ثمرة

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Calculus 1 – Derivative

(Section 3.1)

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YouTube: Precalculusq8 Twitter: Precalculusq8

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

Difference Rule: $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

1

$$\frac{d}{dx}(x) = 1$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

3.1 Derivatives of Polynomials and Exponential Functions

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

$\frac{d}{dx}(e) = 0$, $\frac{d}{dt}(\pi) = 0$

$\frac{d}{dz}(2^3) = 0$, $\frac{d}{dt}(-5) = 0$

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

1 $$\boxed{\frac{d}{dx}(x) = 1}$$

2 $$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(x^{-2}) = -2x^{-3}$$

$$\frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-3/2}$$

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Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

$\frac{d}{dx}(3) = 0$, $\frac{d}{dx}(-\frac{1}{3}) = 0$ $\frac{d}{dx}(0) = 0$, $\frac{d}{dx}(e) = 0$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$\frac{d}{dx}(x) = 1x^0 = 1$, $\frac{d}{dx}(x^{-\frac{2}{3}}) = -\frac{2}{3}x^{-\frac{2}{3}-1} = -\frac{2}{3}x^{-5/3}$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

$\frac{d}{dx}(4x^3) = 4(3x^2) = 12x^2$, $\frac{d}{dx}(-\frac{1}{2}x^{-2}) = x^{-3}$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

$\frac{d}{dx}(4e^x) = 4e^x$

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The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$.

(d) $\frac{d}{dr}(r^3) = 3r^2$ ■

e) $$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

رح نشوفها وأيد تأكد إنك عارف شلون اشتقيناها

f) $$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

رح نشوفها وأيد تأكد إنك عارف شلون اشتقيناها

EXAMPLE 2 Differentiate:

(a) $f(x) = \frac{1}{x^2}$

(b) $y = \sqrt[3]{x^2}$

SOLUTION In each case we rewrite the function as a power of x.

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b) $$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{(2/3)-1} = \tfrac{2}{3}x^{-1/3}$$

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EXAMPLE 3 Find equations of the tangent line and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Illustrate by graphing the curve and these lines.

SOLUTION The derivative of $f(x) = x\sqrt{x} = xx^{1/2} = x^{3/2}$ is

$$f'(x) = \tfrac{3}{2}x^{(3/2)-1} = \tfrac{3}{2}x^{1/2} = \tfrac{3}{2}\sqrt{x}$$

So the slope of the tangent line at $(1, 1)$ is $f'(1) = \tfrac{3}{2}$. Therefore an equation of the tangent line is

$$y - 1 = \tfrac{3}{2}(x - 1) \qquad \text{or} \qquad y = \tfrac{3}{2}x - \tfrac{1}{2}$$

The normal line is perpendicular to the tangent line, so its slope is the negative reciprocal of $\tfrac{3}{2}$, that is, $-\tfrac{2}{3}$. Thus an equation of the normal line is

$$y - 1 = -\tfrac{2}{3}(x - 1) \qquad \text{or} \qquad y = -\tfrac{2}{3}x + \tfrac{5}{3}$$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\,\frac{d}{dx}f(x)$$

EXAMPLE 4

(a) $\dfrac{d}{dx}(3x^4) = 3\,\dfrac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

(b) $\dfrac{d}{dx}(-x) = \dfrac{d}{dx}[(-1)x] = (-1)\,\dfrac{d}{dx}(x) = -1(1) = -1$ ■

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The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$(f + g)' = f' + g'$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

EXAMPLE 5

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

■

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EXAMPLE 6 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$

$$= 4x^3 - 12x + 0 = 4x(x^2 - 3)$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0$, $\sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 5.) ■

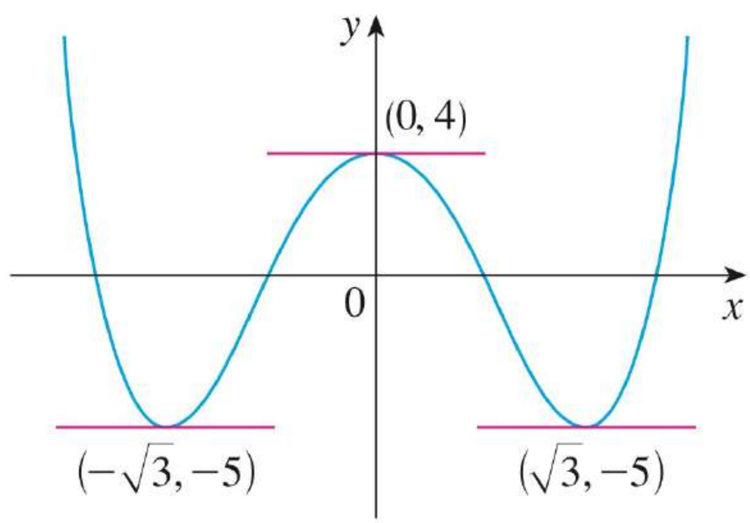

FIGURE 5
The curve $y = x^4 - 6x^2 + 4$ and its horizontal tangents

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Exponential Functions

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

3–32 Differentiate the function.

3. $f(x) = 2^{40}$ **5.** $f(x) = 5.2x + 2.3$

3) $f'(x) = 0$ لأنه ثابت

5) $f'(x) = 5.2$

7. $f(t) = 2t^3 - 3t^2 - 4t$ **9.** $g(x) = x^2(1 - 2x)$

7) $f'(t) = 6t^2 - 6t - 4$

9) $g(x) = x^2 - 2x^3 \Rightarrow g'(x) = 2x - 6x^2$

11. $g(t) = 2t^{-3/4}$ **13.** $F(r) = \dfrac{5}{r^3}$

$-\frac{3}{4} - 1 = \frac{7}{2}$

11) $g'(t) = 2\left(-\frac{3}{4}\right)t \quad = -\dfrac{3}{2} t^{-\frac{7}{4}}$

13) $F(r) = 5r^{-3} \Rightarrow F'(r) = -15r^{-4} = -\dfrac{15}{r^4}$

3–32 Differentiate the function.

15. $R(a) = (3a + 1)^2$

$R(a) = 9a^2 + 6a + 1$

$R'(a) = 18a + 6$

17. $S(p) = \sqrt{p} - p$

$S(p) = p^{\frac{1}{2}} - p$

$S'(p) = \frac{1}{2} p^{\frac{1}{2} - 1} - 1 = \frac{1}{2} p^{-\frac{1}{2}} - 1$

$= \dfrac{1}{2p^{\frac{1}{2}}} - 1 = \dfrac{1}{2\sqrt{p}} - 1$

3–32 Differentiate the function.

19. $y = 3e^x + \frac{4}{\sqrt[3]{x}}$

$$y = 3e^x + 4(x)^{-1/3}$$

$$y' = 3e^x + -\frac{1}{3}\, 4x^{-1/3 - 1}$$

$$y' = 3e^x - \frac{4}{3} x^{-4/3}$$

$$y' = 3e^x - \frac{4}{3\sqrt[3]{x^4}}$$

25. $j(x) = x^{2.4} + e^{2.4}$

$$j'(x) = 2.4x^{1.4} + 0 = 2.4x^{1.4}$$

لأن $e^{2.4}$ يعتبر constant يعني رقم مو نفس e^x

33–36 Find an equation of the tangent line to the curve at the given point.

33. $y = 2x^3 - x^2 + 2, \quad (1, 3)$

$(1, 3)$

$y' = 6x^2 - 2x$

$y'(1) = 6(1) - 2(1) = 4$

$\therefore m = 4$

$\because$ equation of tangent line

$y - y_1 = m(x - x_1)$

$y - 3 = 4(x - 1)$

$y - 3 = 4x - 4$

$y = 4x - 1$

33–36 Find an equation of the tangent line to the curve at the given point.

34. $y = 2e^x + x, \quad (0, 2)$

$$y' = 2e^x + 1$$

$$y'(0) = 2e^0 + 1 = 2 + 1 = 3$$

$$\therefore m = 3$$

$$\therefore y - y_1 = m(x - x_1)$$

equation of tangent line

$$y - 2 = 3(x - 0)$$

$$y = 3x + 2$$

33–36 Find an equation of the tangent line to the curve at the given point.

35. $y = x + \dfrac{2}{x}, \quad (2, 3)$

$y = x + 2x^{-1}$

$y' = 1 + (-1 * 2x^{-2}) = 1 - 2x^{-2}$

$y' = 1 - \dfrac{2}{x^2}$

$y'(2) = 1 - \dfrac{2}{2^2} = 1 - \dfrac{2}{4} = \dfrac{2}{4} = \dfrac{1}{2}$

$\therefore m = 1/2$

equation of tangent line:-

$y - y_1 = m(x - x_1)$

$y - 3 = 1/2 (x - 2)$

$y - 3 = 1/2 x - 1$ $\qquad \therefore y = 1/2 x + 2$

33–36 Find an equation of the tangent line to the curve at the given point.

36. $y = \sqrt[4]{x} - x, \quad (1, 0)$

$$y = x^{1/4} - x$$

$$y' = \frac{1}{4} x^{-3/4} - 1$$

$$y' = \frac{1}{4\sqrt[4]{x^3}} - 1$$

$$y'(1) = \frac{1}{4\sqrt{1}} - 1 = \frac{1}{4} - 1 = -\frac{3}{4}$$

equation of tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{3}{4}(x - 1)$$

$$y = -\frac{3}{4}x + \frac{3}{4}$$

56. For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?

Horizontal tangent means

$\frac{dy}{dx} = 0$, $y' = 0$, $f'(x) = 0$

$f(x) = e^x - 2x$

$f'(x) = e^x - 2$

for H.A , $f'(x) = 0$

$\therefore e^x - 2 = 0 \Rightarrow e^x = 2$

$\ln e^x = \ln 2 \qquad \therefore x = \ln 2$

وإذا تبي قيمة y عوضها قيمة x بالدالة الاصلية

$y = e^{\ln 2} - 2(\ln 2) = 2 - 2\ln 2$

$\therefore$ Tangent is horizontal at $(\ln 2, 2 - 2\ln 2)$

7. [5 + 5 = 10 pts.] Let $f(x) = e^x - 2x$.

(a) Find the point on the curve of f where the tangent line is horizontal.

