Calculus A

مذكرة مراجعة لكمية السكند

(3.1 to 3.11)

Calculus A

Chapter 3: Differential Rules

Sections: 3.1 Derivatives of Polynomial and Exponential Functions

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Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

$$\boxed{1} \qquad \frac{d}{dx}(x) = 1$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

7. [$5+5=10$ pts.] Let $f(x)=e^x-2x$.

(a) Find the point on the curve of f where the tangent line is horizontal.

(b) Using part (a) find an equation of this horizontal tangent line.

(a) Find the point on the curve of f where the tangent line is horizontal.
We have $\boxed{f'(x)=e^x-2}$.
Thus, $\boxed{f'(x)=0}$ when $e^x=2$ i.e., $\boxed{x=\ln 2}$. So the given curve has horizontal tangent when $\boxed{x=\ln 2}$. The corresponding point is $\boxed{(\ln 2, 2-2\ln 2)}$.

(b) Using part (a) find an equation of this horizontal tangent line.
Equation of this tangent line is $\boxed{y=2-2\ln 2}$.

33–36 Find an equation of the tangent line to the curve at the given point.

36. $y = \sqrt[4]{x} - x, \quad (1, 0)$

$$y = x^{1/4} - x$$

$$y' = \frac{1}{4} x^{-3/4} - 1$$

$$y' = \frac{1}{4\sqrt[4]{x^3}} - 1$$

$$y'(1) = \frac{1}{4\sqrt{1}} - 1 = \frac{1}{4} - 1 = -\frac{3}{4}$$

equation of tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{3}{4}(x - 1)$$

$$y = -\frac{3}{4}x + \frac{3}{4}$$

61. Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

$\because$ normal line $= -\frac{1}{m}$

$\therefore f'(x) = \frac{1}{2\sqrt{x}}$

The slope of normal line $= -2\sqrt{x} = m_n$

$L_2 \Rightarrow 2x + y = 1 \Rightarrow y = -2x + 1$

$\therefore m_2 = -2$

$\because L_2 \parallel$ normal of L_1 $\therefore m_2 = m_n$

$\therefore -2 = -2\sqrt{x} \Rightarrow \sqrt{x} = 1 \Rightarrow x = 1$

for finding "y_1" sub x in L_1 we get

$y = \sqrt{1} = 1$ $\therefore (1, 1)$

$\because y - y_1 = m_n (x - x_1) = y - 1 = -2(x-1)$

$\therefore y = -2x + 3$

71. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geqslant 1 \end{cases}$$

Is f differentiable at 1? ~~Sketch the graphs of f and f'~~.

for f to be differentiable at

$x = 1$, then

1) $\lim_{x \to 1} = f(1)$ "continuous"

2) $f'_+(1) = f'_-(1)$

$\because \lim_{x \to 1^-} x^2 + 1 = 2$, $\lim_{x \to 1^+} x + 1 = 2$

$f(1) = 2$ $\therefore f$ is cont on 2

2) $f'_+(1) = 2x \Rightarrow f'_+(1) = 2(1) = 2$

$f'_-(1) = 1$, $\because f'_+(1) \neq f'_-(1)$

$\therefore f$ is not differentiable at $x = 1$

Using Definition of Derivative to Evaluate a Limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

1) $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x}$ 2) $\displaystyle \lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{x - 16}$

خطوات الحل

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$f(x) = e^x$, $f(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = e^0 = 1$

$$\therefore \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Evaluate

$$\lim_{x \to 1} \frac{x^{15} - 1}{x - 1}$$

ما عندك ولا طريقة تقدر تحلها إلا عن طريق طريقة المشتقة

$\because f(x) = x^{15}$ $\quad f(1) = 1$

$f'(x) = 15x^{14}$ $\quad f'(1) = 15$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - a}$$

$$15 = \lim_{x \to 1} \frac{x^{15} - 1}{x - 1}$$

(c) $\lim_{x\to 0} \frac{1-\cos(x)}{\tan(x)}$.

$$= \frac{0}{0}$$

$$\lim_{x\to 0} \frac{1-\cos x}{\tan x} * \frac{1+\cos x}{1+\cos x}$$

$$\lim_{x\to 0} \frac{1-\cos^2 x}{\tan x\,(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{\tan x(1+\cos x)}$$

$$\lim_{x\to 0} \frac{\sin^2 x}{\frac{\sin x}{\cos x}(1+\cos x)} = \lim_{x\to 0} \frac{(\cos x)(\sin^2 x)}{\sin x(1+\cos x)}$$

$$\lim_{x\to 0} \frac{(\cos x)(\sin x)}{1+\cos x} = \frac{(1)(0)}{1+0} = \frac{0}{1} = 0$$

Calculus A

Chapter 3: Differential Rules

Sections: 3.2 The Product and Quotient Rule

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Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

2. [10 pts.] Find **equations** of the tangent lines to the curve $f(x) = \dfrac{x-1}{x+1}$ that are **parallel** to the line $x - 2y = 2$.

We have $f'(x) = \dfrac{2}{(x+1)^2}$. Now, any tangent line to the curve has slope $= \dfrac{2}{(x+1)^2} = \dfrac{1}{2}$. This implies that $x = 1$ or $x = -3$. Therefore, the equation of the first tangent line is $y - 0 = \dfrac{1}{2}(x-1)$ and the equation of the second tangent line is $y - 2 = \dfrac{1}{2}(x+3)$.

1) $f'(x) = \dfrac{2}{(x+1)^2}$, $y = \frac{1}{2}x - 1$

$\therefore y' = 1/2$ $\because$ parallel $\therefore y' = f(x)'$

$\therefore 1/2 = \dfrac{2}{(x+1)^2} \Rightarrow \therefore x_1 = 1$ or $x_2 = -3$

now sub in equ $f(x) = \dfrac{x-1}{x+1}$ to get y_1 and y_2

for $x = 1$, $f(1) = \dfrac{1-1}{1+1} = 0$

for $x = -3$, $f(-3) = \dfrac{-3-1}{-3+1} = +2$

for $(1,0)$:-

$y - y_1 = m(x - x_1) = y - 0 = 1/2(x-1)$

for $(-3, 2)$:- $y - 2 = 1/2(x+3)$

7. [15 pts.] Find the point(s) on the graph of $f(x) = \dfrac{x^2 - 3}{x + 2}$ at which the tangent line is horizontal.

The tangent line is horizontal whenever $f'(x) = 0$ that is when $\dfrac{(x+2)(2x) - (x^2-3)(1)}{(x+2)^2} = 0$ using quotient rule or simply $x^2 + 4x + 3 = 0$ which implies that $x = -1, -3$. Thus the graph of f has horizontal tangent line at the points $(-1, f(-1)) = (-1, -2), (-3, f(-2)) = (-3, -6)$.

7. [10 pts.] Find an equation of the tangent line to the graph of $f(x) = \dfrac{x^2 + 2}{(x+1)e^x}$ at $x = 0$.

