx = -(-1) = 1$$

$$\therefore \int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$\therefore \int_1^3 f(x)\,dx = 1 + 3 = 4$$

$$\therefore \int_1^3 [f(x) - 2g(x)]dx = \int_1^3 f(x)\,dx - 2\int_1^3 g(x)\,dx$$

$$= 4 - 2(5) = -6$$

Answer. We have $\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx = -(-1) + 3 = 4.$

Hence $\int_1^3 (f(x) - 2g(x))\,dx = \int_1^3 f(x)\,dx - 2\int_1^3 g(x)\,dx = 4 - 2(5) = -6.$

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

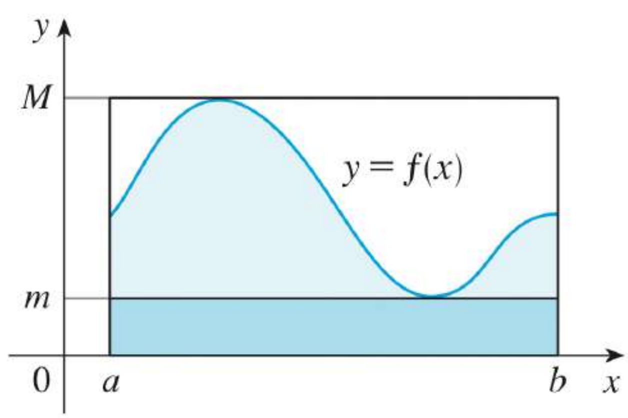


FIGURE 16

Show that (estimate) or use properties of integral

$$4 \leq \int_1^3 \sqrt{x^2+3}\, dx \leq 4\sqrt{3}$$

$$1 \leq x \leq 3$$

$$1 \leq x^2 \leq 9$$

$$4 \leq x^2+3 \leq 12$$

$\sqrt{12} = \sqrt{4*3}$
$= 2\sqrt{3}$

$$2 \leq \sqrt{x^2+3} \leq 2\sqrt{3}$$

$$\int_1^3 2\, dx \leq \int_1^3 \sqrt{x^2+3}\, dx \leq \int_1^3 2\sqrt{3}\, dx$$

$$2(3-1) \leq \int_1^3 \sqrt{x^2+3}\, dx \leq 2\sqrt{3}\,(3-1)$$

$$4 \leq \int_1^3 \sqrt{x^2+3}\, dx \leq 4\sqrt{3}$$

55–58 Use the properties of integrals to verify the inequality without evaluating the integrals.

57. $2 \leqslant \int_{-1}^{1} \sqrt{1+x^2}\, dx \leqslant 2\sqrt{2}$

$$-1 \leq x \leq 1$$ حدود التكامل

$$0 \leq x^2 \leq 1$$

لانه ماله معنى إذا قلت الx
أكبر من الواحد أو تساوي
وبنفس الوقت أهيا أصغر
من الواحد أو تساوي

$$1 \leq x^2+1 \leq 2$$

$$\sqrt{1} \leq \sqrt{x^2+1} \leq \sqrt{2}$$

$$\int_{-1}^{1} 1\, dx \leq \int_{-1}^{1} \sqrt{x^2+1}\, dx \leq \int_{-1}^{1} \sqrt{2}\, dx$$

$$1\,[1-(-1)] \leq \int_{-1}^{1} \sqrt{x^2+1}\, dx \leq \sqrt{2}\,[1-(-1)]$$

$$2 \leq \int_{-1}^{1} \sqrt{x^2+1}\, dx \leq 2\sqrt{2}$$

55–58 Use the properties of integrals to verify the inequality without evaluating the integrals.

56. $\int_0^1 \sqrt{1 + x^2}\, dx \leqslant \int_0^1 \sqrt{1 + x}\, dx$

$$\int_0^1 \sqrt{1+x^2}\, dx \leqslant \int_0^1 \sqrt{1+x}\, dx$$

if $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$

then $\int_a^b f(x)\, dx \geqslant \int_a^b g(x)\, dx$

We know $x^2 \leqslant x$ on $[0,1]$

$$x^2 + 1 \leqslant x + 1$$

$$\sqrt{x^2+1} \leqslant \sqrt{x+1}$$

$$\therefore \int_0^1 \sqrt{1+x^2}\, dx \leqslant \int_0^1 \sqrt{1+x}\, dx$$

Show that (estimate) or

use properties of integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2 x + 3)\, dx \leq \frac{\pi}{3}$$

$$0 \leq \sin^2 x \leq 1$$

$$3 \leq \sin^2 x + 3 \leq 4$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 3 \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2 x + 3 \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4$$

$$3\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2 x + 3 \leq 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$3\,\frac{\pi}{12} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2 x + 3 \leq 4\,\frac{\pi}{12}$$

$$= \frac{\pi}{4} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2 x + 3 \leq \frac{\pi}{3}$$

Calculus A

Chapter 5: Integrals

Sections: 5.3 The Fundamental Theorem of Calculus

A+

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5.3 The Fundamental Theorem of Calculus

نظرية مشتقة التكامل　　　حفظ

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$$\frac{d}{dx}\int_0^{x} f(t)\,dt = f(x)\cdot(1)$$

upper limitمشتقة ال

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x)$$

حفظ　　　رح نشرحها بسكشن الي بعده بالتفصيل

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

$$\int_{-2}^{1} \frac{1}{x}\,dx \neq \Big[\ln|x|\Big]_{-2}^{1}$$

not continuous at $0 \in (-2, 1)$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

find the derivative

$$f(x) = \int_0^x \sqrt{t^2+4}\,dt$$

Where $f(t) = \sqrt{t^2+4}$

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt = \frac{d}{dx}\left[F(t)\right]_0^x\right.$$

$$\frac{d}{dx}\left[F(x) - F(0)\right] = f(x) - 0$$

$$\therefore f(x) = \sqrt{x^2+4}$$

find $\frac{d}{dx}\left[\int_x^4 \sqrt{t^3+5}\; dt\right]$

By the Fundamental Theorem of Calculus

$\because f(t) = \sqrt{t^3+5}$

$$\frac{d}{dx}\left[\int_x^4 f(t)\,dt\right] = \frac{d}{dx}\left[F(t)\right]_x^4$$

$$= \frac{d}{dx}\left[F(4) - F(x)\right] = 0 - f(x)$$

$$= -\sqrt{x^3-5}$$

EXAMPLE 4 Find $\displaystyle \frac{d}{dx}\int_1^{x^4} \sec t\, dt$.

By the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_1^{x^4} \sec t\, dt = \sec(x)^4 \left(\frac{dx^4}{dx}\right)$$

$$= 4x^3 \sec(x)^4$$

7–18 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

7. $g(x) = \int_0^x \sqrt{t + t^3}\, dt$

By the Fundamental Theorem of Calculus

$$g(x) = \int_0^x \sqrt{t+t^3}$$

$$g'(x) = \sqrt{x+x^3}$$

9. $g(s) = \int_5^s (t - t^2)^8\, dt$

By the Fundamental Theorem of Calculus

$$g'(s) = (s - s^2)^8$$

11. $F(x) = \int_x^0 \sqrt{1 + \sec t}\, dt$

$$\left[Hint: \int_x^0 \sqrt{1 + \sec t}\, dt = -\int_0^x \sqrt{1 + \sec t}\, dt \right]$$

By the Fundamental Theorem of Calculus

$$F'(x) = -\sqrt{1+\sec x}$$

12. $R(y) = \int_y^2 t^3 \sin t\, dt$

By the Fundamental Theorem of Calculus

$$R(y) = \int_y^2 t^3 \sin t\, dt = -\int_2^y t^3 \sin t\, dt$$

$$R'(y) = -y^3 \sin y$$

13. $h(x) = \int_1^{e^x} \ln t \, dt$

By FTC :-

$$h'(x) = \ln(e^x) \cdot \frac{de^x}{dx}$$

$$\therefore h'(x) = x \cdot e^x = xe^x$$

17. $y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta \, d\theta$

By FTC :-

$$Y = -\int_{\frac{\pi}{4}}^{\sqrt{x}} \theta \tan \theta$$

$$Y' = \left(-\sqrt{x} \tan\sqrt{x}\right) \frac{d(\sqrt{x})}{dx}$$

$$Y' = \left(-\sqrt{x} \tan\sqrt{x}\right) \left(\frac{1}{2\sqrt{x}}\right)$$

$$Y' = -\frac{\tan\sqrt{x}}{2}$$

18. $y = \int_{\sin x}^{1} \sqrt{1 + t^2}\, dt$

By the Fundamental Theorem of Calculus

$$y = -\int_{1}^{\sin x} \sqrt{1+t^2}\, dt$$

$$y' = -\sqrt{1+\sin^2 x} \quad \frac{d \sin x}{dx}$$

$$y' = -\sqrt{1+\sin^2 x} \quad \cos x$$

$$y' = -\cos x \sqrt{1+\sin^2 x}$$

5. [10 pts.] Find the slope of the tangent line to $F(x) = \int_{\sqrt{x}}^{x^2} \ln(1+2t)dt$ at $x = 1$.

By the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x)$$

$$F'(x) = \ln(1+2x^2)(2x) - \ln(1+2\sqrt{x})\frac{1}{2\sqrt{x}}$$

$$f'(x) = 2x(\ln(1+2x^2)) - \frac{\ln(1+2\sqrt{x})}{2\sqrt{x}}$$

at $x = 1$

$$f'(1) = 2(1)(\ln(1+2(1)^2)) - \frac{\ln(1+2\sqrt{1})}{2\sqrt{1}}$$

$$= 2\ln 3 - \frac{\ln(3)}{2}$$

$$= \frac{4\ln 3}{2} - \frac{\ln 3}{2} = \frac{3\ln 3}{2}$$

We know that $F'(x) = \ln(1+2x^2)2x - \frac{\ln(1+2\sqrt{x})}{2\sqrt{x}}$. Therefore, the slope of the tangent line $m = F'(1) = 2\ln 3 - \frac{\ln 3}{2} = \frac{3\ln 3}{2}$.

7. [10 pts.] Find the inflection points, if any, of the curve

$$y = \int_0^x (1-t)e^t\,dt.$$

By the Fundamental Theorem of Calculus if $y = \int_0^x (1-t)e^t\,dt \Rightarrow y' = (1-x)e^x \Rightarrow y'' = -xe^x$.

The curve y is concave upward when $y'' > 0$; that is, on the interval $(-\infty, 0)$ and concave downward when $y'' < 0$; that is, on the interval $(0, \infty)$. Since f is continuous, therefore the point $(0,0)$ is an inflection point.

By the Fundamental Theorem of Calculus

$$y = \int_0^x (1-t)e^t\,dt \Rightarrow y' = (1-x)e^x \frac{dx}{dt}$$

$$= \quad y' = (1-x)e^x$$

$$y'' = -e^x + (1-x)e^x$$

$$y'' = e^x(-1+1-x) = -xe^x$$

$$f''(c) = 0 \Rightarrow -xe^x = 0 \quad \therefore x = 0 \in D_f$$

	$-\infty$		0		∞
F''		$+$		$-$	
$\cup\ \cap$		$\cup$		$\cap$	

عشان نطلع قيمة الy ..
نعوض قيمة الx بالدالة الاصلية

$$y \int_0^0 (1-t)e^t\,dt = 0$$

inf point $(0,0)$

4. [10 pts.] Find an equation of the tangent line to the graph of the function

$$f(x) = \int_{-1}^{x^3} \frac{\ln(t+2)}{t^2+1}\,dt \quad \text{at } x = -1.$$

Answer. Thanks to the Fundamental Theorem of Calculus, we have $f'(x) = \frac{\ln(x^3+2)}{x^6+1}(3x^2)$ and hence $m_{tan} = f'(-1) = \frac{3\ln(-1+2)}{2} = 0$. In turn, $f(-1) = 0$. Thus, an equation of the tangent line to the graph of f at $x = -1$ is: $y = 0$

By the Fundamental Theorem of Calculus

$$\int_{-1}^{x^3} \frac{\ln(t+2)}{t^2+1}\,dt \Rightarrow f'(x) = \frac{\ln(x^3+2)}{x^6+1}(3x^2)$$

$$f'(-1) = \frac{\ln((-1)^3+2)}{(-1)^6+1}(3(-1)^2) = \frac{3(\ln(1))}{2}$$

$$= \frac{3(0)}{2} = 0$$

$$\therefore\ y - y_1 = f'(-1)(x - x_1)$$

* to find y_1, sub $x = -1$ in $f(x)$

$$f(x) = \int_{-1}^{-1} \frac{\ln(t+2)}{t^2+1}\,dt = 0$$

$$\therefore\ y - 0 = 0(x-(-1)) = \boxed{y = 0}$$

find $f(x)$ if f is continous

function such that:

$$\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt$$

By the Fundamental Theorem of Calculus

By diff. Both side

$$f(x) = (1)\sin x + x\cos x + \frac{f(x)}{1+x^2}$$

$$f(x) - \frac{f(x)}{1+x^2} = \sin x + x\cos x$$

$$f(x)\left[1 - \frac{1}{1+x^2}\right] = \sin x + x\cos x$$

$$f(x)\left[\frac{x^2}{1+x^2}\right] = \sin x + x\cos x$$

$$f(x) = \frac{1+x^2}{x^2}\left[\sin x + x\cos x\right]$$

5. [10 pts.] Let $f(x) = \displaystyle\int_1^{x^2+1} \frac{t^2}{\sqrt{t^2+1}}\,dt$. Find the absolute minimum value of f, if any.

By the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_1^{x^2+1} \frac{t^2}{\sqrt{t^2+1}}\,dt = \frac{(x^2+1)^2}{\sqrt{(x^2+1)^2+1}}(2x)$$

$$F'(x) = 0 \Rightarrow (x^2+1)^2(2x) = 0$$

$x = 0 \in$ Domain

	$-\infty$		0		∞
$f'(x)$		$-$		$+$	
		↘		↗	

$$f(0) = \int_1^{(0)^2+1=1} \frac{t^2}{\sqrt{t^2+1}}\,dt = 0$$

The abs. min. at $\underset{x}{(0},\ \underset{f(x)}{0)}$

Q9. [10 pts.] On what interval is the curve $y = \int_0^x \frac{1-t}{t^2+8}\,dt$ concave downward?