(b) Using part (a) find an equation of this horizontal tangent line.

(a) Find the point on the curve of f where the tangent line is horizontal.

We have $\boxed{f'(x) = e^x - 2}$.

Thus, $\boxed{f'(x) = 0}$ when $e^x = 2$ i.e., $\boxed{x = \ln 2}$. So the given curve has horizontal tangent when $\boxed{x = \ln 2}$. The corresponding point is $\boxed{(\ln 2, 2 - 2\ln 2)}$.

(b) Using part (a) find an equation of this horizontal tangent line.

Equation of this tangent line is $\boxed{y = 2 - 2\ln 2}$.

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57. Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

$$y' = 2e^x + 3 + 15x^2$$

$$\because\ y' = 2$$

$$\therefore\ 2e^x + 3 + 15x^2 = 2$$

$$2e^x + 15x^2 = -1$$

impossible

لو تعوض بأي قيمة مكان
الx مستحيل يعطين
الناتج سالب أصلا

$\therefore\ 2e^x + 15x^2 = -1$ has no solution and no such tangent exists

58. Find an equation of the tangent line to the curve $y = x^4 + 1$ that is parallel to the line $32x - y = 15$.

$L_1 \Rightarrow y = x^4 + 1$

$y_1' = 4x^3$

$L_2 \Rightarrow 32x - y = 15 \Rightarrow y = 32x - 15$

$y_2' = 32$

$\because$ parallel lines $\therefore$ $m_1 = m_2$, $y_1' = y_2'$

$\therefore 4x^3 = 32 \Rightarrow x^3 = 8$

$x = \sqrt[3]{8} = 2$

$y = x^4 + 1 \Rightarrow y = 2^4 + 1 = 17$

$\therefore$ equation of tangent line

$y - y_1 = m(x - x_1) = y - 17 = 32(x - 2)$

$y = 32x - 47$

4 Theorem If f is differentiable at a, then f is continuous at a.

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

for f to be differentiable at $x = a$, then

1) $\lim_{x \to a} = f(a)$ "continuous"

2) $f'_{+}(a) = f'_{-}(a)$

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71. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geqslant 1 \end{cases}$$

Is f differentiable at 1? ~~Sketch the graphs of f and f'~~.

for f to be differentiable at $x = 1$, then

1) $\lim\limits_{x \to 1} = f(1)$ "continuous"

2) $f'_+(1) = f'_-(1)$

$\because \lim\limits_{x \to 1^-} x^2 + 1 = 2$, $\lim\limits_{x \to 1^+} x + 1 = 2$

$f(1) = 2$ $\therefore f$ is cont on 2

2) $f'_+(1) = 2x \Rightarrow f'_+(1) = 2(1) = 2$

$f'_-(1) = 1$, $\because f'_+(1) \neq f'_-(1)$

$\therefore f$ is not differentiable at $x = 1$

72. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} 2x & \text{if } x \leqslant 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geqslant 2 \end{cases}$$

The functions $2x$, $2x - x^2$, $2 - x$ are cont and diff everywhere

But we need to check at $x = 0, 2$

at $x = 0$

$$\lim_{x \to 0^-} 2x = 2(0) = 0 \quad , \quad g(0) = 0$$

$$\lim_{x \to 0^+} 2x - x^2 = 2(0) - (0)^2 = 0$$

$\because \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0)$ $\therefore g$ is cont on $x=0$

$\because g'_-(0) = 2$, $g'_+(0) = 2 - 2(0) = 2$

$\because g(0)$ is cont and $g'_-(0) = g'_+(0)$ then $g(x)$ is differentiable at $x = 0$

at $x = 2$

$$\lim_{x \to 2^-} 2x - x^2 = 2(2) - 2^2 = 0$$

$$\lim_{x \to 2^+} 2 - x = 2 - 2 = 0$$

$$\because \lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2)$$ $\therefore g$ is cont on $x = 2$

$$\because g'_-(2) = 2 - 2x = 2 - 2(2) = -2$$

$$\because g'_+(2) = -1$$

$$\because g'_+(2) \neq g'_-(2)$$

$\therefore g(x)$ is not differentiable at $x = 2$

$\therefore g$ is diff on $\{\mathbb{R} / 2\}$

81. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leqslant 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

Let f is diff. $\therefore f'_-(2) = f'_+(2)$

$\therefore f'_-(2) = 2x = 2(2) = 4$

$f'_+(2) = m$

$\therefore 4 = m$ $\qquad \because f(2) = x^2 = 4$

$(2, 4)$

now sub:-

$f(x) = mx + b \Rightarrow 4 = 4(2) + b$

$\therefore 4 - 8 = b \Rightarrow b = -4$

$\therefore m = 4, \; b = -4$

Using Definition of Derivative to Evaluate a Limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

1) $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x}$

خطوات الحل

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$f(x) = e^x \quad , \quad f(0) = e^0 = 1$

$f'(x) = e^x \quad , \quad f'(0) = e^0 = 1$

$$\therefore \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

83. Evaluate $\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$.

* using definition of derivative

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f(x) = x^{1000}$$

$$f(1) = 1^{1000} = 1$$

$$f'(x) = 1000x^{999}$$

$$f'(1) = 1000(1)^{999} = 1000$$

$$\therefore f'(1) = \lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = 1000$$

١- هل ينفع التعويض المباشر؟

٢- هل ينفع طريقة التحليل (factor)؟

٣-هل تنفع طريقة المرافق؟

٤- هل في المسألة مطلق (absolute value)

٥-هل طريقة squeeze theorem تنفع ؟

٦- هل اللمت رايح لي infinity؟

٧- هل طلب مني بالسؤال طريقة معينة للحل (يعني مثلاً عن طريق التعريف)

إذا ما انفعت ولا طريقة أجرب طريقة المشتقة

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Question 4. Evaluate the following limits:

(a) $\lim_{x\to 1} \frac{x^{2017}-1}{x-1}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - a}$$

$$f(x) = x^{2017} \quad , \quad f(1) = 1^{2017} = 1$$

$$f'(x) = 2017x^{2016} \quad , \quad f'(1) = 2017$$

$$\therefore f'(1) = \lim_{x \to 1} \frac{x^{2017} - 1}{x - 1} = 2017$$

Question 4. Evaluate the following limits:

(d) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16}$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(16) = \lim_{x\to 16} \frac{f(x)-f(16)}{x-a}$$

$$f(x) = \sqrt[4]{x} = x^{1/4} \quad , \quad f(16) = \sqrt[4]{16} = 2$$

$$f'(x) = \frac{1}{4} x^{-3/4} = \frac{1}{4(\sqrt[4]{x})^3}$$

$$f'(16) = \frac{1}{4(\sqrt[4]{16})^3} = \frac{1}{4(2)^3} = \frac{1}{4*8} = \frac{1}{32}$$

$$\therefore f'(16) = \lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16} = \frac{1}{32}$$

(e) $\lim_{x \to 0} \frac{e^x - 1}{x - 1}$

$$\lim_{x \to 0} \frac{e^x - 1}{x - 1} = \frac{e^0 - 1}{0 - 1} = \frac{0}{-1} = 0$$

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Calculus 1 – Product and Quotient Rule

(Section 3.2)

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3.2 The Product and Quotient Rules

The Product Rule

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

EXAMPLE 1

(a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

SOLUTION

(a) By the Product Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(xe^x) \\ &= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\ &= xe^x + e^x \cdot 1 = (x+1)e^x \end{aligned}$$

(b) Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x+1)e^x] \\ &= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) \\ &= (x+1)e^x + e^x \cdot 1 = (x+2)e^x \end{aligned}$$

Further applications of the Product Rule give

$$f'''(x) = (x+3)e^x \qquad f^{(4)}(x) = (x+4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x+n)e^x$$ ■

EXAMPLE 2 Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$\begin{aligned} f'(t) &= \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}\left(\sqrt{t}\right) \\ &= \sqrt{t}\cdot b + (a + bt)\cdot \tfrac{1}{2}t^{-1/2} \\ &= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}} \end{aligned}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$\begin{aligned} f(t) &= a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2} \\ f'(t) &= \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2} \end{aligned}$$

which is equivalent to the answer given in Solution 1. ∎

EXAMPLE 3 If $f(x) = \sqrt{x}\,g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sqrt{x}\,g(x)\right] = \sqrt{x}\,\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}\left[\sqrt{x}\right] \\ &= \sqrt{x}\,g'(x) + g(x)\cdot\tfrac{1}{2}x^{-1/2} \\ &= \sqrt{x}\,g'(x) + \frac{g(x)}{2\sqrt{x}} \end{aligned}$$

So $$f'(4) = \sqrt{4}\,g'(4) + \frac{g(4)}{2\sqrt{4}} = 2\cdot 3 + \frac{2}{2\cdot 2} = 6.5$$

The Quotient Rule

مهم الترتيب شوف وين مكان اشتقاق المقام ووين مكان اشتقاق البسط وواحد قبل الطرح
وواحد بعد الطرح (لا تعكس أماكنهم لأنه الطرح غير إبدالي

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

EXAMPLE 4 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then $y' = ?$

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Table of Differentiation Formulas

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$(cf)' = cf'$	$(f + g)' = f' + g'$	$(f - g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	

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EXAMPLE 5 Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $\left(1, \frac{1}{2}e\right)$.

SOLUTION According to the Quotient Rule, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 + x^2)\dfrac{d}{dx}(e^x) - e^x \dfrac{d}{dx}(1 + x^2)}{(1 + x^2)^2} \\
&= \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \frac{e^x(1 - 2x + x^2)}{(1 + x^2)^2} \\
&= \frac{e^x(1 - x)^2}{(1 + x^2)^2}
\end{aligned}$$

So the slope of the tangent line at $\left(1, \frac{1}{2}e\right)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

This means that the tangent line at $\left(1, \frac{1}{2}e\right)$ is horizontal and its equation is $y = \frac{1}{2}e$. [See Figure 4. Notice that the function is increasing and crosses its tangent line at $\left(1, \frac{1}{2}e\right)$.] ■

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3–26 Differentiate.