We have $f'(x) = \frac{((x+1)e^x)(2x) - (x^2+2)((x+1)e^x + e^x)}{((x+1)e^x)^2}$ using the quotient and product rules for differentiation. Now $f(0) = 2$ and $f'(0) = -4$. Thus an equation of the tangent line is given by $y - 2 = -4(x - 0)$ or simply $y = -4x + 2$.

5. [10 pts.] Find all points on the curve $y = \dfrac{x+3}{x+2}$ where the tangent line is perpendicular to the line $y = x$.

The slope of the line $y = x$ is $\boxed{1.}$

لأن معامل ال x هو ١

$$\frac{dy}{dx} = \frac{(x+2)-(x+3)}{(x+2)^2} = -\frac{1}{(x+2)^2}.$$

Solving for x,

$$-\frac{1}{(x+2)^2} = -1 \Rightarrow |x+2| = 1 \Rightarrow x = -3\,, -1.$$

Hence, the points are $P_1(-3,0)$ and $P_2(-1,2)$.

33–36 Find an equation of the tangent line to the curve at the given point.

36. $y = \sqrt[4]{x} - x, \quad (1, 0)$

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equation of tangent line

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7. [10 pts.] Find an equation of the tangent line to the graph of $f(x) = \dfrac{x^2+2}{(x+1)e^x}$ at $x = 0$.

We have $f'(x) = \frac{((x+1)e^x)(2x)-(x^2+2)((x+1)e^x+e^x)}{((x+1)e^x)^2}$ using the quotient and product rules for differentiation. Now $f(0) = 2$ and $f'(0) = -4$. Thus an equation of the tangent line is given by $y - 2 = -4(x-0)$ or simply $y = -4x + 2$.

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$$\frac{dy}{dx} = \frac{(x+2)-(x+3)}{(x+2)^2} = -\frac{1}{(x+2)^2}.$$

Solving for x,

$$-\frac{1}{(x+2)^2} = -1 \Rightarrow |x+2| = 1 \Rightarrow x = -3, -1.$$

Hence, the points are $P_1(-3,0)$ and $P_2(-1,2)$.

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2. [10 pts.] Find **equations** of the tangent lines to the curve $f(x) = \dfrac{x-1}{x+1}$ that are **parallel** to the line $x - 2y = 2$.

We have $f'(x) = \dfrac{2}{(x+1)^2}$. Now, any tangent line to the curve has slope $= \dfrac{2}{(x+1)^2} = \dfrac{1}{2}$. This implies that $x = 1$ or $x = -3$. Therefore, the equation of the first tangent line is $y - 0 = \dfrac{1}{2}(x-1)$ and the equation of the second tangent line is $y - 2 = \dfrac{1}{2}(x+3)$.

Calculus A

Chapter 3: Differential Rules

Sections: 3.3 Derivatives of Trigonometric Functions

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Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Differentiate.

11. $f(\theta) = \dfrac{\sin\theta}{1+\cos\theta}$

$$f'(\theta) = \frac{\cos\theta(1+\cos\theta) - (-\sin\theta)(\sin\theta)}{(1+\cos\theta)^2}$$

$$f'(\theta) = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{\cos\theta + 1}{(1+\cos\theta)^2}$$

$$f'(\theta) = \frac{1}{(1+\cos\theta)}$$

$$\frac{\cos\theta+1}{(1+\cos\theta)^2}$$

Calculus A

Chapter 3: Differential Rules

Sections: 3.4 The Chain Rule

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حفظ

BASIC PROPERTIES

$$(cf(x))' = c(f'(x))$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$\frac{d}{dx}(c) = 0$$

PRODUCT RULE

$$(f(x)g(x))' = f(x)'g(x) + f(x)g(x)'$$

QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

POWER RULE

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

CHAIN RULE

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

COMMON DERIVATIVES

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(e^x) = e^x$$

CHAIN RULE

حفظ

$$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} f'(x)$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x) e^{f(x)}$$

$$\frac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} \ln a \cdot f'(x)$$

$$\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$$

$$\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$$

$$\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)]$$

$$\frac{d}{dx}(\sec[f(x)]) = f'(x)\sec[f(x)]\tan[f(x)]$$

- If $f(x) = \sqrt{u}$, then $f'(x) = \frac{u'}{2\sqrt{u}}$ ← أو تشتقه عادي
- If $f(x) = \sqrt[3]{u} = u^{1/3}$ then $f'(x) = \frac{1}{3} u^{-2/3} \cdot u'$

3) Let $f(x) = [2 + \sin(1 - x^2)]^4$

find $f'(1)$

$f'(x) = 4[2 + \sin(1 - x^2)]^3 \cos(1 - x^2)(-2x)$

$\therefore f'(1) = 4[2 + \sin(1 - 1^2)]^3 \cos(1 - 1^2)(-2(1))$

$f'(1) = 4[2 - 0]^3 (-2)$

$f'(1) = 4(8)(-2) = -64$

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4. [10 pts.] If $g(1) = 4$, $g'(1) = 5$ and $f'(6) = 4$, find $\dfrac{d}{dx}\Big[f(2x + g(x))\Big]\Big|_{x=1}$.

$$\frac{d}{dx} f(2x + g(x)) = f'(2x + g(x)).(2 + g'(x)).$$

$$\frac{d}{dx} f(2x + g(x))\Big]_{x=1} = f'(2 + g(1)).(2 + g'(1)) = f'(6).7 = (4)(7) = 28.$$

2. [10 pts.] Find an equation of the tangent line at the point $(0,0)$ to the curve

$$1 + \sin(x+y) = xy + \cos(y).$$

We have $\boxed{\dfrac{dy}{dx} = \dfrac{y - \cos(x+y)}{\cos(x+y) - x + \sin y}}$. This implies that the slope of the tangent line is $\boxed{m = \dfrac{dy}{dx}\Big|_{(0,0)} = -1}$. Therefore, an equation of the tangent line is $\boxed{y - 0 = -(x-0)}$.

1. [10 pts.] Find an equation of the tangent line to the curve $y = (x + e^x)^3$ at $x = 0$.

We have $y|_{x=0} = (0+1)^3 = 1$ and $y' = 3(x+e^x)^2(1+e^x)|_{x=0} = (3)(1)(2) = 6$. Thus an equation of the tangent line to the given curve at the given point is give by $y - 1 = 6(x-0)$.