By the FTC, we have $\boxed{f'(x) = \frac{1-x}{x^2+8}}$.

$$\boxed{f''(x) = \frac{-(x^2+8) - 2x(1-x)}{(x^2+8)^2} = \frac{x^2-2x-8}{(x^2+8)^2} = \frac{(x+2)(x-4)}{(x^2+8)^2}}.$$

Observe that, $\boxed{f''(x) > 0 \text{ on } (-\infty,-2) \text{ and } (4,\infty), \text{ and } f''(x) < 0 \text{ on } (-2,4)}$.

By the Concavity test, the graph of f is concave downward on $\boxed{(-2,4)}$.

By the Fundamental Theorem of Calculus

$$y' = \frac{1-x}{x^2+8}$$

$$y'' = \frac{-(x^2+8) - 2x(1-x)}{(x^2+8)^2} = \frac{-x^2-8-2x+2x^2}{(x^2+8)^2}$$

$$= \frac{x^2-2x-8}{(x^2+8)^2} = \frac{(x+2)(x-4)}{(x^2+8)^2}$$

$\therefore f''(x) = 0$, $(x+2)(x-4) = 0$ $\therefore x = -2$, $x = 4$

	$-\infty$ to -2	-2 to 4	4 to ∞
f''	$+$	$-$	$+$
$\cup\ \cap$	$\cup$	$\cap$	$\cup$

f is concave downward on $(-2,4)$

59–63 Find the derivative of the function

59. $g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\, du$

By FTC :-

$$g'(x) = \frac{(3x)^2 - 1}{(3x)^2 + 1}(3) - \frac{(2x)^2 - 1}{(2x)^2 + 1}(2)$$

$$g'(x) = \frac{3(9x^2 - 1)}{(9x^2 + 1)} - \frac{2(4x^2 - 1)}{4x^2 + 1}$$

60. $g(x) = \int_{1-2x}^{1+2x} t \sin t\, dt$

By FTC :-

$$g'(x) = (1 + 2x)(\sin(1+2x)(2) - (1 - 2x)(\sin(1-2x)(-2)$$

$$= (2 + 4x)(\sin(1+2x)) - (4x-2)(\sin(1-2x)$$

59–63 Find the derivative of the function

61. $F(x) = \int_{x}^{x^2} e^{t^2}\, dt$

By FTC :-

$$F'(x) = e^{(x^2)^2}(2x) - e^{(x)^2}(1)$$

$$= 2x\, e^{x^4} - e^{x^2}$$

62. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t\, dt$

By FTC :-

$$F'(x) = \tan^{-1}(2x)\,(2) - \tan^{-1}(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$

$$= 2\tan^{-1}(2x) - \frac{\tan^{-1}\sqrt{x}}{2\sqrt{x}}$$

Calculus A

Chapter 5: Integrals

Sections: 5.4 Indefinite Integrals and the ~~Net Change Theorem~~

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For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

حفظ

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

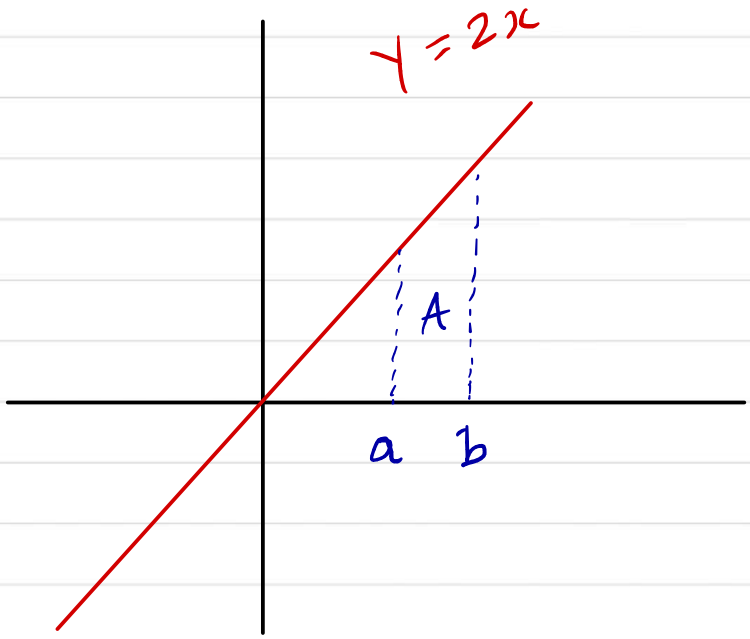
$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2 + 1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

شرح :-



$$\int_1^2 2x\,dx$$

we know $\int 2x\,dx = x^2 + C$

at $x = 1$: $\int 2x\,dx = 1^2 + C$

at $x = 2$: $\int 2x\,dx = 2^2 + C$

* subtract $(2^2 + C) - (1^2 + C)$

$= 4 + C - 1 - C = 3$

حفظ

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

$$\int_1^2 2x\,dx = 2\left[\frac{x^2}{2}\right]_1^2 = \left[x^2\right]_1^2$$

$$= \left[2^2\right] - \left[1^2\right] = 3$$

ملاحظة :

التكامل يتوزع بالجمع والطرح

التكامل لا يتوزع بالضرب والقسمة

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3} x^{3/2} + C$$

$$\int 10^x\,dx = \frac{10^x}{\ln 10} + C$$

$$\int x^{10}\,dx = \frac{x^{11}}{11} + C$$

$$\int e^x\,dx = e^x + C$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\,dx &= 10\int x^4\,dx - 2\int \sec^2 x\,dx \\ &= 10\,\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

EXAMPLE 4 Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

SOLUTION The Fundamental Theorem gives

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= 2\,\frac{x^4}{4} - 6\,\frac{x^2}{2} + 3\tan^{-1}x\Big]_0^2 \\ &= \tfrac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big]_0^2 \\ &= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0 \\ &= -4 + 3\tan^{-1}2\end{aligned}$$

1–4 Verify by differentiation that the formula is correct.

1. $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx = -\frac{\sqrt{1+x^2}}{x} + C$

$$= -\left(\frac{\frac{1}{2}(1+x^2)^{-1/2}(2x)\,x - \sqrt{1+x^2}}{x^2}\right) + 0$$

$$= -\left(\frac{\frac{x^2}{\sqrt{1+x^2}} - \sqrt{1+x^2}}{x^2}\right)$$

$$= -\left(\frac{\frac{x^2}{\sqrt{1+x^2}} - \frac{(\sqrt{1+x^2})(\sqrt{1+x^2})}{\sqrt{1+x^2}}}{x^2}\right)$$

$$= -\left(\frac{\frac{x^2-(1+x^2)}{\sqrt{1+x^2}}}{x^2}\right) = -\left(\frac{x^2-1-x^2}{x^2\sqrt{1+x^2}}\right)$$

$$= -\left(\frac{-1}{x^2\sqrt{1+x^2}}\right) = \frac{1}{x^2\sqrt{1+x^2}}$$

1–4 Verify by differentiation that the formula is correct.

3. $\int \tan^2 x \, dx = \tan x - x + C$

$$\frac{d}{dx}\left(\tan x - x + C\right)$$

$$= \sec^2 x - 1 + 0 = \sec^2 x - 1$$

$$\because \sec^2 x = \tan^2 x + 1$$

$$\therefore \sec^2 x - 1 = \tan^2 x$$

EXAMPLE 5 Evaluate the integral $\int_1^3 e^x\,dx$.

SOLUTION The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

SOLUTION An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} \qquad \blacksquare$$

EXAMPLE 7 Evaluate $\int_3^6 \frac{dx}{x}$.

SOLUTION The given integral is an abbreviation for

$$\int_3^6 \frac{1}{x}\,dx$$

An antiderivative of $f(x) = 1/x$ is $F(x) = \ln|x|$ and, because $3 \leqslant x \leqslant 6$, we can write $F(x) = \ln x$. So

$$\int_3^6 \frac{1}{x}\,dx = \ln x\Big]_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2 \qquad \blacksquare$$

EXAMPLE 8 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

5–18 Find the general indefinite integral.

7. $\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right) dx$

$$= \int 5\,dx + \frac{2}{3}\int x^2\,dx + \frac{3}{4}\int x^3\,dx$$

$$5x + \frac{2}{3}\,\frac{x^3}{3} + \frac{3}{4}\,\frac{x^4}{4} + C$$

$$5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$$

5–18 Find the general indefinite integral.

9. $\int (u + 4)(2u + 1)\, du$

$$\int (2u^2 + u + 8u + 4)\, du$$

$$\left[2\frac{u^3}{3} + \frac{u^2}{2} + 8\frac{u^2}{2} + 4u + C \right]$$

$$\left[\frac{2u^3}{3} + \frac{9u^2}{2} + 4u + C \right]$$

19–44 Evaluate the integral.

19. $\displaystyle\int_1^3 (x^2 + 2x - 4)\,dx$

$$= \left[\frac{x^3}{3} + 2\left(\frac{x^2}{2}\right) - 4x \right]_1^3$$

$$= \left[\frac{x^3}{3} + x^2 - 4x \right]_1^3$$

$$= \left[\frac{3^3}{3} + 3^2 - 4(3) \right] - \left[\frac{1^3}{3} + 1^2 - 4(1) \right]$$

$$= \frac{27}{3} + 9 - 12 - \left[\frac{1}{3} + 1 - 4 \right]$$

$$= 9 + 9 - 12 - \left[\frac{4}{3} - 4 \right]$$

$$= 6 - \left[\frac{4}{3} - \frac{12}{3} \right] = 6 - \left(-\frac{8}{3}\right)$$

$$\therefore 6 + \frac{8}{3} = \frac{18}{3} + \frac{8}{3} = \frac{26}{3}$$

14. $\int \left(\frac{1+r}{r}\right)^2 dr$

$$\int \frac{(1+r)^2}{r^2}\,dr = \int \frac{1^2+2r+r^2}{r^2}\,dr$$

$$\int \left(\frac{1}{r^2} + \frac{2r}{r^2} + \frac{r^2}{r^2}\right) dr$$

$$\int \left(r^{-2} + 2\frac{1}{r} + 1\right) dr$$

$$\frac{r^{-1}}{-1} + 2\ln|r| + r + C$$

$$-\frac{1}{r} + 2\ln|r| + r + c$$

21. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt$

$$\left[\frac{4}{5}\left(\frac{t^4}{4}\right) - \frac{3}{4}\left(\frac{t^3}{3}\right) + \frac{2}{5}\left(\frac{t^2}{2}\right)\right]_0^2$$

$$\left[\frac{t^4}{5} - \frac{t^3}{4} + \frac{t^2}{5}\right]_0^2$$

$$\left[\frac{(2)^4}{5} - \frac{(2)^3}{4} + \frac{(2)^2}{5}\right] - \left[\frac{0^4}{5} - \frac{0^3}{4} + \frac{0^2}{5}\right]$$

$$= \left[\frac{16}{5} + \frac{4}{5} - \frac{8}{4}\right] = \frac{20}{5} - \frac{4}{2}$$

$$= 4 - 2 = 2$$

25. $\displaystyle\int_{\pi/6}^{\pi} \sin\theta \, d\theta$

$$= \left[-\cos\theta\right]_{\frac{\pi}{6}}^{\pi} = -\left[\cos\theta\right]_{\frac{\pi}{6}}^{\pi}$$

$$= -\left[\cos(\pi) - \cos\left(\frac{\pi}{6}\right)\right]$$

$$= -\left[-1 - \frac{\sqrt{3}}{2}\right] = 1 + \frac{\sqrt{3}}{2}$$

26. $\displaystyle\int_{-5}^{5} e \, dx$

$$\left[ex\right]_{-5}^{5} = e\left[x\right]_{-5}^{5}$$

$$= e\left[5-(-5)\right] = 10e$$

23. $\int_1^9 \sqrt{x}\,dx$

$$= \int_1^9 x^{1/2} = \left[\frac{x^{3/2}}{3/2} \right]_1^9$$

$$= \frac{2}{3} \left[x^{3/2} \right]_1^9 = \frac{2}{3} \left[(\sqrt{x})^3 \right]_1^9$$

$$= \frac{2}{3} \left[(\sqrt{9})^3 - (\sqrt{1})^3 \right]$$

$$= \frac{2}{3} \left[(3)^3 - (1)^3 \right]$$

$$= \frac{2}{3} [27 - 1] = \frac{2}{3} (26) = \frac{52}{3}$$

29. $\displaystyle\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx$

$$= \int_1^4 \left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right) = \int_1^4 \left(2x^{-1/2} + x^{3/2}\right)$$

$$= \left[4x^{1/2} + \frac{2}{5}x^{5/2}\right]_1^4$$

$$\left[4(4)^{1/2} + \frac{2}{5}(4)^{5/2}\right] - \left[4(1)^{1/2} + \frac{2}{5}(1)^{5/2}\right]$$

$$\left[4(2) + \frac{2}{5}(\sqrt{4})^5\right] - \left[4 + \frac{2}{5}\right]$$

$$\left[8 + \frac{2}{5}(32)\right] - \left[\frac{20}{5} + \frac{2}{5}\right]$$

$$8 + \frac{64}{5} - \frac{22}{5} = 8 + \frac{42}{5} = \frac{40}{5} + \frac{42}{5}$$

$$= \frac{82}{5}$$

35. $\displaystyle\int_1^2 \frac{v^3 + 3v^6}{v^4}\,dv$

$$\int_1^2 \frac{v^3}{v^4} + \frac{3v^6}{v^4} = \int_1^2 (v^{-1} + 3v^2)$$

$$\left[\ln|v| + 3\left(\frac{v^3}{3}\right)\right]_1^2 = \left[\ln|v| + v^3\right]_1^2$$

$$\ln|2| + 2^3 - [\ln 1 + 1]$$

$$\ln(2) + 8 - 0 - 1 = \ln(2) + 7$$

36. $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\, dz$

$$\int_1^{18} \frac{\sqrt{3}}{\sqrt{z}} = \sqrt{3}\int_1^{18} z^{-1/2}$$

$$\sqrt{3}\left[\frac{z^{1/2}}{\frac{1}{2}}\right]_1^{18} = 2\sqrt{3}\left[z^{1/2}\right]_1^{18}$$

$$= 2\sqrt{3}\left(18^{1/2} - 1^{1/2}\right) = 2\sqrt{3}\left(\sqrt{18} - 1\right)$$