3. $f(x) = (3x^2 - 5x)e^x$

$$f'(x) = (6x-5)e^x + (3x^2-5x)e^x$$

$$= e^x(6x-5+3x^2-5x)$$

$$= e^x(3x^2+x-5)$$

4. $g(x) = (x + 2\sqrt{x})\,e^x$

$$g'(x) = \left(1+2\frac{1}{2\sqrt{x}}\right)e^x + (x+2\sqrt{x})e^x$$

$$g'(x) = \left(1+\frac{1}{\sqrt{x}}\right)e^x + (x+2\sqrt{x})e^x$$

$$g'(x) = e^x\left(1+\frac{1}{\sqrt{x}}+x+2\sqrt{x}\right)$$

6. $y = \dfrac{e^x}{1-e^x}$ $\quad = \dfrac{e^x}{(1-e^x)^2}$

3–26 Differentiate.

5. $y = \dfrac{x}{e^x}$

$$y' = \frac{1e^x - (e^x)x}{(e^x)^2} = \frac{e^x - e^x x}{e^{2x}}$$

$$= \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}$$

7. $g(x) = \dfrac{1+2x}{3-4x}$

$$g'(x) = \frac{(0+2)(3-4x) - (-4)(1+2x)}{(3-4x)^2}$$

$$= \frac{6-8x+4+8x}{(3-4x)^2} = \frac{10}{(3-4x)^2}$$

8. $G(x) = \dfrac{x^2-2}{2x+1}$

$$= \frac{2x^2+2x+4}{(2x+1)^2}$$

3–26 Differentiate.

9. $H(u) = (u - \sqrt{u})(u + \sqrt{u})$

$H(u) = u^2 + u\sqrt{u} - u\sqrt{u} - u$

$H(u) = u^2 - u$

$H'(u) = 2u - 1$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$

$J(v) = v^{-1} + v - 2v^{-3} - 2v^{-1}$

$J'(v) = -v^{-2} + 1 + 6v^{-4} + 2v^{-2}$

$= 1 + v^{-2} + 6v^{-4} = 1 + \frac{1}{v^2} + \frac{6}{v^4}$

12. $f(z) = (1 - e^z)(z + e^z)$

$f'(z) = 1 - ze^z - 2e^{2z}$

3–26 Differentiate.

14. $y = \dfrac{\sqrt{x}}{2 + x}$

$$y' = \frac{\left(\frac{1}{2}x^{-1/2}\right)(2+x) - (1)(\sqrt{x})}{(2+x)^2}$$

$$y' = \frac{x^{-1/2} + \frac{1}{2}x^{1/2} - x^{1/2}}{(2+x)^2}$$

$$y' = \frac{-1/2\, x^{1/2} + x^{-1/2}}{(2+x)^2}$$

$$y' = \frac{x^{-1/2}\left(-\frac{1}{2}x + 1\right)}{(2+x)^2}$$

$$y' = \frac{\left(\frac{-x}{2} + \frac{2}{2}\right)}{x^{1/2}(2+x)^2} = \frac{2-x}{2x^{1/2}(2+x)^2}$$

3–26 Differentiate.

15. $y = \dfrac{t^3 + 3t}{t^2 - 4t + 3}$

SOLUTION

$$y' = \frac{t^4 - 8t^3 + 6t^2 + 9}{(t^2 - 4t + 3)^2}$$

17. $y = e^p\left(p + p\sqrt{p}\right)$

SOLUTION

$$y' = e^p\left(1 + \tfrac{3}{2}\sqrt{p} + p + p\sqrt{p}\right)$$

23. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x}$

$$f'(x) = \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$$

27–30 Find $f'(x)$ and $f''(x)$.

27. $f(x) = (x^3 + 1)e^x$

$$f'(x) = (3x^2)e^x + e^x(x^3+1)$$

$$f'(x) = e^x(3x^2+x^3+1)$$

$$f''(x) = e^x(3x^2+x^3+1) + (6x+3x^2)e^x$$

$$f''(x) = e^x(3x^2+x^3+1+6x+3x^2)$$

$$f''(x) = e^x(x^3+6x^2+6x+1)$$

27–30 Find $f'(x)$ and $f''(x)$.

28. $f(x) = \sqrt{x}\, e^x$

$$f'(x) = \left(\frac{1}{2}x^{-1/2}\right)e^x + e^x x^{1/2}$$

$$f'(x) = e^x\left(\tfrac{1}{2}x^{-1/2} + x^{1/2}\right)$$

$$f''(x) = e^x\left(\tfrac{1}{2}x^{-1/2} + x^{1/2}\right) + \cdots$$

$$\left(-\tfrac{1}{4}x^{-3/2} + \tfrac{1}{2}x^{-1/2}\right)e^x$$

$$f''(x) = e^x\left(\tfrac{1}{2}x^{-1/2} + x^{1/2} - \frac{1}{4}x^{-3/2} + \tfrac{1}{2}x^{-1/2}\right)$$

$$f''(x) = e^x\left(x^{-1/2} + x^{1/2} - \tfrac{1}{4}x^{-3/2}\right)$$

$$f''(x) = e^x\left(\frac{1}{\sqrt{x}} + \sqrt{x} - \frac{1}{4\sqrt{x^3}}\right)$$

3–26 Differentiate.

11. $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

$F(y) = (y^{-2} - 3y^{-4})(y + 5y^3)$

$$F'(y) = 5 + \frac{14}{y^2} + \frac{9}{y^4}$$

33–34 Find equations of the tangent line and normal line to the given curve at the specified point.

33. $y = 2xe^x, \quad (0, 0)$

$$y' = 2e^x + e^x(2x)$$

$$y' = 2e^x(1 + x)$$

at $(0, 0)$

$\therefore y'(0) = 2e^0(1+0) = 2$

$\because y - y_1 = m(x - x_1) = y - 0 = 2(x - 0)$

$\therefore$ equation of tangent line at $(0,0)$ is $y = 2x$

$\because$ equation of normal tangent :–

$$y - y_1 = \frac{-1}{m}(x - x_1) = y - 0 = -\frac{1}{2}(x - 0)$$

$$\therefore y = -\frac{1}{2}x$$

$\therefore$ Tangent line $y = 2x$, normal line $y = -\frac{1}{2}x$

47. If $g(x) = xf(x)$, where $f(3) = 4$ and $f'(3) = -2$, find an equation of the tangent line to the graph of g at the point where $x = 3$.

$g'(x) = f(x) + x f'(x)$

$g'(3) = f(3) + 3f'(3)$

$g'(3) = 4 + 3(-2)$

$\therefore g'(3) = -2$

$g(3) = 3f(3) = 3(4) = 12$

$\therefore (3, 12)$

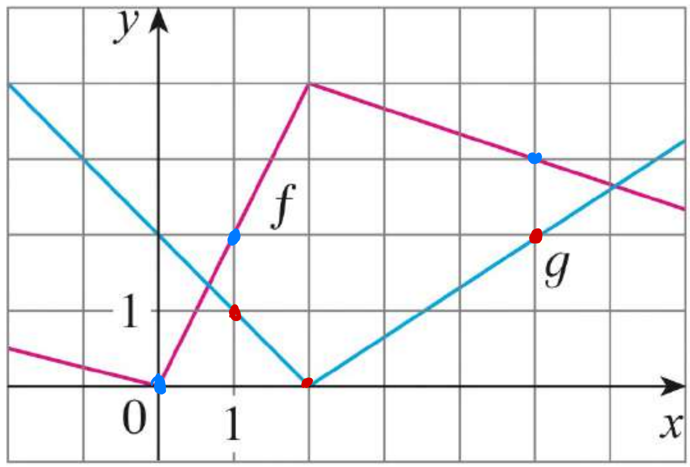
equation of tangent line

$y - y_1 = m(x - x_1) = y - 12 = -2(x - 3)$

$\therefore y = -2x + 18$

49. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(a) Find $u'(1)$. (b) Find $v'(5)$.



$$u'(1) = f'(1)\,g(1) + g'(1)\,f(1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore f'(1) = \frac{Y_2 - Y_1}{x_2 - x_1} = \frac{2-0}{1-0} = 2$$

$$\therefore g'(1) = \frac{Y_2 - Y_1}{x_2 - x_1} = \frac{0-2}{2-0} = -1$$

$$\therefore u'(1) = (2)(1) + (2)(-1) = 2 - 2 = 0$$

2. [10 pts.] Find **equations** of the tangent lines to the curve $f(x) = \frac{x-1}{x+1}$ that are **parallel** to the line $x - 2y = 2$.

We have $f'(x) = \frac{2}{(x+1)^2}$. Now, any tangent line to the curve has slope $= \frac{2}{(x+1)^2} = \frac{1}{2}$. This implies that $x = 1$ or $x = -3$. Therefore, the equation of the first tangent line is $y - 0 = \frac{1}{2}(x-1)$ and the equation of the second tangent line is $y - 2 = \frac{1}{2}(x+3)$.

1) $f'(x) = \frac{2}{(x+1)^2}$, $y = \frac{1}{2}x + 1$

$\therefore y' = 1/2$ $\because$ parallel $\therefore y' = f(x)'$

$\therefore 1/2 = \frac{2}{(x+1)^2} \Rightarrow \therefore x_1 = 1$ or $x_2 = -3$

now sub in equ $f(x) = \frac{x-1}{x+1}$ to get y_1 and y_2

for $x = 1$, $f(1) = \frac{1-1}{1+1} = 0$

for $x = -3$, $f(-3) = \frac{-3-1}{-3+1} = +2$

for $(1, 0)$:-

$y - y_1 = m(x - x_1) = y - 0 = 1/2(x-1)$

for $(-3, 2)$:- $y - 2 = 1/2(x+3)$

8. [10 pts.] Find an equation of the tangent line to the graph of the function $f(x) = \frac{e^x}{x+1} + x + 3$ at $x = 0$.

We have $f'(x) = \frac{xe^x}{(x+1)^2} + 1$. It is clear that $f(0) = 4$ and $f'(0) = 1$. Therefore, an equation of the tangent line to the graph of the function f at $x = 0$ is given by $y = x + 4$.

for equation of line I need

1) slope

2) point

1) $f'(x) = \frac{xe^x}{(x+1)^2} + 1$

$f'(0) = 1 \leftarrow$ slope

2) $f(\overset{x}{0}) = \frac{e^0}{0+1} + 0 + 3 = \overset{y}{4}$

$(x, y) = (0, 4) \leftarrow$ point

$y = x + 4$

$y - y_1 = m(x - x_1) = y - 4 = 1(x - 0) \Rightarrow$

7. [10 pts.] Find the points on the graph: $y = \dfrac{x}{x^2+1}$ where the tangent line is horizontal.