3.5 Implicit Differentiation

$\frac{dy}{dx}$ نستخدم طريقة الاشتقاق الضمني (implicit Differentiation) لايجاد المشتقة
لما يكون صعب عليي أفصل الy عن x مثال :

examples:-

- $x^2 + y^2 = 25$
- $2xy + \sin x = 6x \sin(x) - 1$
- $\sin(x+y) = x$

طرق الاشتقاق

1) $x^2 \xrightarrow{\frac{d}{dx}} 2x$

2) $y^2 \xrightarrow{\frac{d}{dx}} 2y\,y'$ $\qquad y' = \frac{dy}{dx}$

3) $y e^{\sin x} \xrightarrow{\frac{d}{dx}} y' e^{\sin x} + y\, e^{\sin x} \cdot \cos x$

$\frac{dy}{dx} = y'$

طريقة حل الاشتقاق الضمني :

١- نشتق الطرفين بالنسبة للمتغير x

٢- نخلي $\frac{dy}{dx}$ بطرف بروحها

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

$$x^2 + y^2 = 25$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dy}(25)$$

$$(2x + 2yy') = 0$$

$$2yy' = -2x \Rightarrow y' = \frac{-2x}{2y}$$

$$y' = -\frac{x}{y}$$

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$y' = \frac{dy}{dx} = -\frac{3}{4}$$

x y

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

(b)
$$f'(x) = x\frac{1}{1+\left(\sqrt{x}\right)^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

3.6 Derivatives of Logarithmic Functions

$$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}}$$

$$\frac{d}{dx}(\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

في بعض المسائل رح نستخدم خواص الln أو log كثر ما نقدر عشان تسهل علينا عملية الاشتقاق

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- $\ln(AB) = \ln A + \ln B$
- $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
- $\ln A^x = x \ln A$

↳ ex: $\ln x^4 = 4 \ln x$, $\ln x^{\sin x} = (\sin x) \ln x$

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

ex: $x^{\sin x}$, $x^{\sqrt{x}}$, $\sqrt{x}^{\ln x}$ ....

نستخدم الطريقة هذي لما :
١- دالة أس دالة .
٢- دالة معقدة علينا

EXAMPLE 7 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$ ■

طريقة حل اشتقاق دالة أس دالة :

$$\frac{d}{dx}\left[f(x)\right]^{g(x)}$$

1) $y = f(x)^{g(x)}$

2) $\ln y = \ln f(x)^{g(x)}$ أخذ ln

3) $\ln y = g(x) \ln f(x)$ خواص ال ln

4) $\frac{1}{y} y' = g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x)$ اشتق الطرفين

5) $y' = y \left(g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x) \right)$ أبسط إذا في مجال للتبسيط

بعدين أعوض بقيمة y

3) Let $f(x) = [2+\sin(1-x^2)]^4$

find $f'(1)$

$$f'(x) = 4[2+\sin(1-x^2)]^3 \cos(1-x^2)(-2x)$$

$$\therefore f'(1) = 4[2+\sin(1-1^2)]^3 \cos(1-1^2)(-2(1))$$

$$f'(1) = 4[2-0]^3 (-2)$$

$$f'(1) = 4(8)(-2) = -64$$

4. [10 pts.] If $g(1) = 4$, $g'(1) = 5$ and $f'(6) = 4$, find $\frac{d}{dx}\Big[f(2x+g(x))\Big]\Big|_{x=1}$.

$$\frac{d}{dx} f(2x+g(x)) = f'(2x+g(x)).(2+g'(x)).$$

$$\frac{d}{dx} f(2x+g(x))\Big]_{x=1} = f'(2+g(1)).(2+g'(1)) = f'(6).7 = (4)(7) = 28.$$

2. [10 pts.] Find an equation of the tangent line at the point $(0,0)$ to the curve

$$1+\sin(x+y) = xy+\cos(y).$$

We have $\boxed{\dfrac{dy}{dx} = \dfrac{y-\cos(x+y)}{\cos(x+y)-x+\sin y}}$. This implies that the slope of the tangent line is $\boxed{m = \dfrac{dy}{dx}|_{(0,0)} = -1}$. Therefore, an equation of the tangent line is $\boxed{y-0 = -(x-0)}$.

1. [10 pts.] Find an equation of the tangent line to the curve $y = (x+e^x)^3$ at $x = 0$.

We have $y|_{x=0} = (0+1)^3 = 1$ and $y' = 3(x+e^x)^2(1+e^x)|_{x=0} = (3)(1)(2) = 6$. Thus an equation of the tangent line to the given curve at the given point is give by $y-1 = 6(x-0)$.

1) find the equation of tangent line at $(0,3)$ --

$$y^3 + x^2 y + x^2 - 3y^2 = 0$$

$$3y^2 y' + x^2 y' + 2xy + 2x - 6yy'$$

at $(\overset{x}{0}, \overset{y}{3})$

$$3(3)^2(y') + (0)(y') + 2(0)(3) + 2(0) - 6(3)(y') = 0$$

$$27y' - 18y' = 0 \Rightarrow 9y' = 0$$

$$y' = 0$$

$$y' = 0 = m \qquad y - y_1 = m(x - x_1)$$

$$y - 3 = 0(x - 0) \qquad \therefore y = 3$$

Q3. [10 pts.] Find an equation of the tangent line at the point $(0,1)$ to the curve with equation

$$2y + x^2y + x\sin(x^2) = e^{xy} + 1.$$

$$2y + x^2y + x\sin(x^2) = e^{xy} + 1.$$

Differentiating both sides wrt x we get:

$$2y' + 2xy + x^2y' + \sin(x^2) + x\cos(x^2)(2x) = (y + xy')e^{xy}.$$

At $(0,1)$ we get

$$y' = \frac{1}{2}.$$

Thus, an equation of the tangent line is $y - 1 = \frac{1}{2}x$.

Q4. [10 pts.] Find an equation of the tangent line to the curve $y = \sin(xy^2)$ at the point $(\pi/2, 1)$.

Differentiating implicitly with respect to x, we get $\boxed{y' = \cos(xy^2)\left[y^2 + 2xyy'\right]}$.

At the point $(\pi/2, 1)$,

$\boxed{y' = 0}$. Thus, an equation of the tangent line to the given cure at the point $(\pi/2, 1)$ is $\boxed{y = 1}$.

b) Find $\frac{dy}{dx}$ if: $y = \frac{x}{\sin^{-1}(2x)}$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\sin^{-1}2x = \frac{1}{\sqrt{1-(2x)^2}} * 2$$

مشتقة داخل القوس

$$y' = \frac{(1)(\sin^{-1}(2x)) - \left(\frac{1}{\sqrt{1-4x^2}}\right)(2) * x}{(\sin^{-1}(2x))^2}$$

$$y' = \frac{(\sin^{-1}(2x)) - \left(\frac{2x}{\sqrt{1-4x^2}}\right)}{(\sin^{-1}(2x))^2} = \frac{\frac{(\sin^{-1}(2x))\sqrt{1-4x^2} - 2x}{\sqrt{1-4x^2}}}{(\sin^{-1}(2x))^2}$$

$$y' = \frac{(\sin^{-1}(2x))\sqrt{1-4x^2} - 2x}{\sqrt{1-4x^2}\;(\sin^{-1}(2x))^2}$$

$$y' = \frac{\sin^{-1}(2x) - \frac{2x}{\sqrt{1-4x^2}}}{(\sin^{-1}(2x))^2} = \frac{\sin^{-1}(2x)\sqrt{1-4x^2} - 2x}{\sqrt{1-4x^2}\,(\sin^{-1}(2x))^2}.$$

7) find $\frac{dy}{dx}$

$$e^{xy} + 2y - 3x = \sin(y)$$

$$e^{xy}(y + xy') + 2y' - 3 = \cos(y)y'$$

$$ye^{xy} + xy'e^{xy} + 2y' - 3 = \cos(y)y'$$

$$y'(xe^{xy} + 2 - \cos y) = 3 - ye^{xy}$$

$$y' = \frac{3 - e^{xy}y}{xe^{xy} - \cos y + 2}$$

find $\frac{dy}{dx}$ $\quad y = x^{\sin x}$

$$y = x^{\sin x}$$

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{1}{x} \sin x$$

$$y' = \left(\cos x \ln x + \frac{\sin x}{x}\right) y$$

$$\therefore y' = \left(\cos x \ln x + \frac{\sin x}{x}\right) x^{\sin x}$$

c) Find $\dfrac{dy}{dx}$ if: $y = (3x^2 - 2x + 1)^{\tan x}$.