We know $\sqrt{18} = \sqrt{9*2} = 3\sqrt{2}$

$$2\sqrt{3}\left(3\sqrt{2} - 1\right) = 6\sqrt{2}\sqrt{3} - 2\sqrt{3}$$

$$= 6\sqrt{6} - 2\sqrt{3}$$

21. $\int_{-2}^{3} (x^2 - 3)\, dx$

$$= \left[\frac{x^3}{3} - 3x \right]_{-2}^{3}$$

$$= \left[\frac{(3)^3}{3} - 3(3) \right] - \left[\frac{(-2)^3}{3} - 3(-2) \right]$$

$$= \left[\frac{27}{3} - 9 \right] - \left[\frac{-8}{3} + 6 \right]$$

$$= [9 - 9] - \left[\frac{-8}{3} + \frac{18}{3} \right]$$

$$= 0 - \left[\frac{10}{3} \right]$$

$$= -\frac{10}{3}$$

27. $\int_0^{\pi} (5e^x + 3\sin x)\,dx$

$$5\int_0^{\pi} e^x + 3\int_0^{\pi} \sin x$$

$$5\left[e^x\right]_0^{\pi} + 3\left[-\cos x\right]_0^{\pi}$$

$$5\left[e^{\pi} - e^0\right] - 3\left[\cos\pi - \cos 0\right]$$

$$5\left[e^{\pi} - 1\right] - 3\left[-1 - 1\right]$$

$$5e^{\pi} - 5 - 3(-2)$$

$$5e^{\pi} - 5 + 6 = 5e^{\pi} + 1$$

37. $\int_0^1 (x^e + e^x)\,dx$

$$\left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1$$

$$\left[\frac{1^{e+1}}{e+1} + e^1\right] - \left[\frac{0^{e+1}}{e+1} + e^0\right]$$

$$= \frac{1}{e+1} + e - [0 + 1]$$

$$= \frac{1}{e+1} + e - 1 = \frac{1}{e+1} + \frac{(e+1)(e-1)}{e+1}$$

$$\frac{1 + e^2 - 1}{e+1} = \frac{e^2}{e+1}$$

41. $\displaystyle\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}}$

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{dr}{\sqrt{1-r^2}} = \left[\sin^{-1}(r)\right]_0^{\sqrt{3}/2}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin(0)$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

30. $\displaystyle\int_0^1 \frac{4}{1+p^2}\, dp$

$$= 4\int_0^1 \frac{1}{1+p^2}\, dp = 4\left[\tan^{-1} p\right]_0^1$$

$$= 4\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= 4\left[\frac{\pi}{4} - 0\right] = \pi$$

34. $\int_0^1 (5x - 5^x)\, dx$

$$= \left[5\frac{x^2}{2} - \frac{5^x}{\ln 5} \right]_0^1$$

$$= \left[5\frac{1}{2} - \frac{5}{\ln 5} \right] - \left[5\frac{0}{2} - \frac{5^0}{\ln 5} \right]$$

$$= \frac{5}{2} - \frac{5}{\ln 5} - \frac{1}{\ln 5} = \frac{5}{2} - \frac{4}{\ln 5}$$

35. $\int_0^1 (x^{10} + 10^x)\, dx$

$$= \frac{1}{11} + \frac{9}{\ln 10}$$

Integrating an Absolute Value

مافي تكامل حق المطلق

ولكن كل الي نقدر نسويه أن نعيد تعريفه

نطلع أصفار المطلق ولازم نتأكد من أصفار المطلق تنتمي للفترة الي أبي أكاملها

5.	$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

$$\int_0^4 |x-1|\,dx$$

$x - 1 = 0$

$x = 1 \in (0,4)$

$-\quad +$

$0 \quad 1 \quad 4$

$$\int_0^4 |x-1|\,dx = \int_0^1 -(x-1)\,dx + \int_1^4 +(x-1)\,dx$$

$$\int_{0}^{3} |x^2+2x-3| \, dx$$

$x^2+2x-3=0$ * نعيد تعريف المطلقة

$(x-1)(x+3)=0$

$x=1 \in (0,3)$ But $x=-3 \notin (0,3)$

$$\therefore \int_{0}^{3} |x^2+2x-3| \, dx = -\int_{0}^{1} x^2+2x-3 \, dx + \int_{1}^{3} x^2+2x-3 \, dx$$

$$= -\left[\frac{x^3}{3}+x^2-3x\right]_0^1 + \left[\frac{x^3}{3}+x^2-3x\right]_1^3$$

$$= -\left(\frac{1^3}{3}+1-3\right)-(0+0-0)+\left(\frac{27}{3}+9-9\right)-\left(\frac{1}{3}+1-3\right)$$

$$= +\frac{5}{3}+9+\frac{5}{3}=\frac{37}{3}$$

44. $\int_0^2 |2x - 1|\, dx$

$2x - 1 = 0$

$x = \frac{1}{2}$

$-$ $+$

0 $\frac{1}{2}$ 2

$$|2x-1| = \begin{cases} -(2x-1) & x < 1/2 \\ 2x-1 & x > 1/2 \end{cases}$$

$$\therefore \int_0^2 |2x-1|\,dx = \int_0^{1/2} |2x-1|\,dx + \int_{1/2}^{2} |2x-1|\,dx$$

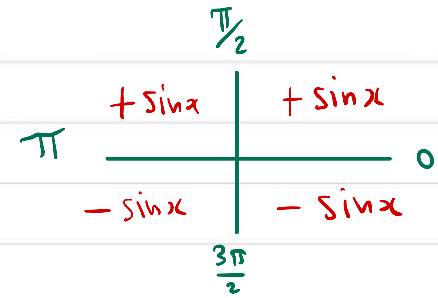
$$= \int_0^{1/2} 1-2x \; dx + \int_{1/2}^{2} 2x-1 \; dx$$

$$= [x - x^2]_0^{1/2} + [x^2 - x]_{1/2}^{2}$$

$$= [\tfrac{1}{2} - \tfrac{1}{4}] - [0 - 0] + [4 - 2] - [\tfrac{1}{4} - \tfrac{1}{2}]$$

$$= \frac{1}{4} + 2 + \frac{1}{4} = \frac{5}{2}$$

46. $\displaystyle\int_0^{3\pi/2} |\sin x|\, dx$



$$|\sin x| = \begin{cases} \sin x & 0 < x < \pi \\ -\sin x & \pi < x < \frac{3\pi}{2} \end{cases}$$

$$\int_0^{\frac{3\pi}{2}} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{3\pi/2} -\sin x\, dx$$

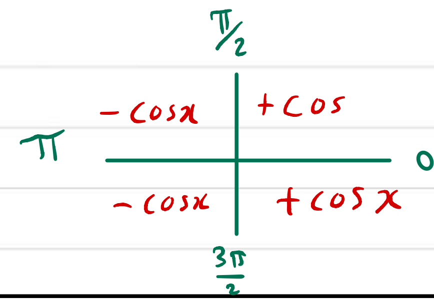
$$= \Big[-\cos x\Big]_0^{\pi} - \Big[-\cos x\Big]_{\pi}^{\frac{3\pi}{2}}$$

$$-\left[\cos \pi - \cos 0\right] + \left[\cos \frac{3\pi}{2} - \cos \pi\right]$$

$$= -\left[-1 - (1)\right] + \left[\,0 - (-1)\right]$$

$$= 2 + 1 = 3$$

$$\int_0^{\pi} |\cos x|\, dx$$



$$\int_0^{\frac{\pi}{2}} \cos x\, dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x\, dx$$

$$[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$[\sin \tfrac{\pi}{2} - \sin 0] - [\sin \pi - \sin \pi/2]$$

$$[\ 1 - 0\ ] - [\ 0 - 1\ ]$$

$$= 1 + 1 = 2$$

Calculus A

Chapter 5: Integrals

Sections: 5.5 The Substitution Rule

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YouTube: Precaclulusq8

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

شلون أختار u : ١) أختار الu للدالة الي مشتقتها موجودة معاها

٢) جرب داخل الدوال

٣) جرب الدالة الي مرفوعة أكبر أس

٤) جرب الدالة الموجودة بالمقام

خطوات الحل

١- أختار الu
٢- اشتقها بالنسبة للمتغير
٣- لازم التكامل يصير بمتغير واحد الحين بس
٤- ييب التكامل الغير محدد
٥- شيل u وحط مكانها الدالة الاصلية (مع تكاملها إذا كان محدد)

---

* special case **46.** $\int x^2\sqrt{2+x}\,dx$ $= \int (u-2)^2\sqrt{u}\,du$

$u = 2 + x \Rightarrow (u-2) = x \Rightarrow (u-2)^2 = x^2$

$du = dx$

1–6 Evaluate the integral by making the given substitution.

1. $\int \cos 2x \, dx, \quad u = 2x$

$u = 2x$ $\qquad\qquad$ $du = 2\,dx$

$$\frac{du}{2} = dx$$

$$\int \cos(2x)\, dx = \int \cos(u) \frac{du}{2}$$

$$= \frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin u + C$$

$$= \frac{1}{2} \sin 2x + C$$

2. $\int xe^{-x^2}\,dx, \quad u = -x^2$

$u = -x^2$

$du = -2x\,dx$

$\dfrac{du}{-2} = x\,dx$

$$\int x e^{-x^2} dx = \int e^u \frac{-du}{2}$$

$$= \frac{-1}{2}\int e^u\,du = \frac{-1}{2} e^u + C$$

$$= -\frac{1}{2} e^{-x^2} + C$$

1–6 Evaluate the integral by making the given substitution.

6. $\int \sqrt{2t+1}\, dt, \quad u = 2t + 1$

$u = 2t + 1$ $\qquad$ $du = 2\,dt$

$\dfrac{du}{2} = dt$

$$= \int \sqrt{2t+1}\; dt = \int \sqrt{u}\, \frac{du}{2} = \frac{1}{2} \int u^{1/2}\, dt$$

$$= \frac{1}{2}\left(\frac{2}{3} u^{3/2}\right) + C = \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3}(2t+1)^{3/2} + C = \frac{(3t+1)^{3/2}}{3} + C$$

3. $\int x^2\sqrt{x^3+1}\,dx, \quad u = x^3+1$

$u = x^3 + 1$ $\qquad du = 3x^2\,dx$

$\frac{du}{3} = x^2\,dx$

$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,\frac{du}{3}$

$= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right)$

$= \frac{2}{9}u^{3/2} = \frac{2}{9}(x^3+1)^{3/2} + C$

4. $\int \sin^2\theta \cos\theta \, d\theta, \quad u = \sin\theta$

$u = \sin\theta$ $\qquad$ $du = \cos\theta \, d\theta$

$$\int \sin^2\theta \cos\theta \, d\theta = \int u^2 \, du$$

$$= \frac{u^3}{3} + C = \frac{(\sin\theta)^3}{3} + C$$

15. $\int \cos^3\theta \sin\theta \, d\theta$

$u = \cos\theta$

$$= -\frac{\cos^4\theta}{4} + C$$

$$\int \cos^3(5x)\sin(5x)\,dx$$

$u = \cos(5x)$ $\quad\quad du = -5\sin(5x)\,dx$

$-\frac{1}{5}du = \sin(5x)\,dx$

$$\int u^3\left(-\frac{1}{5}\right)du = -\frac{1}{5}\int u^3\,du$$

$$-\frac{1}{5}\left[\frac{u^4}{4}\right] + C = -\frac{1}{20}u^4 + C$$

$$= -\frac{1}{20}\left(\cos^4(5x)\right) + C$$

5. $\int \frac{x^3}{x^4 - 5}\,dx, \quad u = x^4 - 5$

$u = x^4 - 5$ $\qquad du = 4x^3\,dx$

$$\frac{du}{4} = x^3\,dx$$

$$\int \frac{x^3}{x^4-5}\,dx = \int \frac{1}{u}\,\frac{du}{4}$$

$$= \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C$$

$$\frac{1}{4}\ln|x^4 - 5| + C$$

9. $\int (1 - 2x)^9 \, dx$

$$u = 1 - 2x \qquad du = -2\,dx$$

$$\frac{du}{-2} = dx$$

$$\int (1-2x)^9 \, dx = \int u^9 \left(-\frac{du}{2}\right)$$

$$-\frac{1}{2}\int u^9 \, du = -\frac{1}{2}\left[\frac{u^{10}}{10}\right] + C$$

$$= -\frac{1}{20} u^{10} + C = -\frac{1}{20}(1-2x)^{10} + C$$

16. $\int e^{-5r}\,dr$

$u = -5r$ $\qquad du = -5\,dr$

$-\dfrac{du}{5} = dr$

$$\int e^{-5r} = \int e^{u}\left(-\frac{du}{5}\right)$$

$$-\frac{1}{5}\int e^{u}\,du = -\frac{1}{5}e^{u} + C$$

$$= -\frac{1}{5}e^{-5r} + C$$

17. $\displaystyle\int \frac{e^u}{(1-e^u)^2}\,du$

$t = 1 - e^u \qquad dt = -e^u\,du$

$-dt = e^u\,du$

$$\int \frac{e^u}{(1-e^u)^2}\,du = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2}\,dt = -\left[\frac{t^{-1}}{-1}\right] + C$$

$$= t^{-1} + C = \frac{1}{t} + C = \frac{1}{1-e^u} + C$$

19. $\displaystyle\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

$$u = 3ax + bx^3 \qquad du = (3a + 3bx^2)\,dx$$

$$du = 3(a + bx^2)\,dx$$

$$\frac{du}{3} = (a + bx^2)\,dx$$

$$\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx = \int \frac{1}{\sqrt{u}}\,\frac{1}{3}\,du$$

$$= \frac{1}{3}\int u^{-1/2}\,du \qquad = \frac{1}{3}\left[\frac{2}{1}u^{1/2}\right] + C$$