We have $\boxed{\dfrac{dy}{dx} = \dfrac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}}$.

Thus, $\dfrac{dy}{dx} = 0$ when $\boxed{x = \pm 1}$. So the given curve has horizontal tangents when $\boxed{x = \pm 1}$.
The corresponding points are $\boxed{(\pm 1, \pm 1/2)}$.

1) $\because$ Tangent line Horizontal

$\therefore \dfrac{dy}{dx} = 0$ ← أقدر أطلع قيمة x

$$\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2} = 0$$

الدالة رح تساوي صفر لما البسط يساوي الصفر

$1 - x^2 = 0$

$1 = x^2$

$x = \pm 1$

2) $f(1)$, $f(-1)$ ← عشان أطلع قيمة y

3) (x_1, y_1), (x_2, y_2)

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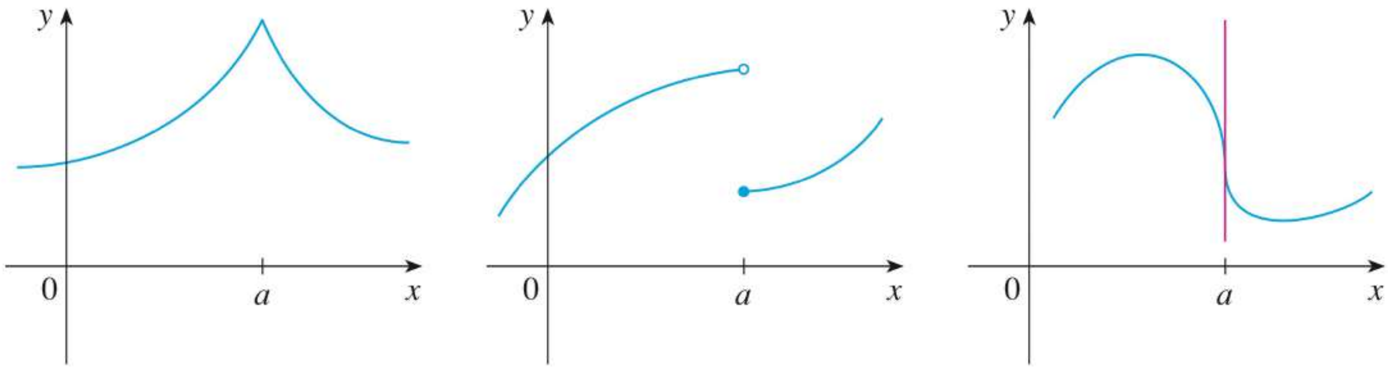
How Can a Function Fail To Be Differentiable?

يعني متى تكون not differentiable



(a) A corner (b) A discontinuity (c) A vertical tangent

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64. The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

and
$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

✓ (a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5-x} & \text{if } x \geqslant 4 \end{cases}$$

✓ (b) Sketch the graph of f.

✓ (c) Where is f discontinuous?

✓ (d) Where is f not differentiable?

---

a) $f'_-(4) = -1$

$$f'_+(4) = \frac{0(5-x) - (-1)(1)}{(5-x)^2}$$

$$= \frac{1}{(5-x)^4} = \frac{1}{(5-4)^2} = 1$$

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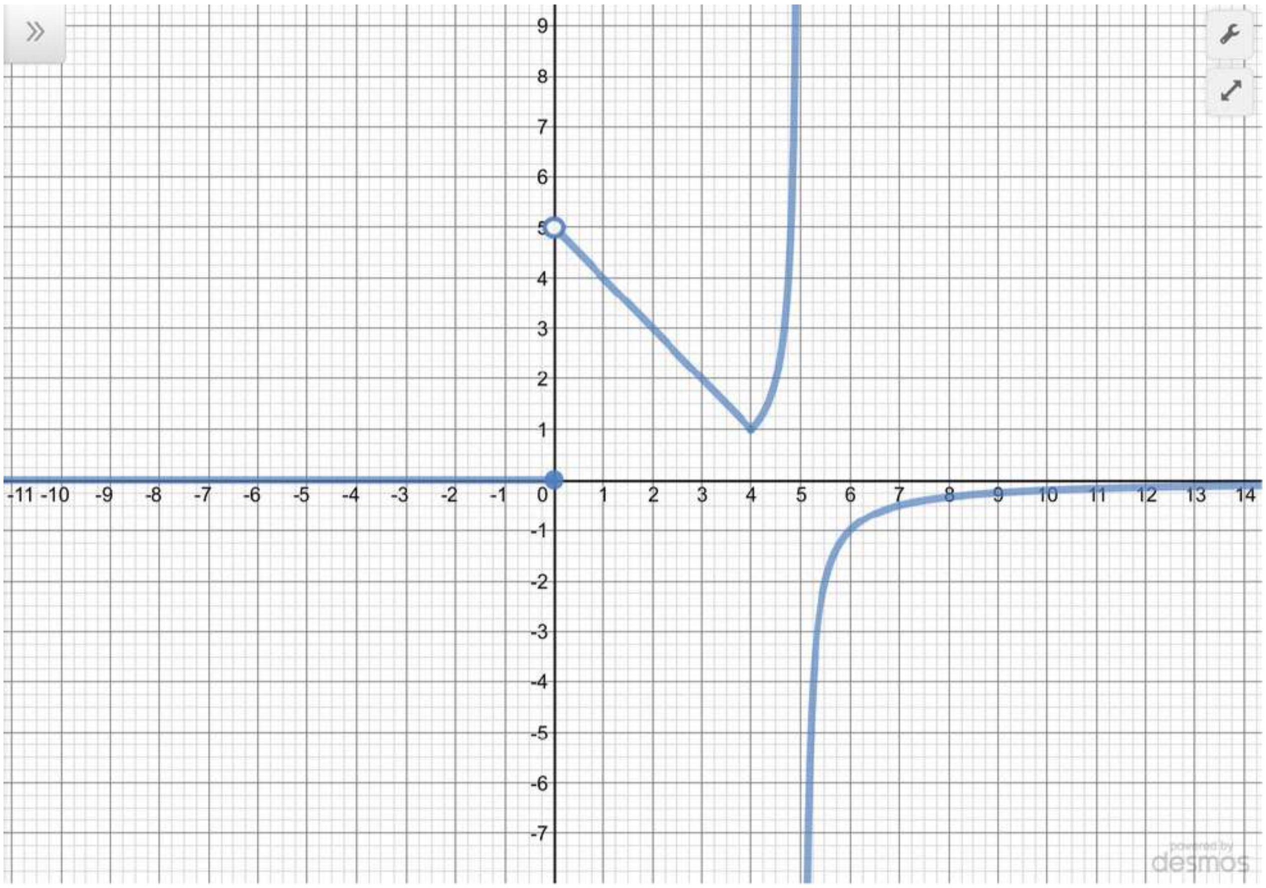

(c) Where is f discontinuous? $x = 0, 5$

(d) Where is f not differentiable? $x = 0, 4, 5$

corner

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Kuwait University

Calculus 1 – Derivatives of Trigonometric

(Section 3.3)

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Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)\,g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

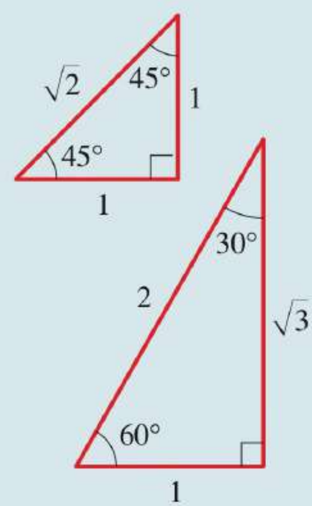
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

خواص
اللمت

SPECIAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

The following values of the trigonometric functions are obtained from the special triangles.

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	0	1	0	—	1	—
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	—	1	—	0



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3.3 Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Derivatives of Trigonometric Functions

حفظ

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE 1 Differentiate $y = x^2 \sin x$.

SOLUTION Using the Product Rule and Formula 4, we have

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) \\ &= x^2 \cos x + 2x \sin x\end{aligned}$$

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Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	0	1	0	—	1	—
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	—	1	—	0

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Remark:-   تذكر خواص اللمت

PROPERTIES OF LIMITS

These properties require that the limit of $f(x)$ and $g(x)$ exist

$$\lim_{x \to a}[cf(x)] = c\lim_{x \to a} f(x)$$

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a}[f(x)\, g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$$

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EXAMPLE 4 Find the 27th derivative of $\cos x$.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

لأن حسب حلي للمشتقة ... المشتقة
الرابعة رجعتني لنفس السؤال
فاستنتج إن كل ٤ مرات تشتق الدالة
ترجع لنفس الدالة الي بديت فيها (هذي
خاصية من خواص الدوال المثلثية في
الاشتقاق)

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$

$$f^{(4)}(x) = f^{(8)}(x) = f^{(12)}(x) \ldots f^{(24)}(x)$$

$$f^{(24)}(x) = \cos x$$

EXAMPLE 1

(a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Answer in section 3.2

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1–16 Differentiate.

1. $f(x) = x^2 \sin x$

$f'(x) = (2x)(\sin x) + (\cos x)(x^2)$

2. $f(x) = x \cos x + 2 \tan x$

$f'(x) = 1 \cos x + (-\sin x) x + 2\sec^2 x$

$f'(x) = \cos x - x \sin x + 2\sec^2 x$

5. $y = \sec\theta \tan\theta$

$y' = \sec\theta \tan\theta \tan\theta + \sec\theta \cdot \sec^2\theta$

$y' = \sec\theta \tan^2\theta + \sec^3\theta$

$y' = \sec\theta(\tan^2\theta + \sec^2\theta)$

1–16 Differentiate.