Taking the logarithm of both sides we get: $\ln y = \tan x \; \ln(3x^2 - 2x + 1)$.

Differentiating w.r.t x yields: $\dfrac{y'}{y} = \sec^2 x \; \ln(3x^2 - 2x + 1) + \dfrac{(6x - 2) \; \tan x}{(3x^2 - 2x + 1)}$.

Or, $\dfrac{y'}{y} = \dfrac{\sec^2 x \; \ln(3x^2 - 2x + 1) \; (3x^2 - 2x + 1) + (6x - 2) \; \tan x}{(3x^2 - 2x + 1)}$.

Thus $y' = \left((3x^2 - 2x + 1)^{\tan x}\right) \left(\dfrac{\sec^2 x \; \ln(3x^2 - 2x + 1) \; (3x^2 - 2x + 1) + (6x - 2) \; \tan x}{(3x^2 - 2x + 1)}\right)$.

Q2. [10 pts.] Find $\dfrac{dy}{dx}$, where $y = (\sin x)^{\ln x}$.

Taking the logarithm of both sides we get: $\ln y = \ln(\sin x)^{\ln x} = \ln x \ln(\sin x)$.

Differentiating wrt x yields: $\dfrac{y'}{y} = \dfrac{1}{x} \ln(\sin x) + \dfrac{\cos x}{\sin x} \ln x$. Thus

$$y' = \left(\frac{1}{x} \ln(\sin x) + \frac{\cos x}{\sin x} \ln x\right) (\sin x)^{\ln x}.$$

3) $f(x) = (x^2+1)^{\ln x}$, find y'

Let $y = (x^2+1)^{\ln x}$

$$\ln y = \ln (x^2+1)^{\ln x}$$

$$\ln y = (\ln x)(\ln (x^2+1))$$

Product rule

$$\frac{y'}{y} = \frac{1}{x}(\ln(x^2+1)) + (\ln x)\left(\frac{2x}{x^2+1}\right)$$

$$y' = \left(\frac{\ln x^2+1}{x} + \frac{2x\ln x}{x^2+1}\right) y$$

$$y' = \left(\frac{\ln x^2+1}{x} + \frac{2x\ln x}{x^2+1}\right)(x^2+1)^{\ln x}$$

Question 2. Differentiate the followings:

(a) Find $\frac{dy}{dx}$ if $y = x^{\sin(x)}$

(b) Find $\frac{dy}{dx}$ if $y = (3x^2 - 2x + 1)^{\tan(x)}$

(c) Find $\frac{dy}{dx}$ if $y = (x^2 + 1)^{\ln(x)}$

(d) Find $\frac{dy}{dx}$ if $e^{xy} + 2y - 3x = \sin(y)$

(e) Find $\frac{dy}{dx}$ if $y = x^2y^2 + x\sin(y) = y$

a) let $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \sin x \frac{1}{x} + \cos x \ln x$$

$$y' = y\left(\sin x \frac{1}{x} + \cos x \ln x\right)$$

$\frac{\sin x}{x}$ ✓

$$\frac{dy}{dx} = x^{\sin x}\left[(\sin x)\frac{1}{x} + \cos x \ln x\right)$$

(b) Find $\frac{dy}{dx}$ if $y = (3x^2 - 2x + 1)^{\tan(x)}$

---

b) let $y = (3x^2 - 2x + 1)^{\tan x}$

$$\ln y = \ln(3x^2 - 2x + 1)^{\tan x}$$

$$\ln y = \tan x \ln(3x^2 - 2x + 1)$$

$$\frac{y'}{y} = \sec^2 x \ln(3x^2 - 2x + 1) + \left(\frac{3x^2 - 2x + 1}{6x - 2}\right) \tan x$$

$$y' = y\left[\sec^2 x \ln(3x^2 - 2x + 1) + \left(\frac{3x^2 - 2x + 1}{6x - 2}\right) \tan x\right]$$

$$\frac{dy}{dx} = (3x^2 - 2x + 1)^{\tan x} \left[\sec^2 x \ln(3x^2 - 2x + 1) + \left(\frac{3x^2 - 2x + 1}{6x - 2}\right) \tan x\right]$$

(c) Find $\frac{dy}{dx}$ if $y = (x^2 + 1)^{\ln(x)}$

---

c) let $y = (x^2+1)^{\ln(x)}$

$$\ln y = \ln (x^2+1)^{\ln(x)}$$

$$\ln y = \ln x \ \ln(x^2+1)$$

$$\frac{y'}{y} = \frac{1}{x} \ln(x^2+1) + \left(\frac{2x}{x^2+1}\right) \ln x$$

$$y' = y\left[\frac{\ln(x^2+1)}{x} + \frac{2x \ln x}{x^2+1}\right]$$

$$y' = (x^2+1)^{\ln(x)}\left[\frac{\ln(x^2+1)}{x} + \frac{2x \ln x}{x^2+1}\right]$$

(d) Find $\frac{dy}{dx}$ if $e^{xy} + 2y - 3x = \sin(y)$

---

d) $e^{xy}(1y + xy') + 2yy' - 3 = \cos(y)y'$

$ye^{xy} + y'xe^{xy} + 2yy' - 3 = y'\cos(y)$

$y'xe^{xy} + 2yy' - y'\cos(y) = 3 - ye^{xy}$

$y'(xe^{xy} + 2y - \cos(y) = 3 - ye^{xy}$

$$y' = \frac{3 - ye^{xy}}{(xe^{xy} + 2y - \cos(y))}$$

(e) Find $\frac{dy}{dx}$ if $y = x^2y^2 + x\sin(y) = y$

e) $2xy^2 + x^2(2yy') + \sin(y) + x\cos(y)y' = y'$

$2x^2yy' + xy'\cos(y) - y' = -2xy^2 - \sin(y)$

$y'(2x^2y + x\cos(y) - 1 = -2xy^2 - \sin(y)$

$$y' = \frac{-2xy^2 - \sin(y)}{2x^2y + x\cos(y) - 1} = -\frac{(2xy + \sin y)}{2x^2y + x\cos(y) - 1}$$

Question 3. Find an equation of the tangent and normal lines to the following curves at the given points:

(a) $\sin(x+y) = (x+2)(y+2)$ at $P(2,-2)$

equ of tangent: $y - y_1 = m(x - x_1)$

ضربت الاقواس
ببعض .. أسهل
لي لما اشتقها

$$\sin(x+y) = xy + 2x + 2y + 4$$

$$\cos(x+y)\cdot(1+y') = 1y + xy' + 2 + 2y'$$

$$\cos(x+y) + y'\cos(x+y) = y + xy' + 2 + 2y'$$

$$y'\cos(x+y) - 2y' - xy' = y + 2 - \cos(x+y)$$

$$y'(\cos(x+y) - 2 - x) = y + 2 - \cos(x+y)$$

$$y' = \frac{y+2-\cos(x+y)}{(\cos(x+y) - 2 - x)}$$

$\because (2,-2)$

$$y' = \frac{-2+2-\cos(2-2)}{(\cos(2-2) - 2 - 2)} = \frac{0-1}{1-2-2} = \frac{-1}{-3} = \frac{1}{3}$$

$P(2,-2)$, $y' = \frac{1}{3}$

equ of tangent: $y - y_1 = y'(x - x_1)$

$$y - (-2) = \frac{1}{3}(x-2)$$

$$y + 2 = \frac{1}{3}(x-2)$$

equ of normal : $y - y_1 = y'_n(x - x_1)$

$$y'_n = -\frac{1}{y'} = -\frac{1}{\frac{1}{3}} = -3$$

$$y - (-2) = -3(x-2)$$

$$y + 2 = -3(x-2)$$

(b) $y^3 + x^2y + x^2 - 3y^2 = 0$ at $P(0,3)$

$$3y^2y' + 2xy + x^2y' + 2x - 6yy' = 0$$

$$3y^2y' + x^2y' - 6yy' = -2xy - 2x$$

$$y'(3y^2 + x^2 - 6y) = -2xy - 2x$$

$$y' = \frac{-2xy - 2x}{3y^2 + x^2 - 6y}$$

$$y' = -\frac{2xy - 2x}{3y^2 - 6y + x^2}$$ رتبتها

@ $P(0,3)$

$$y' = -\frac{2(0)(3) - 0}{3(3)^2 - 6(3) + 0} = 0$$

equ of tang: $y - 3 = 0(x - 0) \Rightarrow y - 3 = 0$

equ of Normal: undefined لان رح يطلع المقام بصفر

$\rightarrow y_n' = \frac{-1}{y'} = \frac{-1}{0} = \phi$

(c) $2y + x^2y + x\sin(x^2) = e^{xy} + 1$ at $P(0,1)$

$$2y' + 2xy + x^2y' + \sin(x^2) + x(\cos(x^2))2x = e^{xy}(y + xy')$$

$$2y' + 2xy + x^2y' + \sin(x^2) + 2x^2\cos(x^2) = ye^{xy} + e^{xy}xy'$$

$$2y' + x^2y' - e^{xy}xy' = ye^{xy} - 2xy - \sin(x^2) - 2x^2\cos(x^2)$$

$$y'(2 + x^2 - xe^{xy}) = ye^{xy} - 2xy - \sin(x^2) - 2x^2\cos(x^2)$$

$$y' = \frac{ye^{xy} - 2xy - \sin(x^2) - 2x^2\cos(x^2)}{2 + x^2 - xe^{xy}}$$

$\because (0,1)$

$$y' = \frac{1e^{0 \to 1} - 0 - 0 - 0}{2 + 0 - 0} = \frac{1}{2}$$

equ of Tang: $y - 1 = \frac{1}{2}(x - 0) = y - 1 = \frac{1}{2}x$

equ of Normal: $y - 1 = -2(x - 0) = y - 1 = -2x$

(d) $y = x^{\ln(x)}$ at $P(1,1)$

$$y = x^{\ln x} \qquad \ln y = \ln x^{\ln x}$$

$$\ln y = \ln x \ln x$$

$$\frac{y'}{y} = \frac{1}{x}\ln x + \frac{1}{x}\ln x$$

$$\frac{y'}{y} = \frac{\ln x}{x} + \frac{\ln x}{x} = \frac{2\ln x}{x}$$

$$y' = y\left(\frac{2\ln x}{x}\right)$$

$$y' = \frac{x^{\ln x} \cdot 2\ln x}{x}$$

$\because (1,1)$

$$\frac{1^{\ln 1} \cdot 2\ln 1}{1} = \frac{1^{0} \cdot 0}{1} = \frac{0}{1} = 0$$

equ of Tang : $y-1 = 0(x-0) = y-1 = 0$
$y = 1$

equ of Normal : undefined لان رح يطلع المقام بصفر

(e) $y = \sin(xy^2)$ at $P(\frac{\pi}{2}, 1)$

$$y' = \cos(xy^2) \cdot (y^2 + 2xyy')$$

$$y' = y^2\cos(xy^2) + 2xyy'\cos(xy^2)$$

$$y' - 2xyy'\cos(xy^2) = y^2\cos(xy^2)$$

$$y'(1 - 2xy\cos(xy^2)) = y^2\cos(xy^2)$$

$$y' = \frac{y^2\cos(xy^2)}{1 - 2xy\cos(xy^2)}$$

$\because (\frac{\pi}{2}, 1)$

$$\frac{1\cos(\frac{\pi}{2} * 1)}{1 - 2\frac{\pi}{2} * 1\cos(\frac{\pi}{2} * 1)} = \frac{0}{1} = 0$$

equ of Tang: $y - 1 = 0(x - \frac{\pi}{2}) = y - 1 = 0$, $y = 1$

equ of Normal: undefined

لان رح يطلع المقام بصفر

Question 1. Show that the left-hand-side is equal to the right-hand-side:

(a) $\frac{d}{dx}\left[\sin^{-1}(2x)\right] = \frac{2}{\sqrt{1-4x^2}}$

(b) $\frac{d}{dx}\sinh(x) = \cosh(x)$

(c) $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$

Let $y = \sin^{-1}(2x)$

$\therefore \sin y = 2x$

$z = \sqrt{1-4x^2}$

$z = \sqrt{1^2 - (2x)^2} = \sqrt{1-4x^2}$

now we diff with respect to x

$\therefore \sin y = 2x$

$(\cos y)\, y' = 2$ اشتقيناها اشتقاق ضمني

$\therefore y' = \frac{2}{\cos y} = \frac{2}{\frac{\sqrt{1-4x^2}}{1}}$

$\therefore \frac{d}{dx}(\sin^{-1} 2x) = y' = \frac{2}{\sqrt{1-4x^2}}$

3. [10 pts.] Let $y = (\sqrt{x})^{\sin x}$. Find $\dfrac{dy}{dx}$.

Taking logarithms of both sides of the equation and using the Laws of Logarithms to simplify, we get:

$$\boxed{\ln y = \ln(\sqrt{x})^{\sin x} = \sin x \ln \sqrt{x} = \frac{1}{2} \sin x \ln x}.$$

Differentiating implicitly with respect to x gives:

$$\boxed{\frac{y'}{y} = \frac{1}{2}\left(\frac{\sin x}{x} + \cos x \ln x\right).}$$

Therefore, $\boxed{\dfrac{dy}{dx} = \dfrac{(\sqrt{x})^{\sin x}}{2}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)}$.