$$= \frac{2}{3}u^{1/2} + C \qquad = \frac{2}{3}\sqrt{3ax + bx^3} + C$$

21. $\int \frac{(\ln x)^2}{x}\,dx$

$u = \ln x$ $\qquad du = \frac{1}{x}\,dx$

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du$$

$$\frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

23. $\int \sec^2\theta \tan^3\theta \, d\theta$

$u = \tan\theta \qquad du = \sec^2\theta \, d\theta$

$$\int \sec^2\theta \tan^3\theta \, d\theta = \int u^3 \, du$$

$$= \frac{u^4}{4} + C \qquad = \frac{\tan^4\theta}{4} + C$$

25. $\int e^x \sqrt{1 + e^x}\, dx$

$u = 1 + e^x$ $\qquad du = e^x dx$

$$\int e^x \sqrt{1+e^x}\, dx = \int \sqrt{u}\, du$$

$$\frac{2}{3} u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$$

28. $\int e^{\cos t} \sin t \, dt$

$u = \cos t \qquad du = -\sin t \, dt$

$-du = \sin t \, dt$

$$\int e^{\cos t} \sin t \, dt = \int e^{u} (-du)$$

$$= -\int e^{u} \, du = -e^{u} + C$$

$$= -e^{\cos t} + C$$

30. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x}\,dx$

$u = \tan x \qquad du = \sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan^2 x} = \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du = \left[\frac{u^{-1}}{-1}\right] + C$$

$$= -\frac{1}{u} + C = -\frac{1}{\tan x} + C$$

$$= -\cot x + C$$

44. $\displaystyle\int \frac{x}{1+x^4}\,dx$

$$\int \frac{x}{1+(x^2)^2}$$

$u = x^2$ $\qquad\qquad$ $du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int \frac{1}{1+u^2}\left(\frac{du}{2}\right)$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C$$

$$\frac{1}{2}\tan^{-1}x^2 + C$$

46. $\int x^2\sqrt{2+x}\,dx$

$u = 2 + x \Rightarrow du = 1\,dx$

$u - 2 = x$

$(u-2)^2 = x^2$

$\therefore \int x^2 \sqrt{2+x^2}\,dx = \int (u-2)^2 \sqrt{u}\,du$

$\int (u^2 - 4u + 4)\,u^{1/2}\,du$

$\int \left(u^{5/2} - 4u^{3/2} + 4u^{1/2}\right) du$

$= \frac{2}{7}u^{7/2} - 4\left(\frac{2}{5}u^{5/2}\right) + 4\left(\frac{2}{3}u^{3/2}\right) + C$

$= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C$

47. $\int x(2x+5)^8\,dx$

$u = 2x + 5$ $\qquad$ $du = 2\,dx$

$x = \dfrac{u-5}{2}$ $\qquad$ $\dfrac{du}{2} = dx$

$$\int x(2x+5)^8\,dx = \int \left(\frac{u-5}{2}\right) u^8 \frac{1}{2}\,du$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int (u-5)\,u^8\,du = \frac{1}{4}\int u^9 - 5u^8\,du$$

$$= \frac{1}{4}\left[\frac{u^{10}}{10} - 5\left(\frac{u^9}{9}\right)\right] + C$$

$$= \frac{1}{4}\left[\frac{(2x+5)^{10}}{10} - \frac{5(2x+5)^9}{9}\right] + C$$

55. $\int_0^1 \sqrt[3]{1+7x}\,dx$

$$u = 1 + 7x \qquad du = 7\,dx$$

$$\frac{du}{7} = dx$$

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int u^{1/3} \cdot \frac{du}{7}$$

$$= \frac{1}{7}\int u^{1/3}\,du = \frac{1}{7}\left[\frac{3}{4}u^{4/3}\right]$$

$$= \frac{3}{28}\left[u^{4/3}\right] = \frac{3}{28}\left[(1+7x)^{4/3}\right]_0^1$$

$$\frac{3}{28}\left[(1+7(1))^{4/3} - (1+7(0))^{4/3}\right]$$

$$\frac{3}{28}[8]^{4/3} - (1) = \frac{3}{28}[16-1] = \frac{45}{28}$$

59. $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

$$u = \frac{1}{x} \qquad du = \frac{-1}{x^2}\,dx$$

$$-du = \frac{1}{x^2}\,dx$$

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int e^u(-du)$$

$$= -\int e^u\,du = -e^u = -\left[e^{1/x}\right]_1^2$$

$$= -\left[e^{1/2} - e^1\right] = e - e^{1/2}$$

60. $\int_0^1 xe^{-x^2}\,dx$

$u = -x^2$ $\qquad du = -2\,dx$

$$\frac{du}{-2} = x\,dx$$

$$\int_0^1 xe^{-x^2} = \int e^u \left(-\frac{du}{2}\right) = -\frac{1}{2}\int e^u\,du$$

$$-\frac{1}{2}\left[e^{-x^2}\right]_0^1 = -\frac{1}{2}\left[e^{-1} - e^0\right]$$

$$= -\frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2} - \frac{1}{2e}$$

62. $\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$

$u = \sin x \qquad du = \cos x\, dx$

$$\int_0^{\pi/2} \cos x \sin(\sin x)\, dx = \int \sin(u)\, du$$

$$= -\cos u = -\left[\cos(\sin x)\right]_0^{\pi/2}$$

$$= -\left[\cos\left(\sin \frac{\pi}{2}\right) - \cos(\sin 0)\right]$$

$$= -\left[\cos(1) - \cos(0)\right]$$

$$= -\cos(1) + 1 = 1 - \cos(1)$$

64. $\int_0^a x\sqrt{a^2 - x^2}\, dx$

$$u = a^2 - x^2 \qquad du = -2x\,dx$$

$$\frac{du}{-2} = x\,dx$$

$$\int_0^a x\sqrt{a^2 - x^2}\, dx = \int \sqrt{u}\left(-\frac{du}{2}\right)$$

$$= -\frac{1}{2}\int u^{1/2}\, du = -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]$$

$$= -\frac{1}{3}u^{3/2} = -\frac{1}{3}\left[(a^2 - x^2)^{3/2}\right]_0^a$$

$$= -\frac{1}{3}\left[(a^2 - a^2)^{3/2} - (a^2 - 0)^{3/2}\right]$$

$$= -\frac{1}{3}(-a^3) = \frac{1}{3}a^3$$

71. $\displaystyle\int_0^1 \frac{e^z + 1}{e^z + z}\,dz$

$u = e^z + z \qquad du = e^z + 1\,dz$

$$\int_0^1 \frac{e^z+1}{e^z+z}\,dz = \int \frac{du}{u} = \ln|u|$$

شلنا المطلق لانه حدود التكامل بالمنطقة الموجبة للx فماله داعي مطلق نحط

$$\left[\ln(e^z+z)\right]_0^1 = \left[\ln(e^1+1) - (\ln(e^0+0)\right)$$

$$= \ln(e+1) - \ln 1 = \ln(1+e)$$

Examples (Even Functions):	***Examples (Odd Functions):***
$x^2 - 2$	$x^3 - x$
5	$\sqrt[5]{x}$
$x^2\lvert x\rvert$	$x^3\lvert x\rvert$
$\dfrac{x^4+1}{3x^8}$	$\dfrac{x^2+5}{x^3+2x}$
$\dfrac{x^3-2x}{x^5}$	$\dfrac{x^3-x^9}{x^4}$

Example: **Determine whether each function is Even, Odd or Neither.**

1. $f(x) = x^5 + x$

$(-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$ odd

2. $f(x) = 1 - x^4$

$1 - (-x)^4 = 1 - x^4 = f(x)$ even

3. $f(x) = 2x - x^2$

$2(-x) - (-x)^2 = -2x - x^2 = -(2x + x^2)$ Neither

4. $f(x) = |x| + 2$

$|-x| + 2 = |x| + 2 = f(x)$ even

5. $f(x) = 3$

$f(x) = 3 = f(-x)$ even

6. $f(x) = \dfrac{x}{x^2 + x^6}$

$\dfrac{-x}{(-x)^2 + (-x)^6} = \dfrac{-x}{x^2 + x^6} = -f(x)$ odd

7. $f(x) = \dfrac{x^2}{x^4 + x}$

$\dfrac{(-x^2)}{(-x)^4 + (-x)} = \dfrac{x^2}{x^4 - x}$ Neither

Symmetry

The next theorem uses the Substitution Rule for Definite Integrals (6) to simplify the calculation of integrals of functions that possess symmetry properties.

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Even-Odd Identities:

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

EXAMPLE 11 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$

(b) $\displaystyle\int_{-1}^{1} (x^2 + \cos x)\tan x\,dx.$

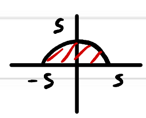
(b) $\displaystyle\int_{-1}^{1} (x^2 + \cos x)\tan x\,dx$. Since $\boxed{f(x) = (x^2 + \cos x)\tan x \text{ is odd, } \displaystyle\int_{-1}^{1} (x^2 + \cos x)\tan x\,dx = 0}$.

38. $\int_{-5}^{5} \left(x - \sqrt{25 - x^2}\right) dx$

$$\int_{-5}^{5} x \, dx - \int_{-5}^{5} \sqrt{25-x^2} \, dx$$

$$\left[\frac{x^2}{2}\right]_{-5}^{5} = \left[\frac{25}{2} - \frac{25}{2}\right] = 0$$

or $\int_{-5}^{5} x \, dx = 0$ odd function



$$\int_{-5}^{5} \sqrt{25-x^2} \, dx$$

$y = \sqrt{25-x^2}$

$y^2 = 25 - x^2$

$x^2 + y^2 = 25$

$= \frac{1}{2} * \pi * 5^2 = \frac{25}{2}\pi$ $\quad \therefore r = 5$

$$\int_{-5}^{5} x \, dx - \int_{-5}^{5} \sqrt{25-x^2} \, dx = 0 - \frac{25\pi}{2} = -\frac{25\pi}{2}$$

9. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

(a) $\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx.$

(a) $\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx.$ Let $\boxed{u=\sqrt{x}+1 \Rightarrow du=\frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du}$.

Therefore, $\boxed{\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx = 2\int \frac{1}{u^2}\,du = \frac{-2}{u}+C = \frac{-2}{\sqrt{x}+1}+C}$.

IV. $\displaystyle\int_{-1}^{1} x^2 \tan x\,dx =$

a) 0.

b) 1.

c) -1.

d) 2.

e) None of the above.

7. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

I. $\displaystyle\int \sin^3(x)\cos(x)\,dx.$

II. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx.$

7. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

I. $\displaystyle\int \sin^3(x)\cos(x)\,dx.$

Let $\boxed{u = \sin x, \text{ then } du = \cos x dx}$.

$$\boxed{\int \sin^3(x)\cos(x)\,dx = \int u^3 du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C}.$$

II. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx.$

Let $\boxed{u = \ln x, \text{ then } du = \dfrac{dx}{x}}$.

$$\boxed{\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{du}{u} = \big[\ln|u|\big]_1^2 = \ln 2 - \ln 1 = \ln 2}.$$

7. [10 + 10 = 20 pts.] Evaluate each of the following integrals:

(i) $\displaystyle\int \frac{\cos(\pi x)}{(1+\sin(\pi x))^3}\,dx.$

(ii) $\displaystyle\int_1^e \frac{\ln(x^2)}{x}\,dx.$

Answer.

Let $u = 1+\sin(\pi x)$, then $du = \pi\cos(\pi x)dx$. Therefore, $\displaystyle\int \frac{\cos(\pi x)}{(1+\sin(\pi x))^3}\,dx = \frac{1}{\pi}\int \frac{1}{u^3}\,du = \frac{1}{\pi}\left(\frac{u^{-3+1}}{-3+1}\right) + C = -\frac{1}{2\pi(1+\sin(\pi x))^2} + C.$

(ii) $\displaystyle\int_1^e \frac{\ln(x^2)}{x}\,dx.$

Let $u = \ln x$, then $du = \dfrac{1}{x}dx$.

Therefore, $\displaystyle\int_1^e \frac{\ln(x^2)}{x}\,dx = 2\int_1^e \frac{\ln x}{x}\,dx = 2\int_0^1 u\,du = \left[u^2\right]_0^1 = 1.$

7. [10 + 10 pts.] Evaluate the integral

(a) $\displaystyle\int \frac{dx}{x(\ln x)^5}$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2\sec^2 x}{1+\tan x} dx$

(a) We use the substitution $u = \ln x$, then $\displaystyle\int \frac{dx}{x(\ln x)^5} = \int \frac{du}{u^5} = \frac{u^{-4}}{-4} + C = \frac{-1}{4(\ln x)^4} + C$.

(b) We use the substitution $u = 1 + \tan x$, then $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2\sec^2 x}{1+\tan x} dx = \int_1^2 \frac{2du}{u} = [2\ln u]_1^2 = 2\ln 2 - 2\ln 1 =$ $2\ln 2$.

EXAMPLE 5 Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Also $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\begin{aligned}
\int \sqrt{1+x^2}\, x^5\, dx &= \int \sqrt{1+x^2}\, x^4 \cdot x\, dx \\
&= \int \sqrt{u}\,(u-1)^2 \cdot \tfrac{1}{2}\, du = \tfrac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\, du \\
&= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du \\
&= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\cdot\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C \\
&= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C \qquad \blacksquare
\end{aligned}$$

Q8. [5+5=10 pts.] Evaluate each of the following integrals.