6. $g(\theta) = e^{\theta}(\tan\theta - \theta)$

المتغيّر

لأن $g(\theta)$

θ في هذه المسألة حطينا x ليش؟

$$g(\theta) = e^{\theta}\tan\theta - e^{\theta}\theta$$

$$g'(\theta) = e^{\theta}\tan\theta + e^{\theta}\sec^2\theta - (e^{\theta}\theta + e^{\theta}1)$$

$$g'(\theta) = e^{\theta}\tan\theta + e^{\theta}\sec^2\theta - e^{\theta}\theta - e^{\theta}$$

$$g'(\theta) = e^{\theta}(\tan\theta + \sec^2\theta - \theta - 1)$$

$$\because \sec^2\theta - 1 = \tan^2\theta$$

$$\therefore g'(\theta) = e^{\theta}(\tan\theta + \tan^2\theta - \theta)$$

1–16 Differentiate.

8. $f(t) = \dfrac{\cot t}{e^t}$

$$f'(t) = \frac{(-\csc^2 t)\, e^t - (\cot t)(e^t)}{(e^t)^2}$$

$$f'(t) = \frac{e^t(-\csc^2 t - \cot)}{(e^t)^2}$$

$$f'(t) = \frac{(-\csc^2(t) - \cot(t))}{e^t}$$

$$f'(t) = \frac{-[\csc^2(t) + \cot(t)]}{e^t}$$

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1–16 Differentiate.

9. $y = \dfrac{x}{2 - \tan x}$

$$y' = \frac{(1)(2-\tan x) - (0 - \sec^2 x)(x)}{(2-\tan x)^2}$$

$$y' = \frac{2 - \tan x + x\sec^2 x}{(2-\tan x)^2}$$

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta}$

$$f'(\theta) = \frac{\cos\theta(1+\cos\theta) - (-\sin\theta)(\sin\theta)}{(1+\cos\theta)^2}$$

$$f'(\theta) = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{\cos\theta + 1}{(1+\cos\theta)^2}$$

$$\frac{\cos\theta+1}{(1+\cos\theta)^{\cancel{2}}}$$

$$f'(\theta) = \frac{1}{(1+\cos\theta)}$$

31. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

(c) Show that your answers to parts (a) and (b) are equivalent.

$$f'(x) = \frac{(\sec^2 x)\sec x - [(\sec x \tan x)((\tan x) - 1)]}{(\sec x)^2}$$

$$f'(x) = \frac{(\sec^2 x)\sec x - [(\sec x \tan^2 x) - \sec x \tan x]}{(\sec x)^2}$$

$$f'(x) = \frac{\sec^3 x - \sec x \tan^2 x + \sec x \tan x}{\sec^2 x}$$

$$f'(x) = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x}$$

$$f'(x) = \frac{\sec^2 x - \tan^2 x + \tan x}{\sec x}$$

$\sec^2 x = 1 + \tan^2 x$

$$f'(x) = \frac{1 + \tan^2 x - \tan^2 x + \tan x}{\sec x}$$

$$f'(x) = \frac{1 + \tan x}{\sec x}$$

31. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

(c) Show that your answers to parts (a) and (b) are equivalent.

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31 b) $$\frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}}$$

$$= \frac{\frac{\sin x}{\cos x} - \frac{\cos x}{\cos x}}{\frac{1}{\cos x}} = \frac{\frac{\sin x - \cos x}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\sin x - \cos x}{\frac{\cos x}{\cos x}} = \frac{\sin x - \cos x}{1} = \sin x - \cos x$$

$$f'(x) = \cos x - (-\sin x) = \cos x + \sin x$$

إجابة مطلوب الأول

31 c) $$f'(x) = \boxed{\frac{1+\tan x}{\sec x}} = \frac{1}{\sec x} + \frac{\tan x}{\sec x}$$

$$= \cos x + \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \cos x + \frac{\sin x}{\frac{\cos x}{\cos x}} = \cos x + \sin x =$$

21–24 Find an equation of the tangent line to the curve at the given point.

22. $y = e^x \cos x, \quad (0, 1)$

$$y' = e^x \cos x + (-\sin x)e^x$$

$$y'(0) = e^0 \cos 0 - \sin 0 \, e^0$$

$$y'(0) = 1 - 0 = 1$$

$\therefore$ equation of tangent line

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 0)$$

$$\therefore y = x + 1$$

33–34 For what values of x does the graph of f have a horizontal tangent?

34. $f(x) = e^x \cos x$

$f'(x) = e^x \cos x + (-\sin x) e^x$

$= e^x \cos x - e^x \sin x$

$= e^x (\cos x - \sin x)$

$\because$ f has horizental tangent line

$\therefore e^x (\cos x - \sin x) = 0$

$\because e^x \neq 0$

$\cos x - \sin x = 0$

$\cos x = \sin x$

when $x = \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4} + k\pi$

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$f'(x) = \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{(1 + \tan x) \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x (\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$\because \tan^2 x + 1 = \sec^2 x$

$\therefore \tan^2 x - \sec^2 x = -1$

$$= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$$

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Thamrah

Kuwait University

Calculus 1 – The Chain Rule

(Section 3.4)

For Contact and Support:

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حفظ

BASIC PROPERTIES

$$(cf(x))' = c(f'(x))$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

$$\frac{d}{dx}(c) = 0$$

PRODUCT RULE

$$(f(x)g(x))' = f(x)'g(x) + f(x)g(x)'$$

QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

POWER RULE

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

CHAIN RULE

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

COMMON DERIVATIVES

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(e^x) = e^x$$

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Section 3-9 : Chain Rule

We've taken a lot of derivatives over the course of the last few sections. However, if you look back they have all been functions similar to the following kinds of functions.

$$R\left(z\right)=\sqrt{z} \qquad f\left(t\right)=t^{50} \qquad y=\tan\left(x\right) \qquad h\left(w\right)=\mathbf{e}^{w} \qquad g\left(x\right)=\ln x$$

These are all fairly simple functions in that wherever the variable appears it is by itself. What about functions like the following,

$$R\left(z\right)=\sqrt{5z-8} \qquad f\left(t\right)=\left(2t^{3}+\cos\left(t\right)\right)^{50} \qquad y=\tan\left(\sqrt[3]{3x^{2}}+\tan\left(5x\right)\right)$$

$$h\left(w\right)=\mathbf{e}^{w^{4}-3w^{2}+9} \qquad g\left(x\right)=\ln\left(x^{-4}+x^{4}\right)$$

None of our rules will work on these functions and yet some of these functions are closer to the derivatives that we're liable to run into than the functions in the first set.

Let's take the first one for example. Back in the section on the definition of the derivative we actually used the definition to compute this derivative. In that section we found that,

$$R'\left(z\right)=\frac{5}{2\sqrt{5z-8}}$$

If we were to just use the power rule on this we would get,

$$\frac{1}{2}\left(5z-8\right)^{-\frac{1}{2}}=\frac{1}{2\sqrt{5z-8}}$$

which is not the derivative that we computed using the definition. It is close, but it's not the same. So, the power rule alone simply won't work to get the derivative here.

Let's keep looking at this function and note that if we define,

$$f\left(z\right)=\sqrt{z} \qquad g\left(z\right)=5z-8$$

then we can write the function as a composition.

$$R\left(z\right)=\left(f\circ g\right)\left(z\right)=f\left(g\left(z\right)\right)=\sqrt{5z-8}$$

and it turns out that it's actually fairly simple to differentiate a function composition using the **Chain Rule**. There are two forms of the chain rule. Here they are.

Chain Rule

Suppose that we have two functions $f\left(x\right)$ and $g\left(x\right)$ and they are both differentiable.

1. If we define $F\left(x\right)=\left(f\circ g\right)\left(x\right)$ then the derivative of $F\left(x\right)$ is,

$$F'\left(x\right)=f'\left(g\left(x\right)\right)\ g'\left(x\right)$$

2. If we have $y=f\left(u\right)$ and $u=g\left(x\right)$ then the derivative of *y* is,

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

Example 1 Use the Chain Rule to differentiate $R(z)=\sqrt{5z-8}$.

Solution

We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$f(z)=\sqrt{z} \qquad\qquad g(z)=5z-8$$

$$f'(z)=\frac{1}{2\sqrt{z}} \qquad\qquad g'(z)=5$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z))\,g'(z) \\ &= f'(5z-8)\,g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}(5) \\ &= \frac{1}{2\sqrt{5z-8}}(5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

And this is what we got using the definition of the derivative.

In general, we don't really do all the composition stuff in using the Chain Rule. That can get a little complicated and in fact obscures the fact that there is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition.

Let's take the function from the previous example and rewrite it slightly.

$$R(z)=\underbrace{(5z-8)}_{\text{inside function}}\overbrace{\;}^{}\underbrace{^{\frac{1}{2}}}_{\text{outside function}}$$

$$R'(z)=\overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}}\quad\underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

In general, this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x)=\underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}}\quad\underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}}\quad\underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

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Example 2 Differentiate each of the following.

(a) $f(x) = \sin(3x^2 + x)$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

(e) $y = \sec(1 - 5x)$

(f) $P(t) = \cos^4(t) + \cos(t^4)$

Solution

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(3x^2 + x)}_{\substack{\text{leave inside} \\ \text{function alone}}} \underbrace{(6x + 1)}_{\substack{\text{times derivative} \\ \text{of inside function}}}$$

Or with a little rewriting,

$$f'(x) = (6x + 1)\cos(3x^2 + x)$$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned} f'(t) &= 50(2t^3 + \cos(t))^{49}(6t^2 - \sin(t)) \\ &= 50(6t^2 - \sin(t))(2t^3 + \cos(t))^{49} \end{aligned}$$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

Identifying the outside function in the previous two was fairly simple since it really was the "outside" function in some sense. In this case we need to be a little careful. Recall that the outside function is the last operation that we would perform in an evaluation. In this case if we were to evaluate this function the last operation would be the exponential. Therefore, the outside function is the exponential function and the inside function is its exponent.

Here's the derivative.

$$\begin{aligned} h'(w) &= \mathbf{e}^{w^4 - 3w^2 + 9}(4w^3 - 6w) \\ &= (4w^3 - 6w)\mathbf{e}^{w^4 - 3w^2 + 9} \end{aligned}$$

Remember, we leave the inside function alone when we differentiate the outside function. So, the derivative of the exponential function (with the inside left alone) is just the original function.