Calculus A

Chapter 3: Differential Rules

Sections: 3.9 Related Rates

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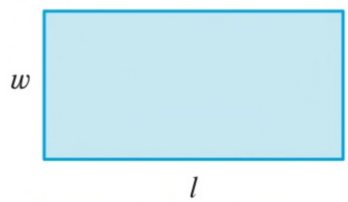
S=Surface Area = A = Area. P = Perimeter = Circumference = C. Volume = V

Rectangle

$A = lw$

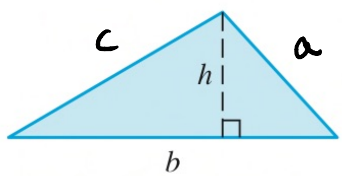
$P = 2l + 2w$



Triangle

$A = \frac{1}{2}bh$

$P = a+b+c$

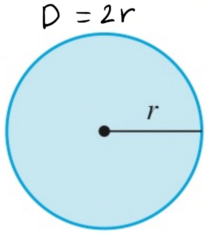


Circle

$A = \pi r^2$

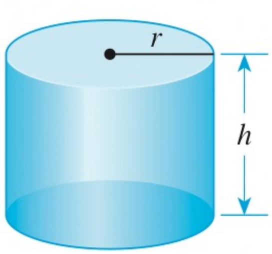
$C = 2\pi r$

$D = 2r$



Cylinder

$V = \pi r^2 h$

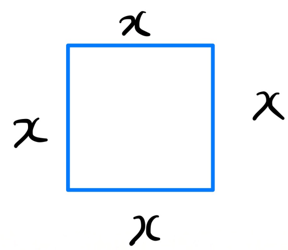


$S = 2\pi r h + 2\pi r^2$

Square

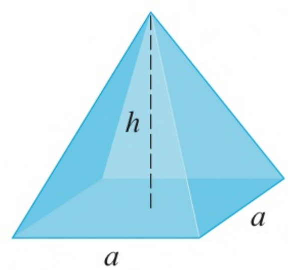
$A = x^2$

$P = 4x$



Pyramid

$V = \frac{1}{3}ha^2$

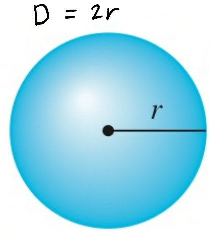


Sphere

$V = \frac{4}{3}\pi r^3$

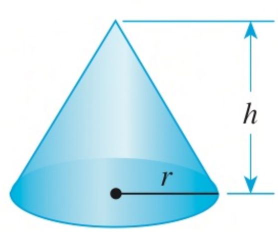
$A = 4\pi r^2$

$D = 2r$



Cone

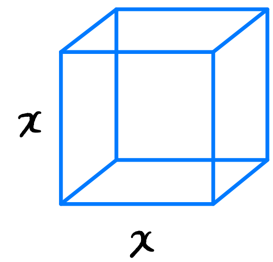
$V = \frac{1}{3}\pi r^2 h$



$S = \pi r\sqrt{r^2+h^2}$

Cube

$V = x^3$



$S = 6x^2$

RECTANGULAR SOLID

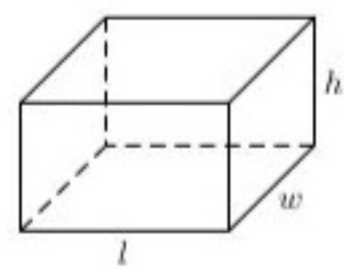
l = length, w = width,

h = height

Volume: $V = lwh$

Surface Area:

$S = 2lw + 2lh + 2wh$



STEPS:

1. As you read the problem pull out essential information & make a diagram if possible.

اكتب المعطيات كلها والمطلوب من السؤال

2. Write down any known rate of change & the rate of change you are looking for, e.g.

$$\frac{dV}{dt} = 3 \quad \& \quad \frac{dr}{dt} = ?$$

3. Be careful with signs...if the amount is decreasing, the rate of change is negative.

4. Pay attention to whether quantities are fixed or varying. For example, if a ladder is 12 meters long you can just call it 12. And if a radius is changing a changing rate, just call it r. You will plug in values for varying quantities at the end.

القانون المناسب الي يربط بين المعطيات والمطلوب

6. Set up an equation involving the appropriate quantities.

7. Differentiate with respect to t using implicit differentiation.

8. Plug in known items (you may need to find some quantities using geometry).

9. Solve for the item you are looking for, most often this will be a rate of change.

10. Express your final answer in a full sentence with units that answers the question asked.

$\frac{d}{dt} = -$

أي كمية يعطيني إياها قاعده تتناقص مع الوقت
رح أحط ماينس جدامه
مثال :
السرعة تتناقص ، تقلص الحجم ، الخ...

Q2. [10 pts.] The surface area of a sphere increases at the rate of 48π cm^2/sec. Find the rate of change of the volume when the surface area is 36 cm^2.

We have $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2 = 36$ cm^2; $r = \dfrac{3}{\sqrt{\pi}}$. Since $\dfrac{dA}{dt} = 8\pi r\dfrac{dr}{dt} = 48\pi$ cm^2/sec, then $\dfrac{dr}{dt} = \dfrac{6}{r} = 2\sqrt{\pi}$ cm/sec. Differentiating V w.r.t t we get: $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. So $\dfrac{dV}{dt} = 4\pi\dfrac{9}{\pi}2\sqrt{\pi} = 72\sqrt{\pi}$ cm^3/sec.

The length of a rectangle is increasing at rate of 10 cm/s and its width is decreasing of rate of 4 cm/s. When the length is 40 cm and the width is 15 cm . How fast is the area of the rectangle changing?

---

given: rectangle, $\dfrac{dL}{dt} = 10$ cm/s , $\dfrac{dw}{dt} = -4$ cm/s

$L = 40$ cm , $W = 15$ cm

$\therefore A = LW$

$$\frac{dA}{dt} \overset{?}{=} \frac{dL}{dt} W + L\frac{dw}{dt}$$

$$= 10 * 15 + 40(-4)$$

$$= 150 - 160 = -10 \text{ cm}^2/\text{s}$$

The area of rectangle is decreasing by 10 cm^2/s

Question 4. The followings are word problems for rectangular shapes. Use related rates to solve them.