(a) $\int (\cos x)(\sin(\sin x))\ dx$ (b) $\int_0^{\pi/4} 4(1+\tan x)^2 \sec^2 x\ dx$

(a) Let $\boxed{u = \sin x, \text{ then } du = \cos x\, dx}$. Thus,

$$\int (\cos x)(\sin(\sin x))\ dx = \boxed{\int \sin(u)\ du = -\cos u + C = -\ \cos(\sin x) + C}.$$

(b) Let $\boxed{u = 1 + \tan x, \text{ then } du = \sec^2 x\, dx}$. Thus,

$$\int_0^{\pi/4} 4(1+\tan x)^2 \sec^2 x\ dx = \boxed{\int_1^2 4u^2\ du = \frac{4u^3}{3}\Big]_1^2 = 28/3}.$$

EXAMPLE 6 Calculate $\int \tan x\, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 6 can also be written as

Calculus A

Chapter 6: Application of Integration

Sections: 6.1 Area Between Curves

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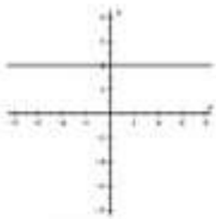
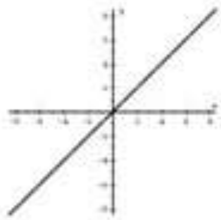
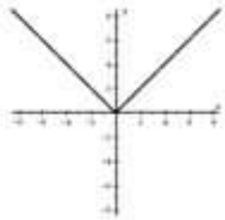
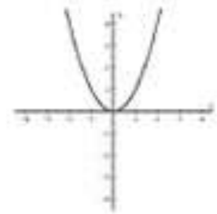
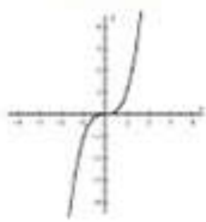
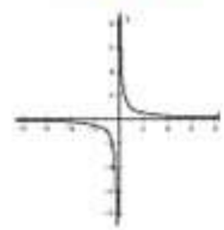
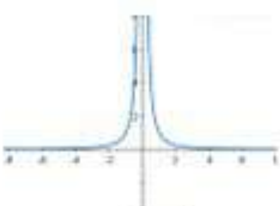
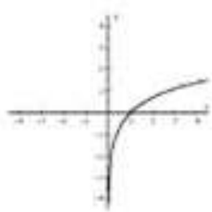
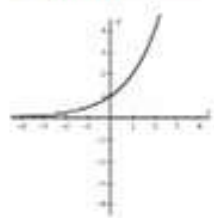
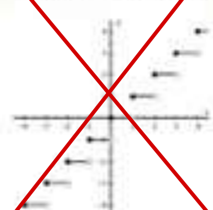
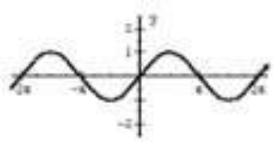
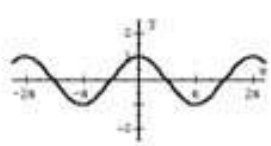
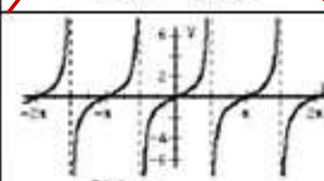
2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Constant  ✓ $f(x) = c$	Linear  ✓ $f(x) = x$	Absolute Value  ✓ $f(x) = \|x\|$	Quadratic  ✓ $f(x) = x^2$
Square Root  ✓ $f(x) = \sqrt{x}$	Cubic  ✓ $f(x) = x^3$	Cube Root  ✓ $f(x) = \sqrt[3]{x}$	Reciprocal/Inverse/ Rational  $f(x) = \frac{1}{x}$
Rational  $f(x) = \frac{1}{x^2}$	Logarithmic  ✓ $f(x) = \ln(x)$	Exponential  ✓ $f(x) = e^x$	Greatest Integer (Step Function)  $f(x) = [[x]]$
Trigonometric Functions →	 ✓ $f(x) = \sin(x)$	 ✓ $f(x) = \cos(x)$	 $f(x) = \tan(x)$

SOME FUNCTIONS AND THEIR GRAPHS

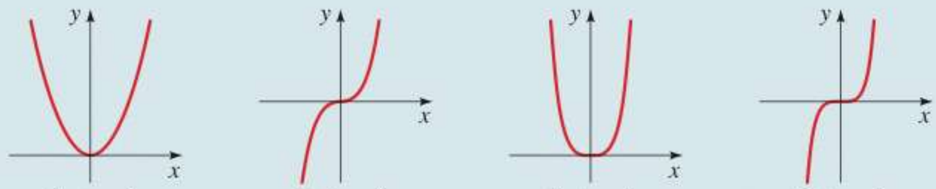
Linear functions

$f(x) = mx + b$

$f(x) = b$ $\qquad$ $f(x) = mx + b$

Power functions

$f(x) = x^n$



$f(x) = x^2$ $\qquad$ $f(x) = x^3$ $\qquad$ $f(x) = x^4$ $\qquad$ $f(x) = x^5$

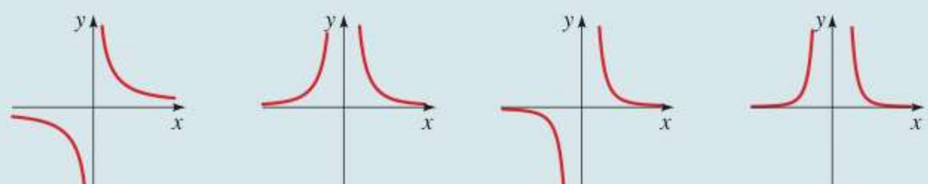
Root functions

$f(x) = \sqrt[n]{x}$

$f(x) = \sqrt{x}$ $\qquad$ $f(x) = \sqrt[3]{x}$ $\qquad$ $f(x) = \sqrt[4]{x}$ $\qquad$ $f(x) = \sqrt[5]{x}$

Reciprocal functions

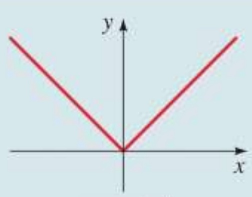
$f(x) = \dfrac{1}{x^n}$



$f(x) = \dfrac{1}{x}$ $\qquad$ $f(x) = \dfrac{1}{x^2}$ $\qquad$ $f(x) = \dfrac{1}{x^3}$ $\qquad$ $f(x) = \dfrac{1}{x^4}$

Absolute value function

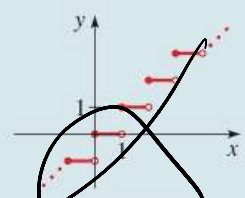
$f(x) = |x|$



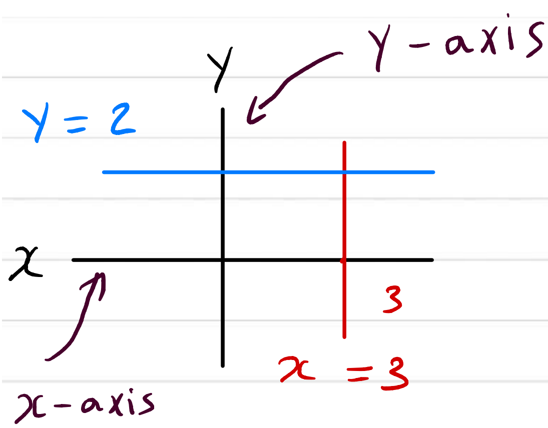
$f(x) = |x|$

Greatest integer function

$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$

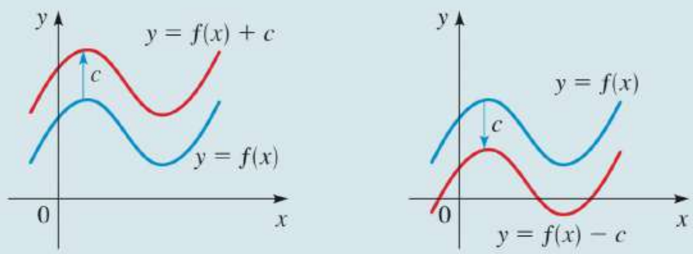


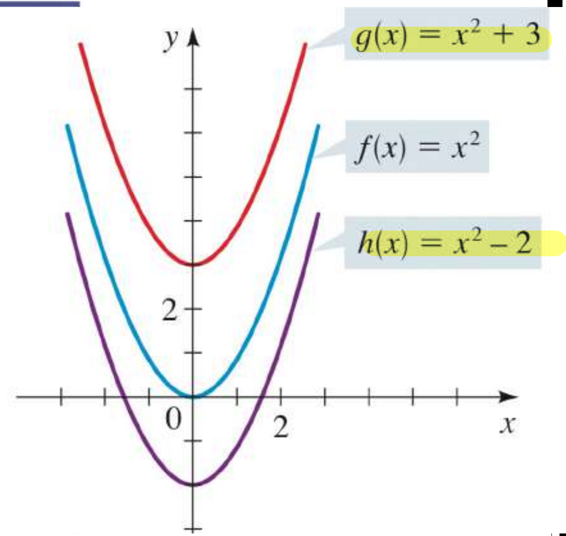
VERTICAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



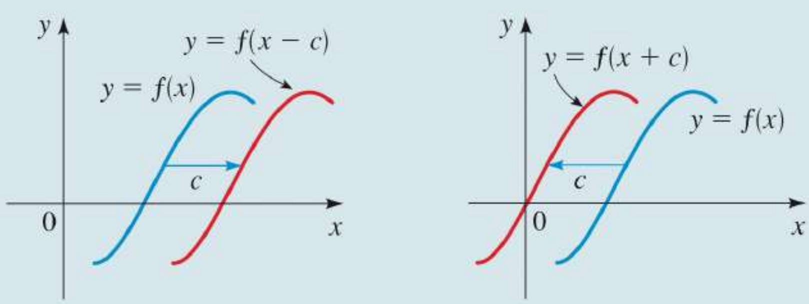


HORIZONTAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



The graphs of g and h are sketched in Figure 2.

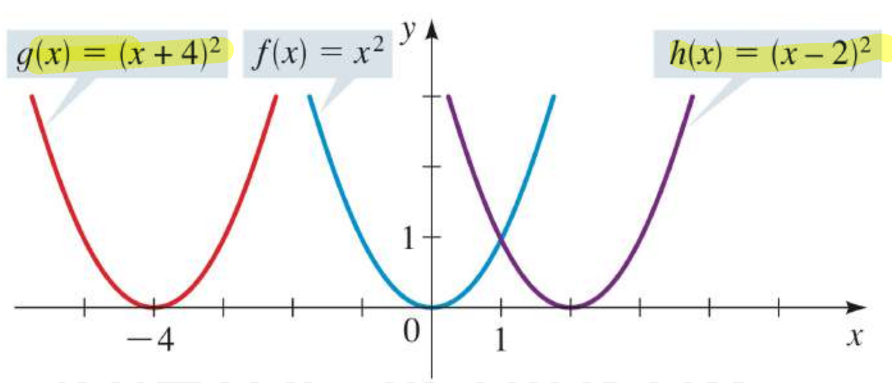
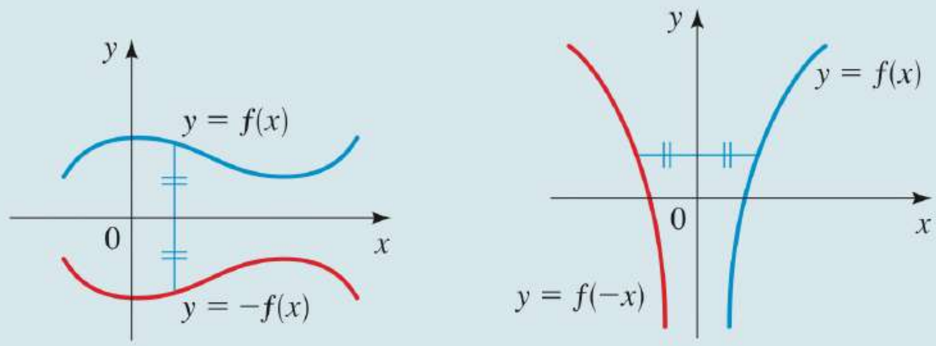