(e) $y = \sec(1-5x)$

In this case the outside function is the secant and the inside is the $1-5x$.

$$\begin{aligned} y' &= \sec(1-5x)\tan(1-5x)(-5) \\ &= -5\sec(1-5x)\tan(1-5x) \end{aligned}$$

In this case the derivative of the outside function is $\sec(x)\tan(x)$. However, since we leave the inside function alone we don't get *x*'s in both. Instead we get $1-5x$ in both.

(f) $P(t) = \cos^4(t) + \cos(t^4)$

There are two points to this problem. First, there are two terms and each will require a different application of the chain rule. That will often be the case so don't expect just a single chain rule when doing these problems. Second, we need to be very careful in choosing the outside and inside function for each term.

Recall that the first term can actually be written as,

$$\cos^4(t) = (\cos(t))^4$$

So, in the first term the outside function is the exponent of 4 and the inside function is the cosine. In the second term it's exactly the opposite. In the second term the outside function is the cosine and the inside function is t^4. Here's the derivative for this function.

$$\begin{aligned} P'(t) &= 4\cos^3(t)(-\sin(t)) - \sin(t^4)(4t^3) \\ &= -4\sin(t)\cos^3(t) - 4t^3\sin(t^4) \end{aligned}$$

Example 3 Differentiate each of the following.

(a) $f(x) = [g(x)]^n$

(b) $f(x) = \mathbf{e}^{g(x)}$

Solution

(a) The outside function is the exponent and the inside is $g(x)$.

$$f'(x) = n[g(x)]^{n-1} g'(x)$$

(b) The outside function is the exponential function and the inside is $g(x)$.

$$f'(x) = g'(x)\mathbf{e}^{g(x)}$$

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Example 4 Differentiate each of the following.

(b) $f(z)=\sin\left(z\mathbf{e}^{z}\right)$

(c) $y=\dfrac{\left(x^{3}+4\right)^{5}}{\left(1-2x^{2}\right)^{3}}$

(d) $h(t)=\left(\dfrac{2t+3}{6-t^{2}}\right)^{3}$

(b) $f(z)=\sin\left(z\mathbf{e}^{z}\right)$

$$f'(z)=\cos\left(z\mathbf{e}^{z}\right)\frac{d}{dz}\left[z\mathbf{e}^{z}\right]$$

$$f'(z)=\cos\left(z\mathbf{e}^{z}\right)\left(\mathbf{e}^{z}+z\mathbf{e}^{z}\right)$$

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(c) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

$$y' = \frac{5\left(x^3+4\right)^4\left(3x^2\right)\left(1-2x^2\right)^3 - \left(x^3+4\right)^5(3)\left(1-2x^2\right)^2(-4x)}{\left(\left(1-2x^2\right)^3\right)^2}$$

عوامل مشتركة

$$y' = \frac{\left(x^3+4\right)^4\left(1-2x^2\right)^2\left[5\left(3x^2\right)\left(1-2x^2\right) - \left(x^3+4\right)(3)(-4x)\right]}{\left(1-2x^2\right)^6}$$

$$y' = \frac{(x^3+4)^4(1-2x^2)^2\left[15x^2-30x^4-(-12x^4-48x)\right]}{(1-2x^2)^6}$$

$$y' = \frac{(x^3+4)^4\,(15x^2-18x^4+48x)}{(1-2x^2)^4}$$

$$y' = \frac{(x^3+4)^4\,3x\,(5x-6x^3+16)}{(1-2x^2)^4}$$

$$= \frac{3x\left(x^3+4\right)^4\left(5x-6x^3+16\right)}{\left(1-2x^2\right)^4}$$

(d) $h(t)=\left(\dfrac{2t+3}{6-t^2}\right)^3$

$$h'(t)=3\left(\frac{2t+3}{6-t^2}\right)^2\frac{d}{dt}\left[\frac{2t+3}{6-t^2}\right]$$

$$=3\left(\frac{2t+3}{6-t^2}\right)^2\left[\frac{2(6-t^2)-(2t+3)(-2t)}{(6-t^2)^2}\right]$$

$$=3\left(\frac{2t+3}{6-t^2}\right)^2\left[\frac{2t^2+6t+12}{(6-t^2)^2}\right]$$

24. $f(t)=2^{t^3}$

$$f'(t)=2^{t^3}\ln(2)(3t^2)$$

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Example 5 Differentiate each of the following.

(a) $h(z) = \dfrac{2}{\left(4z + \mathbf{e}^{-9z}\right)^{10}}$

(b) $f(y) = \sqrt{2y + \left(3y + 4y^2\right)^3}$

(d) $g(t) = \sin^3\left(\mathbf{e}^{1-t} + 3\sin(6t)\right)$

(a) $h(z) = \dfrac{2}{\left(4z + \mathbf{e}^{-9z}\right)^{10}}$

$$h(z) = 2\left(4z + \mathbf{e}^{-9z}\right)^{-10}$$

Now, let's start the derivative.

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11} \frac{d}{dz}\left(4z + \mathbf{e}^{-9z}\right)$$

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11}\left(4 - 9\mathbf{e}^{-9z}\right)$$

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(b) $f(y)=\sqrt{2y+\left(3y+4y^2\right)^3}$

$$f'(y)=\frac{1}{2}\left(2y+\left(3y+4y^2\right)^3\right)^{-\frac{1}{2}}\frac{d}{dy}\left(2y+\left(3y+4y^2\right)^3\right)$$

$$=\frac{1}{2}\left(2y+\left(3y+4y^2\right)^3\right)^{-\frac{1}{2}}\left(2+3\left(3y+4y^2\right)^2(3+8y)\right)$$

$$=\frac{1}{2}\left(2y+\left(3y+4y^2\right)^3\right)^{-\frac{1}{2}}\left(2+(9+24y)\left(3y+4y^2\right)^2\right)$$

(d) $g(t)=\sin^3\left(\mathbf{e}^{1-t}+3\sin(6t)\right)$

We'll need to be a little careful with this one.

$$g'(t)=3\sin^2\left(\mathbf{e}^{1-t}+3\sin(6t)\right)\frac{d}{dt}\sin\left(\mathbf{e}^{1-t}+3\sin(6t)\right)$$

$$=3\sin^2\left(\mathbf{e}^{1-t}+3\sin(6t)\right)\cos\left(\mathbf{e}^{1-t}+3\sin(6t)\right)\frac{d}{dt}\left(\mathbf{e}^{1-t}+3\sin(6t)\right)$$

$$=3\sin^2\left(\mathbf{e}^{1-t}+3\sin(6t)\right)\cos\left(\mathbf{e}^{1-t}+3\sin(6t)\right)\left(\mathbf{e}^{1-t}(-1)+3\cos(6t)(6)\right)$$

$$=3\left(-\mathbf{e}^{1-t}+18\cos(6t)\right)\sin^2\left(\mathbf{e}^{1-t}+3\sin(6t)\right)\cos\left(\mathbf{e}^{1-t}+3\sin(6t)\right)$$

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CHAIN RULE

حفظ

$$\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} f'(x)$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

$$\frac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} \ln a \cdot f'(x)$$

$$\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$$

$$\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$$

$$\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)]$$

$$\frac{d}{dx}(\sec[f(x)]) = f'(x)\sec[f(x)]\tan[f(x)]$$

- If $f(x) = \sqrt{u}$, then $f'(x) = \frac{u'}{2\sqrt{u}}$ ← أو تشتقه عادي
- If $f(x) = \sqrt[3]{u} = u^{1/3}$ then $f'(x) = \frac{1}{3} u^{-2/3} \cdot u'$

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3.4 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule can be written either in the prime notation

2 $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine: $$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

Formula for tangent: $$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

To do the chain rule:

1. Differentiate the outer function, keeping the inner function the same.
2. Multiply this by the derivative of the inner function.

1) $f(x) = (4x-3)^5$

$f'(x) = 5(4x-3)(4) = 20(4x-3)$

2) $g(x) = \cos(2x)$

$g'(x) = -\sin(2x)(2) = -2\sin(2x)$

3) $h(x) = (\sin 2x)^4$

$h'(x) = 4(\sin 2x)^3(2\cos 2x)$

$= 8(\sin 2x)^3(\cos 2x)$

$(\sin 2x)^3$

$= \sin^3 2x$

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EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}} \qquad \blacksquare$$

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$$\frac{d}{dx}\sin(x^2)$$

- $y = \sin(u)$
- $u = x^2$

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chain rule : $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$

$$\frac{dy}{du} = \cos(x^2)$$

$$\frac{du}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \cos(x^2)\cdot 2x$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

a) $y = \sin(x^2)$

using chain rule

$$y' = \cos(x^2)\cdot 2x$$

b) $y = \sin^2 x = (\sin(x))^2$

using chain rule

$$y' = 2(\sin(x))\cdot \cos x$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,
$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION Taking $u = g(x) = x^3 - 1$ and $n = 100$ in (4), we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

■

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION First rewrite f: $\quad f(x) = (x^2 + x + 1)^{-1/3}$

Thus
$$f'(x) = -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}\frac{d}{dx}(x^2 + x + 1)$$

$$= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)$$

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EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{2t+1-2t+4}{(2t+1)^2} = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2} =$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

EXAMPLE 6 Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$.

SOLUTION In this example we must use the Product Rule before using the Chain Rule:

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5$$

$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$

$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1)$$

$$= 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x+1)^4 \cdot 2$$

Noticing that each term has the common factor $\boxed{2(2x+1)^4(x^3 - x + 1)^3}$, we could factor it out and write the answer as

$$\frac{dy}{dx} = 2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$$

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EXAMPLE 7 Differentiate $y = e^{\sin x}$.

SOLUTION Here the inner function is $g(x) = \sin x$ and the outer function is the exponential function $f(x) = e^x$. So, by the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cos x$$ ■

More generally, the Chain Rule gives

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

because ln b is a constant. So we have the formula

5 $$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$

because ln b is a constant. So we have the formula

Don't confuse Formula 5 (where x is the *exponent*) with the Power Rule (where x is the *base*):

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

5 $$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$

In particular, if $b = 2$, we get

6 $$\frac{d}{dx}(2^x) = 2^x \ln 2$$

$$2^{x^2} \cdot 2^1 = 2^{x^2+1}$$

7 $$\frac{d}{dx}(2^{x^2}) = 2^{x^2}\ln 2 \cdot (2x) = 2^{x^2+1}\, x \ln(2)$$

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7–46 Find the derivative of the function.