(a) Each side of a square is increasing at a rate of 6 cm/sec. At what rate is the area of the square increasing when the area is 25 cm^2.

given:- square, $\frac{dx}{dt} = 6\ cm/s$

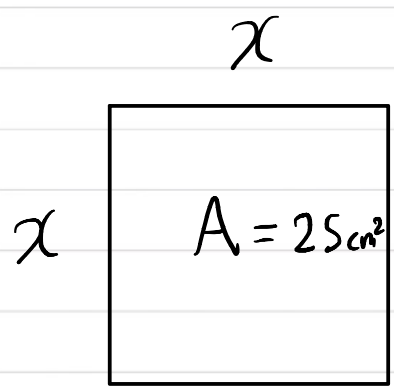
$A = 25\ cm^2$

unknown: $\frac{dA}{dt} = ??$

Formula: $A(x) = x^2$

Diff:-

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$



we need x:-

$\because A = 25\ cm^2$ & $A = x^2$

$\therefore 25\ cm^2 = x^2 \Rightarrow x = 5\ cm$ — ما في طول بالماينس

$\therefore \frac{dA}{dt} = 2(5cm)(6\ cm/s) = 60\ cm^2/s$

(c) Oil spilled from a tanker spreads in a circle whose area increases at a constant rate of 9 km^2/hr. How fast is the radius of the spill increasing when its area is 16 km^2?

given: circle, $\frac{dA}{dt} = 9\ km^2/hr$

$A = 16\ km^2$

unknown: $\frac{dr}{dt} = ??$

Formula: $A(r) = \pi r^2$

Diff:-

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

??

$$\Rightarrow 9 = 2\pi r \frac{dr}{dt}$$

بس ناخذ
الموجب لأنه طول

$$\because A = \pi r^2 \Rightarrow 16 = \pi r^2 \Rightarrow r^2 = \frac{16}{\pi}$$

$$\Rightarrow r = \sqrt{\frac{16}{\pi}} = \frac{\sqrt{16}}{\sqrt{\pi}} = \frac{4}{\sqrt{\pi}}$$

$$\boxed{\frac{\pi}{\sqrt{\pi}} = \frac{\pi}{\pi^{1/2}} = \pi^{1-1/2} = \pi^{1/2} = \sqrt{\pi}}$$

$$\because 9 = 2\pi \frac{4}{\sqrt{\pi}} \frac{dr}{dt} \Rightarrow 9 = 8\sqrt{\pi} \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{9}{8\sqrt{\pi}}$$

Question 5. The followings are word problems for cylindrical shapes. Use related rates to solve them.

(a) If a cylindrical tank of diameter 6m is being filled with water at a rate of 2 m^3/hr, how fast is the water level rising?

[~~10pts~~] We have $\boxed{V = \pi r^2 h}$. This implies that $\boxed{\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}}$. Therefore, $\boxed{\frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt}}$.

Hence, $\boxed{\frac{dh}{dt} = \frac{2}{9\pi}\ \mathrm{m/hr}}$.

Example 1: "The Falling Ladder"

A ladder is sliding down along a vertical wall. If the ladder is 5 meters long and the top is slipping at the constant rate of 0.9 m/s, how fast is the bottom of the ladder moving along the ground when the bottom is 4 meters from the wall?

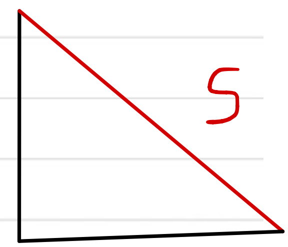
$$\frac{dx}{dt} = \frac{72}{60} \text{ m/s}$$

(a) A 5-meter ladder rests against a vertical wall. If the top of the ladder is sliding down the wall at a rate of 0.9 m/s, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 4m from the ground?

$$x^2 + y^2 = 5^2 \longrightarrow (1)$$

$$\frac{dy}{dt} = -\frac{9}{10} \downarrow$$

when $y = 4$



$$\frac{dx}{dt} = ?$$

$$x^2 + 4^2 = 25$$

$$x^2 = 25 - 16 = 9$$

$$x^2 = 9 \Rightarrow x = 3$$

Diff (1):

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(3)\left(\frac{dx}{dt}\right) + 2(4)\left(-\frac{9}{10}\right) = 0$$

$$6\frac{dx}{dt} - \frac{72}{10} = 0 \Rightarrow 6\frac{dx}{dt} = \frac{72}{10 \times 6}$$

$$6$$

$$\frac{dx}{dt} = \frac{72}{60} = \frac{6}{5}$$

(a) A particle is moving along the curve $y = \sqrt{1+x^3}$ in the $x-y$ plane. If the $y-$coordinate is increasing at a rate of 4 cm/sec at $x = 2$, how fast is the $x-$coordinate changing?

$$y = \sqrt{1+x^3} = (1+x^3)^{1/2}$$

نتذكر قاعدين

نشتق بالنسبة للزمن

$$\frac{dy}{dt} = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)\left(\frac{dx}{dt}\right)$$

$\because$ given: $\frac{dy}{dt} = 4$ cm/s , $x = 2$

$$4 = \frac{1}{2}\left(\frac{1}{\sqrt{1+2^3}}\right)\left(3(2)^2\right)\frac{dx}{dt}$$

$$4 = \frac{1}{2}\,\frac{1}{\sqrt{9}}\,(12)\left(\frac{dx}{dt}\right)$$

$$4 = \frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{3}}\,(\overset{\cancel{4*3}\ 2}{\cancel{12}})\left(\frac{dx}{dt}\right)$$

$$4 = 2\,\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\ \text{cm/s}$$

Calculus A

Chapter 3: Differential Rules

Sections: 3.10 Linear Approximation and Differentials

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أنواع الأسئلة في سكشن 3.10

1) Find the linearization

$$L(x) = f(a) + f'(a)(x - a)$$

2) Use a linear approximation (or differentials) to estimate

$$f(x) \approx f(a) + f'(a)(x - a)$$

"يعني تقريبا

إذا مو عطيك قيمة الـa بالسؤال .. أنت رح تيبب قيمة الـa الي هي رح تكون العدد الاقرب للسؤال

ex: $\sqrt{0.9}$, $x = 0.9$ $\therefore a = 1$

$e^{0.1}$, $x = 0.1$ $\therefore a = 0$

$\sqrt{99.8}$, $x = 99.8$ $\therefore a = 100$

شلون تختار a

١- تكون قيمة قريبة من الـx

٢- تكون عارف قيمة الـf(a)

3) Find the differential of each function.

إذا كان معطيك قيمة ل dx

$$dy = f'(x)\,dx$$

3. $[5+5+5=15 \text{ pts.}]$ Let $f(x)=\tan^{-1}(x+1)$.

(a) Find the linearization of f at $x=0$.

(b) Find the differential of f with $dx=0.1$ at $x=0$.

(c) Use a linear approximation to estimate $\tan^{-1}(1.1)$.

We have $f(0)=\tan^{-1}1=\dfrac{\pi}{4}$ and $f'(x)=\dfrac{1}{1+(x+1)^2}$. Thus we obtain $f'(0)=\dfrac{1}{2}$.

(a) $L(x)=f(0)+f'(0)(x-0)=\dfrac{\pi}{4}+\dfrac{1}{2}(x-0)=\dfrac{\pi}{4}+\dfrac{x}{2}$.

(b) $dy=f'(0)dx=(\dfrac{1}{2})(0.1)=0.05$.

(c) $\tan^{-1}(1.1)\approx L(0.1)=\dfrac{\pi}{4}+\dfrac{0.1}{2}=\dfrac{\pi}{4}+0.05$.