FIGURE 2

REFLECTING GRAPHS

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



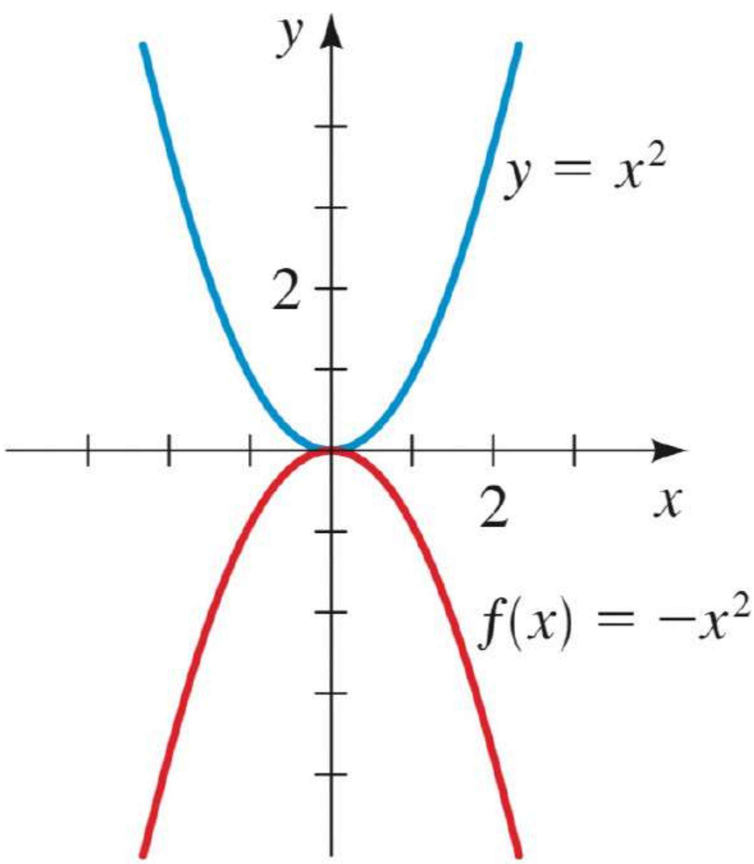


FIGURE 4

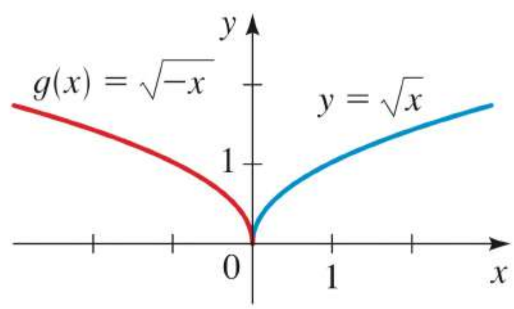
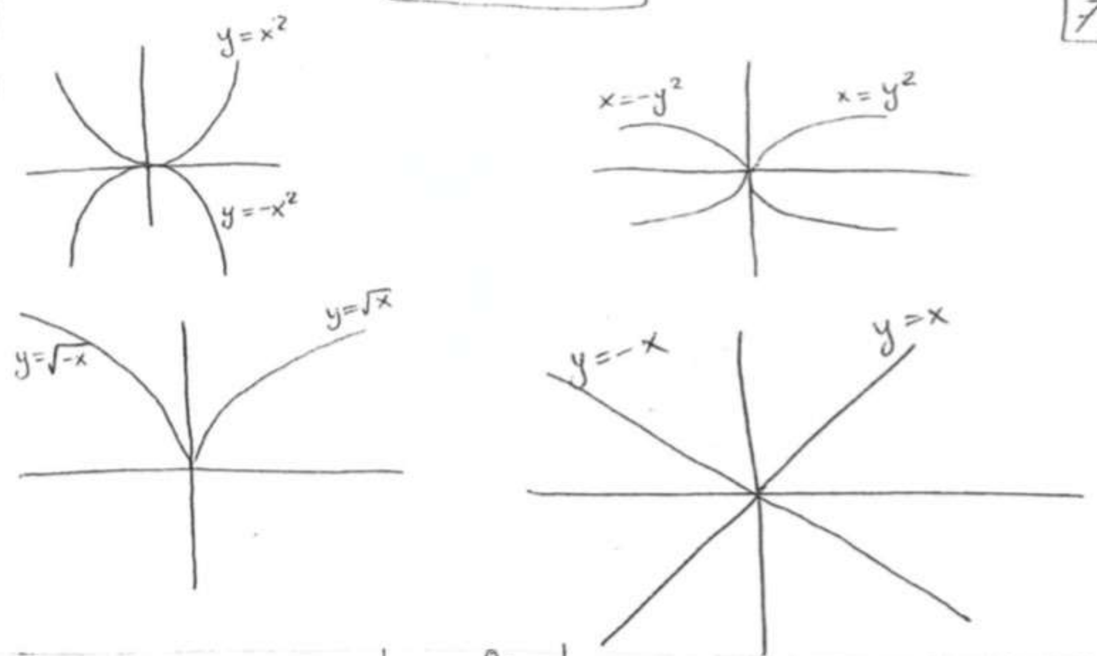
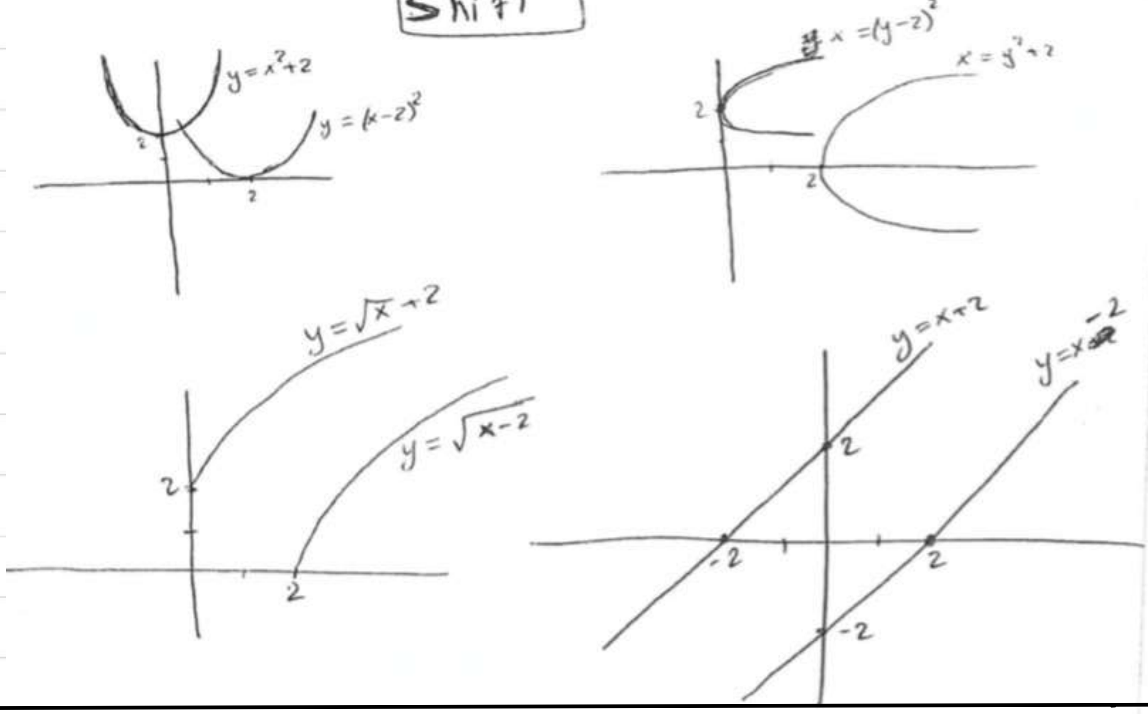


FIGURE 5

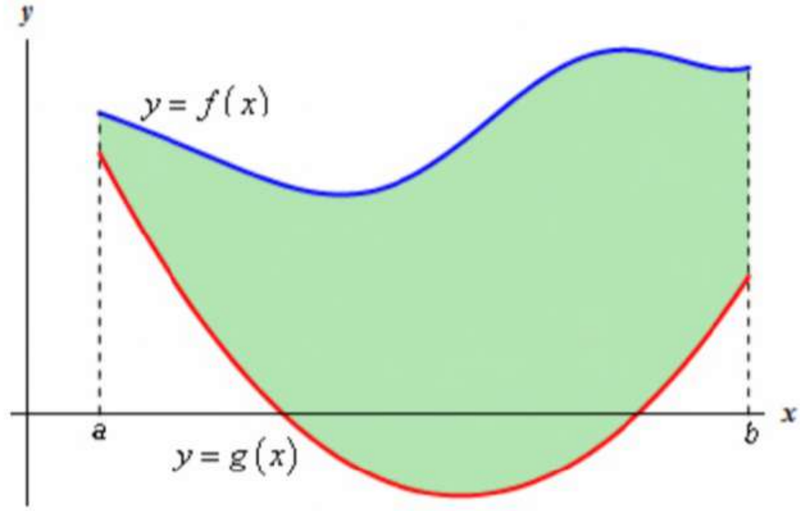
Basic Curves



Shift



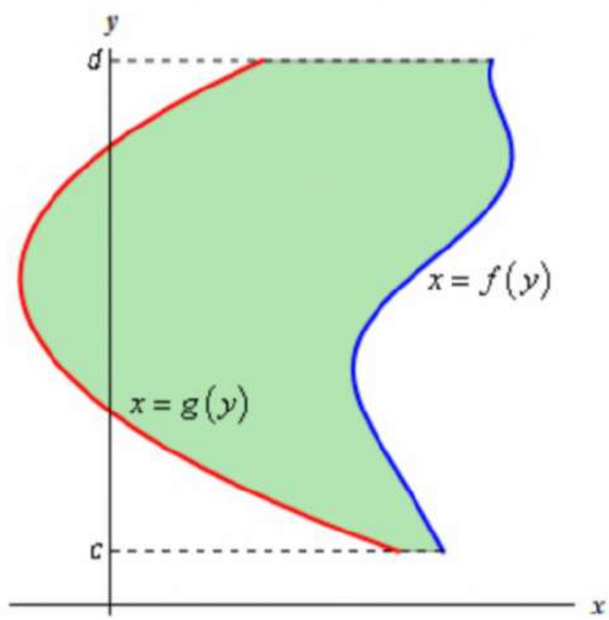
Vertical Shift	$f(x) \pm c$	$+c$ فوق $-c$ تحت
Horizontal Shift	$f(x \pm c)$	$+c$ يسار $-c$ يمين
Reflect x-axis	$-f(x)$	
Reflect y-axis	$f(-x)$	



In the Area and Volume Formulas section of the Extras chapter we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x)\,dx \qquad (1)$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c,d]$ with $f(y) \ge g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y)\,dy \qquad (2)$$

In the first case we will use,

$$A = \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} dx, \qquad a \le x \le b \qquad (3)$$

In the second case we will use,

$$A = \int_c^d \begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix} dy, \qquad c \le y \le d \qquad (4)$$

خطوات الحل :

١) رح نطلع نقطة تقاطع الدالتين

٢) نرسم الدالة عن طريق تعويض(نعوض بقيم التقاطع ونقطة بينهم)

٣) نختار الشريحة الاسهل للتكامل

٤) نطبق قانون مساحة المحصورة

$$A = \int_a^b \left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right) dx, \qquad a \le x \le b \qquad (3)$$

$$A = \int_c^d \left(\begin{matrix}\text{right}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{left}\\ \text{function}\end{matrix}\right) dy, \qquad c \le y \le d \qquad (4)$$

ملاحظات :

الرسم مهم لانه يخليك تعرف حدود التكامل مالتك + تعرف منو الدالة الأكبر

المساحة المحصورة أهيا المساحات الي رح نطلعها في هذا الدرس ، دائما موجبة و دائما رح يكون الشكل مسكر.. يعني بإختصار إذا طلعت المساحة المحصورة سالبة يعني عندك غلط

إذا طلب منك بالامتحان set up the integral يعني ما يبيبك تكامل بس أكتبله التكامل الي تبي تسويه بدون لا تحله

Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

to find the intersection: $y = y$

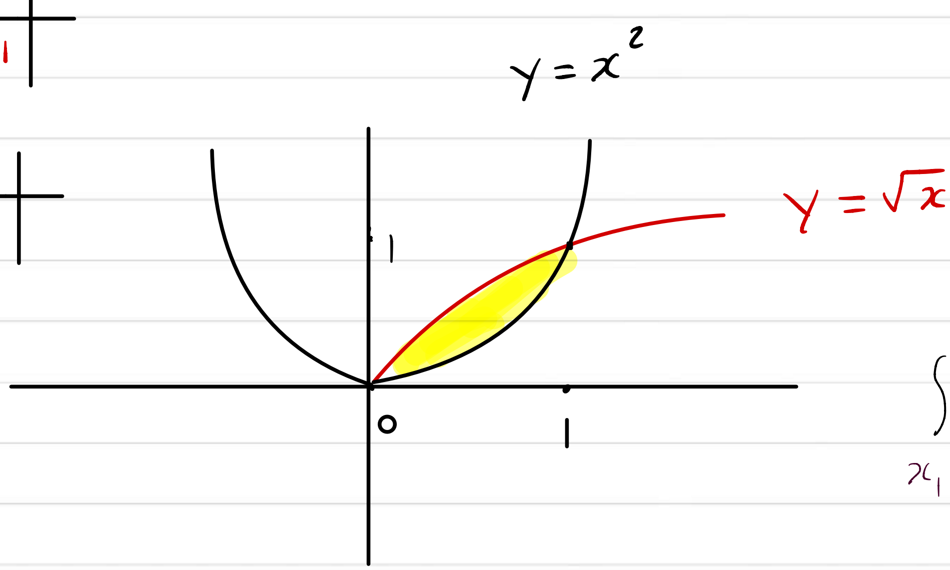
$$x^2 = \sqrt{x} \Rightarrow x^2 - x^{1/2} = 0$$

$$x^{1/2}(x^{3/2} - 1) = 0$$

$\therefore x = 0$ | $(x^{3/2} - 1) = 0$ $\therefore x = 1$

x	0	$\frac{1}{2}$	1
$y=x^2$	0	$\frac{1}{4}$	1

x	0	$\frac{1}{2}$	1
$y=\sqrt{x}$	0	$\frac{1}{\sqrt{2}}$	1



$$\int_{x_1}^{x_2} \square \, dx$$

$$A = \int_0^1 \text{upper} - \text{lower} \; dx = \int_0^1 \sqrt{x} - x^2 \, dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}(x)\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

6. [10 pts.] Sketch the region enclosed by the curves $f(x) = x$ and $g(x) = x^2$ and find its area.

1) points of intersection:-

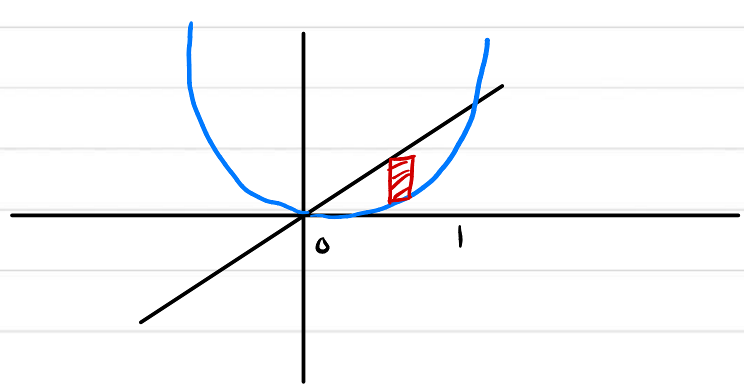
$$f(x) = g(x)$$

$$x = x^2 \Rightarrow x^2 - x = 0$$

$$= x(x-1) = 0 \quad \therefore x = 0 \quad x = 1$$

$$A = \int_0^1 x - x^2 \, dx = \left[\frac{x^2}{2} - \frac{x^3}{2} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



$$\int_{x_1}^{x_2} \square \, dx$$

Q10. [10 pts.] Find the area of the region enclosed by the curves $y = 4 - x^2$ and $y = 3$.