1. $y = \sqrt[3]{1 + 4x}$

$$y = \sqrt[3]{1+4x} = y = u^{1/3}$$

$$\frac{dy}{du} = \frac{1}{3}(u)^{-2/3}$$

$$u = 1+4x \quad , \quad \frac{du}{dx} = 4$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{3}(u)^{-2/3} \cdot 4$$

$$= \frac{4}{3(u)^{2/3}} = \frac{4}{3(1+4x)^{2/3}}$$

$$\frac{dy}{dx} = \frac{4}{3}(1+4x)^{-2/3}$$

2. $y = (2x^3 + 5)^4$

using chain rule

$$\frac{dy}{dx} = 4(2x^3+5)^3 \quad (6x^2)$$

$$\frac{dy}{dx} = 24x^2(2x^3+5)^3$$

3. $y = \tan \pi x$

using chain rule

$$\frac{dy}{dx} = (\sec^2 \pi x)(\pi) = \pi \sec^2(\pi x)$$

4. $y = \sin(\cot x)$

using chain rule

$$\frac{dy}{dx} = \cos(\cot x) \quad (-\csc^2 x)$$

$$= -\csc^2(x) \cos(\cot x)$$

إعادة ترتيب بس

$\longleftarrow$

5. $y = e^{\sqrt{x}}$

using chain rule

$$\frac{dy}{dx} = e^{\sqrt{x}} \cdot \left(\frac{1}{2\sqrt{x}} \right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

7. $F(x) = (5x^6 + 2x^3)^4$

using chain rule

$$F'(x) = 4(5x^6 + 2x^3)^3 (30x^5 + 6x^2)$$

$$F'(x) = 4(x^3(5x^3+2))^3\, 6x^2(5x^3+1)$$

$(x^3)^3$ يعني

$$F'(x) = 4x^9 \cdot 6x^2 (5x^3+2)^3 (5x^3+1)$$

$$F'(x) = 24x^{11}(5x^3+2)^3(5x^3+1)$$

7–46 Find the derivative of the function.

9. $f(x) = \sqrt{5x + 1}$

$$f'(x) = \frac{1}{2\sqrt{5x+1}} \cdot 5$$ using chain rule

$$f'(x) = \frac{5}{2\sqrt{5x+1}}$$

$$f(x) = \sqrt[3]{5x+1}$$

$$f(x) = (5x+1)^{1/3}$$

$\frac{1}{3} - 1 = \frac{1}{3} - \frac{3}{3}$

$= \frac{-2}{3}$

$$f'(x) = \frac{1}{3}(5x+1)^{-2/3}(5)$$ using chain rule

$$f'(x) = \frac{5}{3(5x+1)^{2/3}}$$

10. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}}$

$$f(x) = (x^2 - 1)^{-1/3}$$

using chain rule

$$f'(x) = \frac{-1}{3}(x^2 - 1)^{-1/3 - 1} (2x)$$

$$f'(x) = -\frac{1}{3}(x^2 - 1)^{-4/3} (2x)$$

$$f'(x) = -\frac{2x}{3\sqrt[3]{(x^2-1)^4}}$$

نقدر نبسطها بعد

11. $f(\theta) = \cos(\theta^2)$

using chain rule

$f'(\theta) = -\sin(\theta^2) \quad (2\theta)$

$f'(\theta) = -2\theta \sin(\theta^2)$

12. $g(\theta) = \cos^2\theta$

using chain rule

$g'(\theta) = 2\cos\theta \quad (-\sin\theta)$

$g'(\theta) = -2\cos\theta\sin\theta \quad = -\sin 2\theta$

15. $f(t) = e^{at} \sin bt$

$$f'(t) = (e^{at}\, a) \sin bt + (\cos bt)\, b \cdot e^{at}$$

$$f'(t) = e^{at} a \sin bt + e^{at} \cdot b \cos(bt)$$

$$f'(t) = e^{at} (a \sin bt + b \cos bt)$$

18. $g(x) = (x^2 + 1)^3 (x^2 + 2)^6$

$$g'(x) = \frac{d}{dx} (x^2+1)^3 (x^2+2)^6 + \frac{d}{dx} (x^2+2)^6 (x^2+1)^3$$

$$g'(x) = 3(x^2+1)^2 (2x) (x^2+2)^6 + 6(x^2+2)^5 (2x) (x^2+1)^3$$

$$g'(x) = 6x (x^2+1)^2 (x^2+2)^6 + 12x (x^2+2)^5 (x^2+1)^3$$

$$g'(x) = 6x (x^2+1)^2 (x^2+2)^5 [(x^2+2) + 2(x^2+1)]$$

$$g'(x) = 6x (x^2+1)^2 (x^2+2)^5 [(x^2+2) + 2x^2 + 2]$$

$$g'(x) = 6x (x^2+1)^2 (x^2+2)^5 (3x^2+4)$$

19. $h(t) = (t+1)^{2/3}(2t^2-1)^3$

$$h'(t) = \tfrac{2}{3}(t+1)^{-1/3}(1)(2t^2-1)^3 + \ldots$$

$$(t+1)^{2/3}\,3(2t^2-1)^2(4t)$$

$$= \tfrac{2}{3}(t+1)^{-1/3}(2t^2-1)^3 + 12t(t+1)^{2/3}(2t^2-1)^2$$

$\tfrac{2}{3} - (-\tfrac{1}{3}) = 1$

$$= \frac{2}{3}(t+1)^{-1/3}(2t^2-1)^2\left[(2t^2-1) + 18t(t+1)(1)\right]$$

$12t \div \tfrac{2}{3}$

$12t \cdot \tfrac{3}{2} = 18t$

$$= \frac{2}{3}(t+1)^{-1/3}(2t^2-1)^2\left[2t^2 - 1 + 18t^2 + 18t\right]$$

$$= \frac{2}{3}(t+1)^{-1/3}(2t^2-1)^2\left(20t^2 + 18t - 1\right)$$

21. $y = \sqrt{\dfrac{x}{x+1}}$

$$Y = \left(\frac{x}{x+1}\right)^{1/2}$$

using chain rule

$$Y' = 1/2\left(\frac{x}{x+1}\right)^{-1/2} \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$Y' = 1/2\left(\frac{x}{x+1}\right)^{-1/2}\left(\frac{1(x+1) - (1)x}{(x+1)^2}\right)$$

$$Y' = 1/2\left(\frac{x+1}{x}\right)^{1/2}\left(\frac{x+1-x}{(x+1)^2}\right)$$

$$Y' = 1/2\left(\frac{x+1}{x}\right)^{1/2}\left(\frac{1}{(x+1)^2}\right)$$

$$Y' = \frac{(x+1)^{1/2}}{2x^{1/2}(x+1)^2} = \frac{(x+1)^{-3/2}}{2x^{1/2}} = \frac{1}{2\sqrt{x}(x+1)^{3/2}}$$

22. $y = \left(x + \frac{1}{x}\right)^5$

using chain rule

$$y' = 5\left(x + \frac{1}{x}\right)^4 \left(1 + \left(-\frac{1}{x^2}\right)\right)$$

$$y' = 5\left(x + \frac{1}{x}\right)^4 \left(1 - \frac{1}{x^2}\right)$$

25. $g(u) = \left(\frac{u^3 - 1}{u^3 + 1}\right)^8$

using chain rule

$$g'(u) = 8\left(\frac{u^3 - 1}{u^3 + 1}\right)^7 \left(\frac{3u^2 (u^3+1) - (3u^2)(u^3-1)}{(u^3+1)^2}\right)$$

$$g'(u) = 8\left(\frac{u^3 - 1}{u^3 + 1}\right)^7 \left(\frac{3u^2 \left[u^3+1 - (u^3-1)\right]}{(u^3+1)^2}\right)$$

$$g'(u) = 8\left(\frac{u^3 - 1}{u^3 + 1}\right)^7 \left(\frac{3u^2[2]}{(u^3+1)^2}\right) = 8\left(\frac{u^3 - 1}{u^3 + 1}\right)^7 \left(\frac{6u^2}{(u^3+1)^2}\right)$$

$$= \frac{(u^3-1)^7}{(u^3+1)^7} \frac{8(6u^2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

27. $r(t) = 10^{2\sqrt{t}}$

$r(u) = 10^{u}$

using chain rule

$$\frac{dr}{du} = 10^{u} \ln 10 \cdot \frac{du}{dt}$$

$u = 2\sqrt{t}$, $\frac{du}{dt} = 2 \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{t}}$

$$\therefore \frac{dr}{dt} = \frac{dr}{du} \frac{du}{dt}$$

$$r'(t) = (10^{u} \ln 10)\left(\frac{1}{\sqrt{t}}\right)$$

$$r'(t) = (10^{2\sqrt{t}} \ln 10)\left(\frac{1}{\sqrt{t}}\right)$$

$$r'(t) = \frac{dr}{dt} = \frac{10^{2\sqrt{t}} \ln 10}{\sqrt{t}}$$

28. $f(z) = e^{z/(z-1)}$

$$f'(z) = e^{\frac{z}{z-1}} \frac{d}{dz} \frac{z}{z-1}$$ using chain rule

$$f'(z) = e^{\frac{z}{z-1}} \left(\frac{1(z-1) - (1)(z)}{(z-1)^2} \right)$$

$$f'(z) = e^{\frac{z}{z-1}} \left(\frac{z-1 - z}{(z-1)^2} \right)$$

$$f'(z) = e^{\frac{z}{z-1}} \left(\frac{-1}{(z-1)^2} \right)$$

$$f'(z) = -\frac{e^{\frac{z}{z-1}}}{(z-1)^2}$$

29. $H(r) = \dfrac{(r^2-1)^3}{(2r+1)^5}$

$$H'(r) = \frac{3(r^2-1)^2(2r)(2r+1)^5 - [5(2r+1)^4(2)(r^2-1)^3]}{((2r+1)^5)^2}$$