---

9. $[10 \text{ pts.}]$ Let $f(x)=2^x\cos(x)$.

(a) Find the linearization of f at $a=0$.

(b) Use linear approximation to estimate $f(0.01)$.

---

a) $f(0)=1$ and $f'(x)=2^x\ln 2\cos x-2^x\sin x$. Therefore, $f'(0)=\ln 2$.

Hence,

$$L(x)=1+x\ln 2.$$

b) $f(0.01)\approx L(0.01)=1+0.01\ln 2$.

Q3. [5+5=10 pts.] Let $f(x) = \ln x$.

a) Find the linearization of f at $a = 1$.

b) Use linear approximation to estimate $\ln(1.01)$.

(a) Find the linearization of f at $a = 1$.

$L(x) = f(1) + f'(1)(x - 1) = x - 1.$

(b) Use linear approximation to estimate $\ln(0.01)$.

$f(x) \approx x - 1$, when x is near 1.

Therefore, $\ln(1.01) \approx 1.01 - 1 = 0.01.$

7. [10 pts.] Use **linear approximation** to estimate $f(1.95)$

$$f(x) = \sqrt{x^2+5}$$

$$f(x) = \frac{890}{300}$$

Question 3. Given the function $f(x)$ use Linear Approximation to estimate its value at nearby points.

(a) $f(x) = e^x \cos(x)$, estimate $f(0.1)$.

(b) $f(x) = \ln(x)$, estimate $\ln(1.1)$.

a) We have $\boxed{f'(x) = e^x \cos x - e^x \sin x}$. Also, we have $\boxed{f(0) = 1}$ and $\boxed{f'(0) = 1}$.

Therefore, $\boxed{f(0.1) \simeq f(0) + f'(0)(0.1 - 0) = 1 + 0.1 = 1.1}$.

b) $f(x) \approx f(a) + f'(a)(x-a)$

$x = 1.1$, $a = 1$,

$x - a = 1.1 - 1 = 0.1$

$f(x) = \ln x$, $f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = \frac{1}{1} = 1$

$\therefore \ln(1.1) \approx 0 + 1(0.1)$

$\ln(1.1) \approx 0.1$

(d) $f(x) = e^x$, estimate $e^{0.01}$.

$$f(x) \simeq f(a) + f'(a)(x-a)$$

Let $x = 0.01$

$a = 0$

$\therefore (x-a) = 0.01 - 0 = 0.01$

$f(x) = e^x \qquad f'(x) = e^x$

$f(0) = e^0 = 1 \quad , \quad f'(0) = e^0 = 1$

$\therefore e^{0.01} \simeq 1 + 1(0.01) = 1 + 0.01$

$e^{0.01} \simeq 1 + 0.01 = 1.01$

1–4 Find the linearization $L(x)$ of the function at a.

2. $f(x) = \sin x, \quad a = \pi/6$

$$L(x) = f(a) + f'(a)(x-a)$$

$$a = \frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$f'(x) = \cos x, \quad f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$

5. [10 pts.] Find the linearization of the function $f(x) = x \ln x$ at $a = 1$ and use it to approximate $f(1.1)$.

5. [10 pts.] We have: $\boxed{L(x) = f(1) + f'(1)(x-1) = 0 + f'(1)(x-1)}$.

Since $f'(x) = x(1/x) + \ln x = 1 + \ln x$ and hence $f'(1) = 1$, we get $\boxed{L(x) = x - 1}$.

Thus, $\boxed{f(x) \approx x - 1}$, when x is near 1.

Hence, $\boxed{f(1.1) \approx 1.1 - 1 = 0.1}$.

Calculus A

Chapter 3: Differential Rules

Sections: 3.11 Hyperbolic Functions

A+

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3.11 Hyperbolic Functions

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Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

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Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

1. [10 pts.] If $y = x\sinh(x^2+1) + \arctan(\sqrt{x+1})$, find y'.

$$y' = \sinh(x^2+1) + 2x^2\cosh(x^2+1) + \frac{1}{2(x+2)\sqrt{x+1}}.$$

2) find $\dfrac{dy}{dx}$, where $y = 2^x \cosh x + (\sec^{-1}(x))^5$

$$\frac{dy}{dx} = 2^x \ln 2 \cosh x + 2^x \sinh x + 5\left(sec^{-1} x\right)^4 \left(\frac{1}{x\sqrt{x^2-1}}\right).$$

3) find y'

$$x^2 e^y + y \sinh(x)$$

$$y' = \frac{-2xe^y - \dfrac{ye^x + ye^{-x}}{2}}{x^2 e^y + \sinh x}$$

Q6. [10 pts.] Find an equation of the normal line to the curve: $\cosh(xy) = 2x + y^3$ at $(0, 1)$.

---

we diff.

$\sinh(xy)[1y + xy'] = 2 + 3y^2y'$

at $(0,1)$, we get

$\sinh(0*1)[1*1 + 0*y'] = 2 + 3(1)^2y'$

$0 = 2 + 3y'$

$\Rightarrow 3y' = -2 \Rightarrow y' = \frac{-2}{3} = m$

$\because$ normal line $\therefore m_n = \frac{3}{2}$

equ of normal line

$y - y_0 = m_n(x - x_0)$

$y - 1 = \frac{3}{2}(x - 0)$

Sections 3.10, 3.11

Question 1. Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{\sinh(x)}{e^x}$.

cannot use sandwich theorem on sinhx coshx

$$\lim_{x\to\infty} \frac{\frac{e^x - e^{-x}}{2}}{8e^x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{16e^x} = \frac{\infty - 0}{\infty} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{e^x}{e^x} - \frac{e^{-x}}{e^x}}{\frac{16e^x}{e^x}} = \frac{1 - e^{-x-x}}{16} = \frac{1 - e^{-2x}}{16}$$

$$\therefore \lim_{x\to\infty} \frac{1 - e^{-2x}}{16} = \frac{1 - e^{-\infty}}{16} = \frac{1-0}{16} = \frac{1}{16}$$

Question 2. Differentiate the followings:

(a) Find $\frac{dy}{dx}$ if $x^2e^y + y\sinh(x) = 4$.

(b) Find $\frac{dy}{dx}$ if $y = 2^x\cosh(x) + [\sec^{-1}(x)]^5$

a) $2xe^y + x^2e^yy' + y'\sinh(x) + y\cosh(x) = 0$

$y'(x^2e^y + \sinh(x)) = -2xe^y - y\cosh(x)$

$$y' = -\frac{2xe^y + y\cosh(x)}{(x^2e^y + \sinh(x))}$$

---

b) $2^x\ln(2)\cosh(x) + 2^x\sinh(x)$

$+ 5[\sec^{-1}(x)]^4\left[\frac{1}{|x|\sqrt{x^2-1}}\right]$

$$y' = 2^x\ln(2)\cosh(x) + 2^x\sinh(x) + \frac{5[\sec^{-1}(x)]^4}{|x|\sqrt{x^2-1}}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$