---

to find the intersection:- $y = y$

$$4 - x^2 = 3 \Rightarrow x^2 + 3 - 4 = 0$$

$$x^2 - 1 = 0 \Rightarrow x = 1, \; x = -1$$

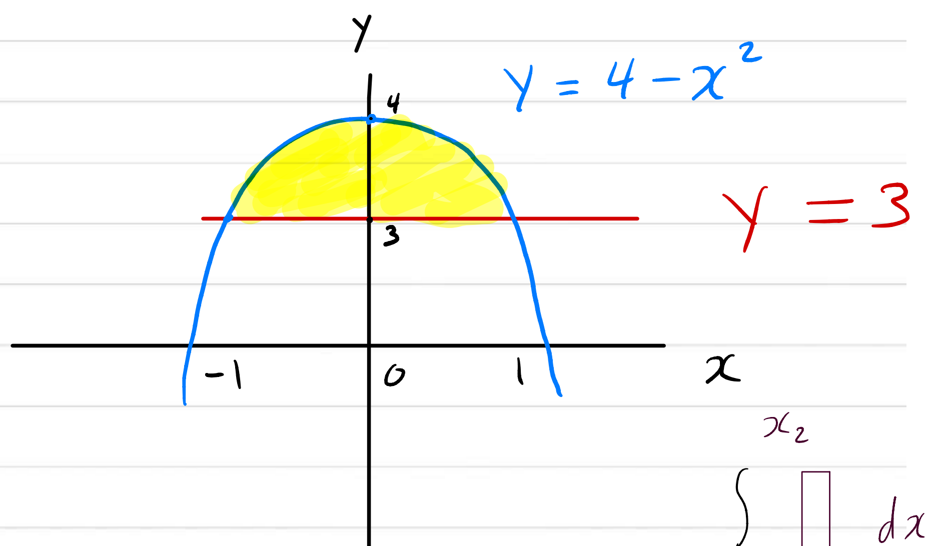
$$A = \int_{-1}^{1} (4 - x^2) - 3 \, dx = \int_{-1}^{1} (1 - x^2) \, dx$$

$y = 4 - x^2$

x	-1	0	1
y	3	4	3

$y = 3$

x	-1	0	1
y	3	3	3



$$\int_{x_1}^{x_2} \boxed{} \, dx$$

$$\boxed{A = \int_{-1}^{1} [(4 - x^2) - 3] \, dx = \int_{-1}^{1} (1 - x^2) \, dx = x - \frac{x^3}{3} \Big]_{-1}^{1} = 4/3.}$$

6. [10 pts.] Let R be the region between the curves $y = x^2$ and $y = \sqrt{x}$, from $x = 0$ to $x = 4$. Sketch the region R, and find its area.

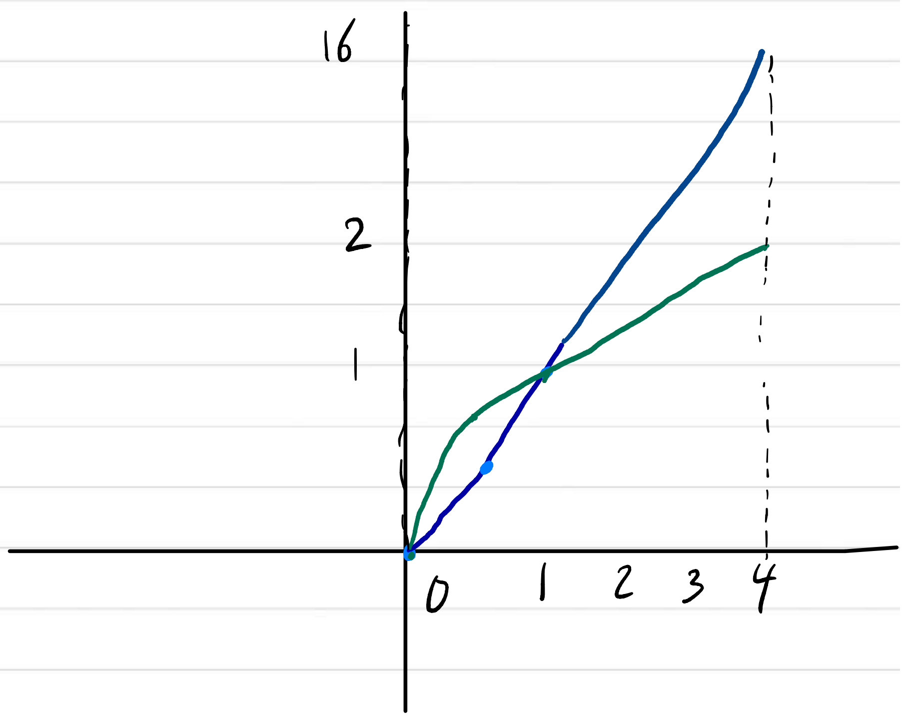
$$Y = Y$$

$$x^2 = \sqrt{x} \quad \Rightarrow \quad x^2 - \sqrt{x} = 0$$

$$x^{1/2}\,(x^{3/2} - 1) = 0 \qquad \therefore\; x = 0,\; x = 1$$

x	0	$\frac{1}{2}$	1	4
$y = x^2$	0	$\frac{1}{4}$	1	16

x	0	$\frac{1}{2}$	1	4
$y = \sqrt{x}$	0	$\frac{1}{\sqrt{2}}$	1	2



$$\int_{x_1}^{x_2} \boxed{}\, dx$$

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx + \int_1^4 x^2 - \sqrt{x}\,dx = \frac{50}{3}$$

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$$A = \int_0^1 (\sqrt{x} - x^2)\,dx + \int_1^4 x^2 - \sqrt{x}\,dx = \frac{50}{3}$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{2}{3}x^{3/2}\right]_1^4$$

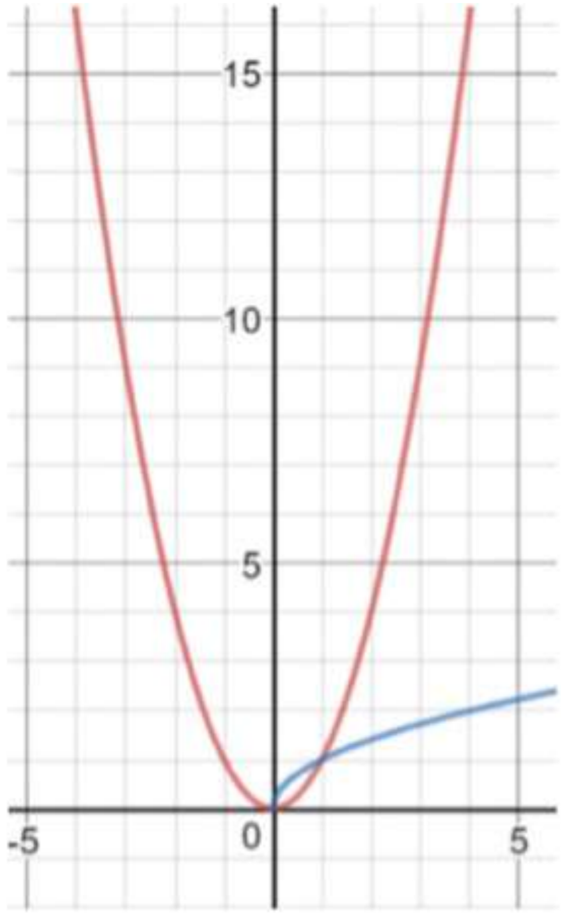
$$= \left[\frac{2}{3}(\sqrt{1})^3 - \frac{1}{3}\right] - [0] +$$

$$\left[\frac{4^3}{3} - \frac{2}{3}(\sqrt{4})^3\right] - \left[\frac{1}{3} - \frac{2}{3}(\sqrt{1})^3\right]$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{64}{3} - \frac{16}{3} - \left(\frac{1}{3} - \frac{2}{3}\right)$$

$$= \frac{1}{3} + \frac{48}{3} + \frac{1}{3} = \frac{50}{3}$$

6. [10 pts.] Let R be the region between the curves $y = x^2$ and $y = \sqrt{x}$, from $x = 0$ to $x = 4$. Sketch the region R, and find its area.



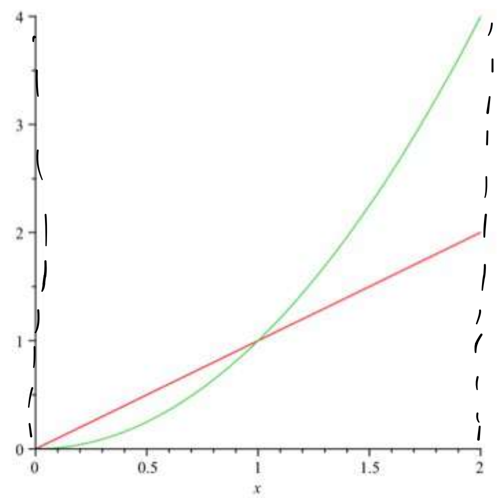
6. [10 pts.] Let R be the region between the curves $y = x^2$ and $y = \sqrt{x}$, from $x = 0$ to $x = 4$. Sketch the region R, and find its area.

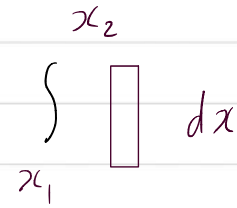
The area of the region R is $\boxed{A = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx + \int_1^4 \left(x^2 - \sqrt{x}\right)\, dx = \frac{1}{3} + \frac{49}{3} = \frac{50}{3}}$.

5. [2 + 8 = 10 pts.] Let R be the region between the curves $y = x$ and $y = x^2$, from $x = 0$ to $x = 2$. Sketch the region R and find its area.

Answer. The points of intersection are $(0,0)$ and $(1,1)$. The area is given by

$$A = \int_0^2 |x - x^2|\, dx = \int_0^1 (x - x^2)\, dx + \int_1^2 (x^2 - x)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = 1.$$





6. [10 pts.] Sketch the region enclosed by the curves $f(x) = \sqrt{x-1}$ and $g(x) = x - 1$ and find its area.

$$f(x) = g(x)$$

$$\sqrt{x-1} = x - 1 \Rightarrow (x-1)^2 = x - 1$$

$$x^2 - 2x + 1 = x - 1$$

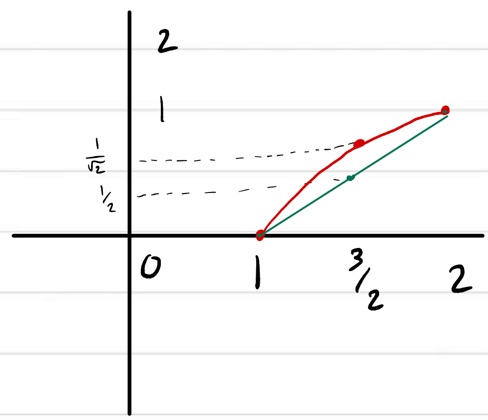
$$x^2 - 2x + 1 - x + 1 = 0$$

$$\int_{x_1}^{x_2} \boxed{}\, dx$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 2 , \quad x = 1$$



$f(x) = \sqrt{x-1}$

x	1	3/2	2
$f(x)$	0	$\frac{1}{\sqrt{2}}$	1

$g(x) = x-1$

x	1	3/2	2
$f(x)$	0	$\frac{1}{2}$	1

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$$A = \int_1^2 \left(\sqrt{x-1} - (x-1)\right) dx = \frac{1}{6}$$

$$A = \int_1^2 \left(\sqrt{x-1} - (x-1)\right) dx = \frac{1}{6}$$

$$\left[\frac{2}{3}(x-1)^{3/2} - \left(\frac{x^2}{2} - x\right)\right]_1^2$$

$$\left[\frac{2}{3}\left(\sqrt{2-1}\right)^3 - \left(\frac{4}{2} - 2\right)\right]$$

$$-\left[\frac{2}{3}\left(\sqrt{1-1}\right)^3 - \left(\frac{1}{2} - 1\right)\right.$$

$$= \frac{2}{3} - \left[\left(\frac{4}{2} - \frac{4}{2}\right) - \left[0 - \frac{1}{2}\right]\right]$$

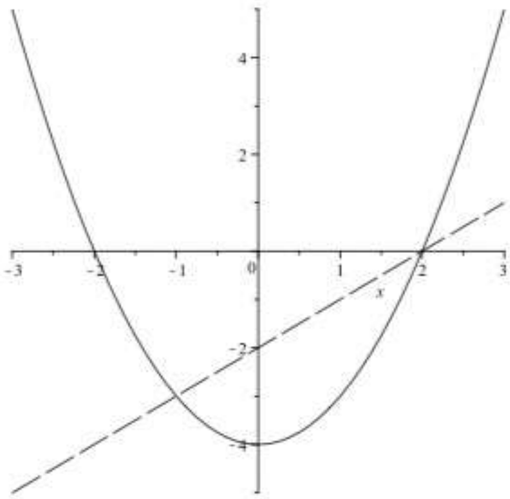
$$= \frac{2}{3} - 0 - \left[\frac{1}{2}\right]$$

$$= \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

8. [$2+8=10$ pts.] Let R be the region enclosed by the curves $y=x-2$ and $y=x^2-4$.

(a) Sketch the region R.

(b) Find the area of the region R.



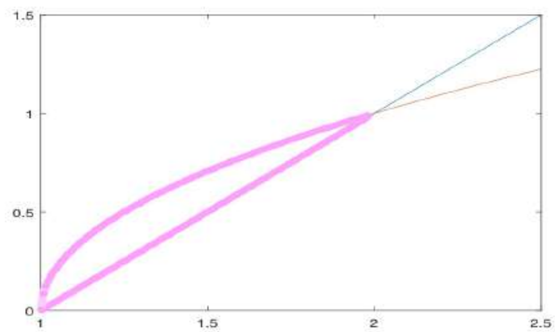
(a)

$\int_{x_1}^{x_2} \square \, dx$

(b) <u>Points of intersection</u>: For points of intersection of the two curves we solve $x^2-4=x-2$. This gives $x^2-x-2=(x+1)(x-2)=0$, which holds for $x=-1$ and $x=2$. Thus, we have two points of intersection: $\boxed{(-1,-3) \text{ and } (2,0)}$.

The area of R is: $\boxed{A=\int_{-1}^{2}((x-2)-(x^2-4))dx=\int_{-1}^{2}(-x^2+x+2)dx=\left[\frac{-x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{2}=\frac{9}{2}}$.

6. [10 pts.] Sketch the region enclosed by the curves $f(x) = \sqrt{x-1}$ and $g(x) = x - 1$ and find its area.

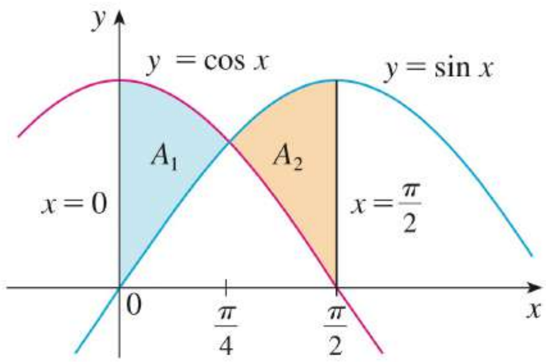


$\int_{x_1}^{x_2} \square \, dx$

$$A = \int_1^2 \left(\sqrt{x-1} - (x-1)\right) dx = \left[\frac{2(x-1)^{\frac{3}{2}}}{3} - \frac{x^2}{2} + x\right]_1^2 = \left(\frac{2}{3} - 2 + 2\right) - \left(0 - \frac{1}{2} + 1\right) = \frac{1}{6}.$$

EXAMPLE 6 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

SOLUTION The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 12.



Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is

$$A = \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

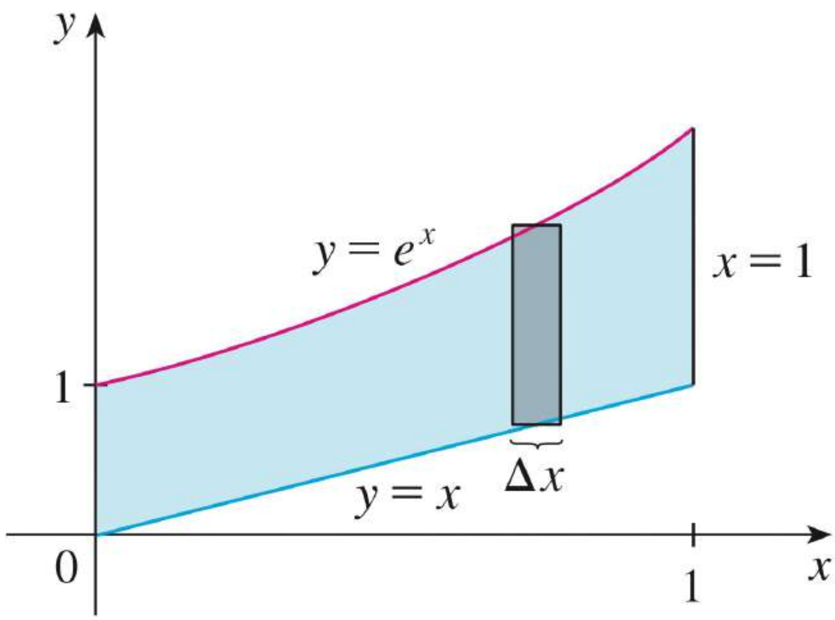
$$= 2\sqrt{2} - 2$$

EXAMPLE 1 Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

SOLUTION The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 (e^x - x)\,dx = e^x - \tfrac{1}{2}x^2\Big]_0^1$$

$$= e - \tfrac{1}{2} - 1 = e - 1.5$$ ■



EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

SOLUTION We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are $(0, 0)$ and $(1, 1)$.

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

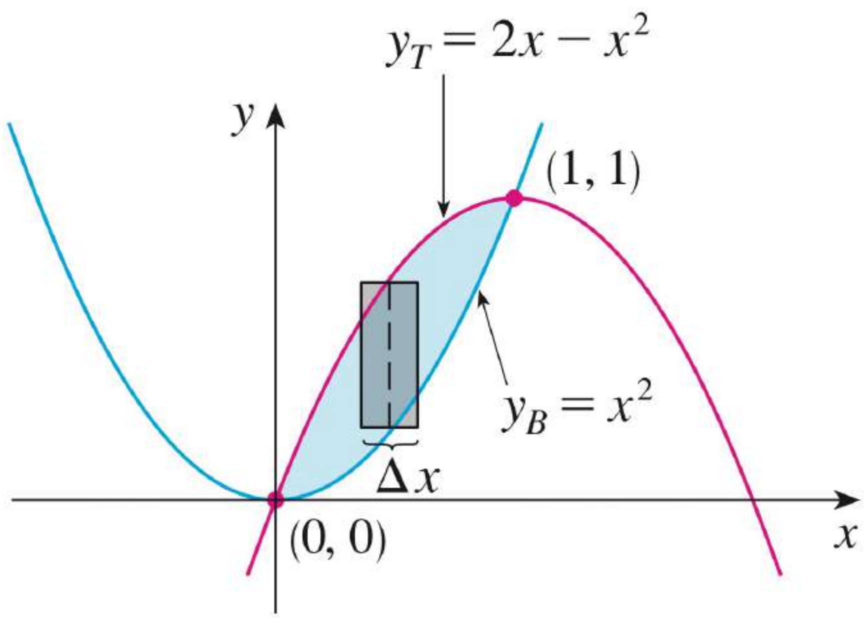
$2x - x^2 = x^2$

$2x - 2x^2 = 0 \Rightarrow 2x(1 - x) = 0$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$



EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

$$y^2 = 2x + 6$$

$$y^2 - 6 = 2x$$

$$x = \frac{1}{2}y^2 - 3 \quad , \quad x = y + 1$$

$$x = x$$

$$\frac{1}{2}y^2 - 3 = y + 1$$

$$\frac{1}{2}y^2 - y - 3 - 1 = 0$$

$$y^2 - 2y - 8 = 0$$

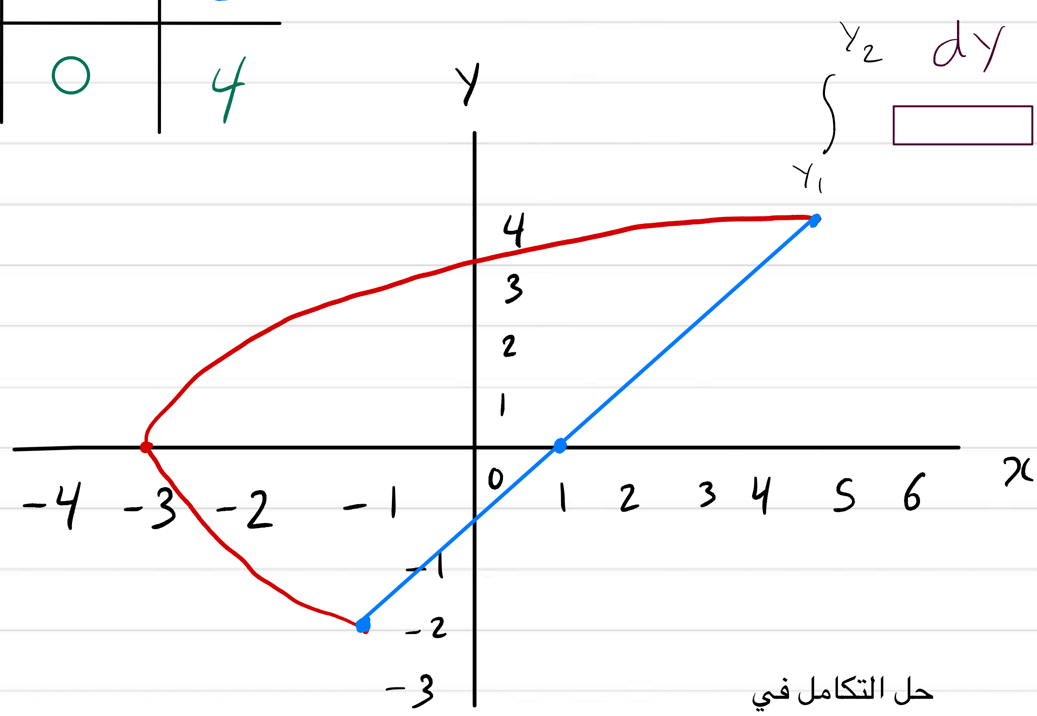
$$(y - 4)(y + 2) = 0 \quad \therefore y = 4, \; y = -2$$

$$x = \frac{1}{2}y^2 - 3$$

x	-1	-3	5
y	-2	0	4

$$x = y + 1$$

x	-1	1	5
y	-2	0	4



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$$A = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy = 18$$

$$A = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right)\, dy = 18$$

$$\left[\frac{y^2}{2} + y\right]_{-2}^{4} - \left[\frac{1}{2}\frac{y^3}{3} - 3y\right]_{-2}^{4}$$

$$= \left[\left(\frac{16}{2} + 4\right) - \left(\frac{4}{2} - 2\right)\right] \ldots$$

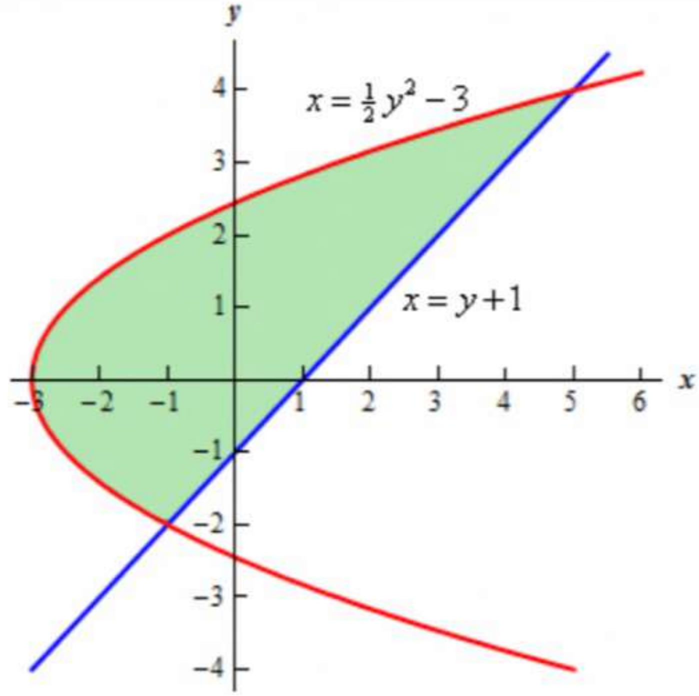
$$- \left[\left(\frac{64}{6} - 12\right) - \left(\frac{-8}{6} + 6\right)\right]$$

$$= [8 + 4) - (2 - 2)]$$

$$- \left[\left(\frac{64}{6} - \frac{72}{6}\right) - \left(\frac{-8}{6} + \frac{36}{6}\right)\right]$$

$$= 12 - \left[\frac{-8}{6} - \frac{28}{6}\right]$$

$$= 12 - \left[\frac{-36}{6}\right] = 12 + 6 = 18$$



The area is,

$$
\begin{aligned}
A &= \int_c^d \begin{pmatrix}\text{right} \\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left} \\ \text{function}\end{pmatrix} dy \\
&= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy \\
&= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\,dy \\
&= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Big|_{-2}^{4} \\
&= 18
\end{aligned}
$$

This is the same that we got using the first formula and this was definitely easier than the first method.

So, in this last example we've seen a case where we could use either formula to find the area. However, the second was definitely easier.

Students often come into a calculus class with the idea that the only easy way to work with functions is to use them in the form $y = f(x)$. However, as we've seen in this previous example there are

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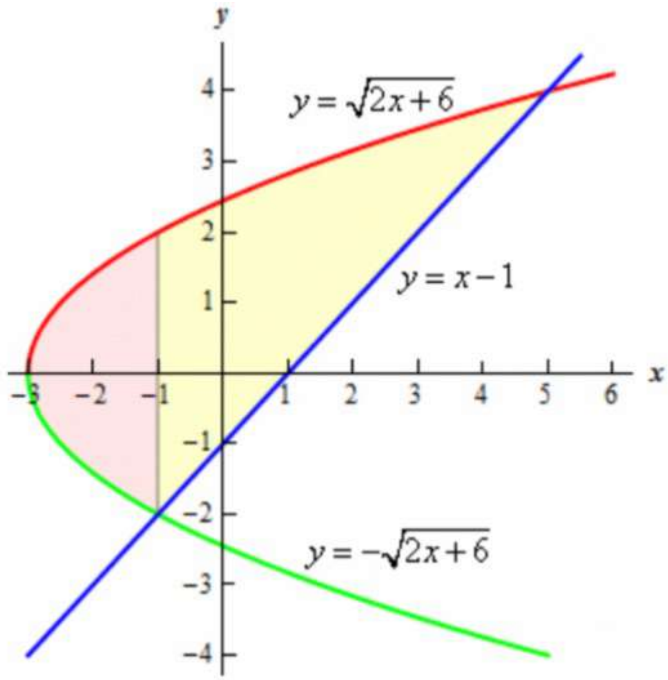

إذا اخترت الشريحة الغلط
الحل رح يصير طويل

The integrals for the area would then be,

$$
\begin{aligned}
A &= \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6} - x + 1\, dx \\
&= \int_{-3}^{-1} 2\sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6}\, dx + \int_{-1}^{5} -x + 1\, dx \\
&= \frac{2}{3} u^{\frac{3}{2}} \Big|_{0}^{4} + \frac{1}{3} u^{\frac{3}{2}} \Big|_{4}^{16} + \left(-\frac{1}{2}x^2 + x\right)\Big|_{-1}^{5} \\
&= 18
\end{aligned}
$$

While these integrals aren't terribly difficult they are more difficult than they need to be.

Recall that there is another formula for determining the area. It is,

$$
A = \int_{c}^{d} \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right) dy, \qquad c \le y \le d
$$

and in our case we do have one function that is always on the left and the other is always on the right. So, in this case this is definitely the way to go. Note that we will need to rewrite the equation of the line since it will need to be in the form $x = f(y)$ but that is easy enough to do. Here is the graph for using this formula.

EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION By solving the two equations we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for x and notice from Figure 15 that the left and right boundary curves are

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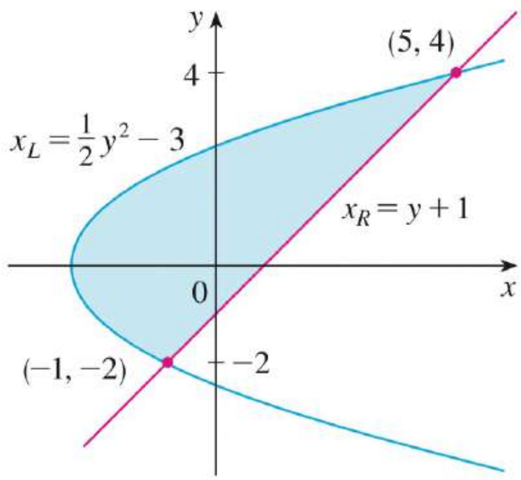
$$x_L = \tfrac{1}{2}y^2 - 3 \qquad \text{and} \qquad x_R = y + 1$$



We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$A = \int_{-2}^{4} (x_R - x_L)\,dy = \int_{-2}^{4} \left[(y + 1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy$$

$$= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \Big]_{-2}^{4}$$

$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right) = 18$$