$$H'(r) = \frac{6r(r^2-1)^2(2r+1)^5 - 10(2r+1)^4(r^2-1)^3}{(2r+1)^{10}}$$

$$H'(r) = \frac{(r^2-1)^2(2r+1)^4\,[6r(2r+1) - 10(r^2-1)]}{(2r+1)^{10}}$$

$$H'(r) = \frac{(r^2-1)^2(2r+1)^4\,[12r^2+6r-10r^2+10]}{(2r+1)^{10}}$$

$$H'(r) = \frac{(r^2-1)^2\,[2r^2+6r+10]}{(2r+1)^6}$$

59. Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

$f'(x) = 2\cos x + 2\sin x \cos x$

$f'(x) = 2\cos x\,(1 + \sin x)$

$\because$ Tangnet line Horizontal

$\therefore f'(x) = 0$

$\therefore 2\cos x\,(1 + \sin x) = 0$

$f'(x) = 0$, when $2\cos x = 0$ or $1 + \sin x = 0$

$2\cos x = 0$	$1 + \sin x = 0$
$\cos x = 0$	$\sin x = -1$
$x = \dfrac{\pi}{2} + 2n\pi$	$x = \dfrac{3\pi}{2} + 2n\pi$
$f(\frac{\pi}{2}) = 2\sin\frac{\pi}{2} + \sin^2\frac{\pi}{2}$	$f(\frac{3\pi}{2}) = 2\sin\frac{3\pi}{2} + \sin^2\frac{3\pi}{2}$
$= 2(1) + 1 = 3$	$= 2(-1) + (-1)^2 = -1$

$\therefore \left(\dfrac{\pi}{2} + 2n\pi, 3\right), \left(\dfrac{3\pi}{2} + 2n\pi, -1\right)$

60. At what point on the curve $y = \sqrt{1 + 2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

$$y = \sqrt{1+2x}$$

$$y' = \frac{1}{2\sqrt{1+2x}}(2) = \frac{1}{\sqrt{1+2x}} \rightarrow (1)$$

$$6x + 2y = 1 \Rightarrow 2y = -6x + 1$$

$$y = -3x + \frac{1}{2}$$

$$\therefore \frac{dy}{dx} = m_2 = -3 \quad , \because \text{two lines perpendicular}$$

$$\therefore m_1 = -\frac{1}{m_2} = -\frac{1}{(-3)} = \frac{1}{3}$$

$$\frac{1}{\sqrt{1+2x}} = \frac{1}{3} \Rightarrow 3^2 = \sqrt{1+2x}^{\,2}$$

$$\Rightarrow 9 = 1 + 2x \Rightarrow 8 = 2x \Rightarrow x = 4$$

$$y = \sqrt{1+2(4)} \Rightarrow \sqrt{9} = 3 \quad \therefore (4, 3)$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

EXAMPLE 8 If $f(x) = \sin(\cos(\tan x))$, then

$$f'(x) = \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x)$$

$$= \cos(\cos(\tan x))[-\sin(\tan x)] \frac{d}{dx} (\tan x)$$

$$= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x$$

Notice that we used the Chain Rule twice. ■

EXAMPLE 9 Differentiate $y = e^{\sec 3\theta}$.

SOLUTION The outer function is the exponential function, the middle function is the secant function, and the inner function is the tripling function. So we have

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \frac{d}{d\theta} (\sec 3\theta)$$

$$= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta} (3\theta)$$

$$= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta$$ ■

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45. $y = \cos\sqrt{\sin(\tan \pi x)}$

$$\frac{d}{dx}\left(\cos\sqrt{\sin\tan\pi x}\right) = -\sin\left(\sqrt{\sin\tan\pi x}\right)\frac{d}{dx}\left(\sqrt{\sin\tan\pi x}\right)$$

$$\frac{d}{dx}\left(\sqrt{\sin\tan\pi x}\right) = \frac{1}{2}\left(\sin(\tan\pi x)\right)^{-1/2}\frac{d}{dx}\left(\sin(\tan\pi x)\right)$$

sin زاوية

$$\frac{d}{dx}\left(\sin(\tan\pi x)\right) = \cos\left((\tan\pi x)\right)\left(\sec^2\pi x\right)\pi$$

$$\therefore \frac{d}{dx} = -\sin\sqrt{\sin(\tan\pi x)}\left(\frac{1}{2}\left(\sin(\tan\pi x)\right)^{-1/2}\right)$$

$$\left(\cos(\tan\pi x)\right)\left(\sec^2\pi x\right)\pi$$

رتبتها !

$$= -\frac{\pi\sec^2(\pi x)\cos(\tan(\pi x))\sin\left(\sqrt{\sin(\tan(\pi x))}\right)}{2\sqrt{\sin(\tan(\pi x))}}$$

71. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

$$r'(x) = f'(g(h(x))) \cdot [g(h(x))]'$$

$$r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

$$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1)$$

$$r'(1) = f'(g(2)) \cdot g'(2) \cdot (4)$$

$$r'(1) = f'(3) \cdot 5 \cdot 4$$

$$r'(1) = 6 \cdot 5 \cdot 4 = 6 \cdot 20 = 120$$

II. If $h(x) = g(f(x))$, where $f(2) = 3$, $f'(2) = 4$, $g(3) = 6$, and $g'(3) = -5$, then $h'(2) =$

a) 12.

b) 24.

c) $\boxed{-20}$. ✓

d) None of the above.

$$h'(x) = g'(f(x)) \cdot f'(x)$$

$$h'(2) = g'(f(2)) \cdot f'(2)$$

$$h'(2) = g'(3) \cdot 4$$

$$h'(2) = -5 \times 4 = -20$$

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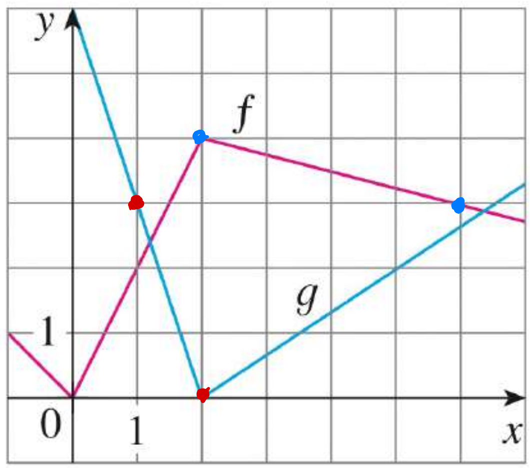
65. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.

(a) $u'(1)$ (b) $v'(1)$ = DNE (c) $w'(1)$ → $w'(1) = -2$

because Not differeable at $g'(2)$

$g(1) = 3$

$f(1) = 2$



(2, 4)

(6, 3)

(1, 3)

(2, 0)

$$u'(x) = f'(g(x)) \cdot g'(x)$$

$$u'(1) = f'(g(1)) \cdot g'(1)$$

$$u'(1) = f'(3) \cdot g'(1)$$

$$f'(3) = \frac{3-4}{6-2} = -\frac{1}{4}$$

$$g'(1) = \frac{0-3}{2-1} = -3$$

$$u'(1) = -\frac{1}{4} \cdot -3 = \frac{3}{4}$$

The Chain Rule can be written either in the prime notation

2

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

II. If $f(x) = 5x$ and $g(x) = 2\cos x$, then $(f \circ g)'(\pi) =$

a) -5. b) $\boxed{0}$. c) 10. d) -10. e) None of the above.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

لما أشتقيت الدالة الخارجية صار عندي عدد
ثابت بدون المتغير بس 5 عشان جذي
مافي مكان أعوض في الg(x)

$$= 5 \cdot (-2\sin x) = -10\sin x$$

$$\therefore (f \circ g)'(\pi) = -10\sin \pi$$

$$= -10(0) = 0$$

@Precalculusq8

Thamrah

Kuwait University

Calculus 1 – Implicit Differentiation

(Section 3.5)

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3.5 Implicit Differentiation

$\frac{dy}{dx}$ نستخدم طريقة الاشتقاق الضمني (implicit Differentiation) لايجاد المشتقة
لما يكون صعب عليي أفصل الy عن x مثال :

examples:-

- $x^2 + y^2 = 25$
- $2xy + \sin x = 6x \sin(x) - 1$
- $\sin(x+y) = x$

طرق الاشتقاق

1) $x^2 \xrightarrow{\frac{d}{dx}} 2x$

2) $y^2 \xrightarrow{\frac{d}{dx}} 2y\,y'$ $\qquad y' = \frac{dy}{dx}$

3) $y e^{\sin x} \xrightarrow{\frac{d}{dx}} y' e^{\sin x} + y\, e^{\sin x} \cdot \cos x$

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$\frac{dy}{dx} = y'$

طريقة حل الاشتقاق الضمني :

١- نشتق الطرفين بالنسبة للمتغير x

٢- نخلي $\frac{dy}{dx}$ بطرف بروحها

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

$$x^2 + y^2 = 25$$

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dy}(25)$$

$$(2x + 2yy') = 0$$

$$2yy' = -2x \Rightarrow y' = \frac{-2x}{2y}$$

$$y' = -\frac{x}{y}$$

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$y' = \frac{dy}{dx} = -\frac{3}{4} \quad x \quad y$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

نفس السؤال بس حلها نفس طريقة سكشن 3.4 chain rule.. بس درباك مو كل مرة تقدر تحل بالchain rule على طول...

ملاحظة : عادتا لما تشوف الy مو بطرف بروحها بسؤال .. فمعناته رح تحل بالاشتقاق الضمني

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

SOLUTION 2

(b) Solving the equation $x^2 + y^2 = 25$ for y, we get $y = \pm\sqrt{25 - x^2}$. The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

So

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$.

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EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x

$$3x^2 + 3y^2y' = 6xy' + 6y$$

$$x^2 + y^2y' = 2xy' + 2y$$

We now solve for y':

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at $(3, 3)$. So an equation of the tangent to the folium at $(3, 3)$ is

$$y - 3 = -1(x - 3) \qquad \text{or} \qquad x + y = 6$$

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EXAMPLE 3 Find y' if $\sin(x+y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x+y) \cdot (1+y') = y^2(-\sin x) + (\cos x)(2yy')$$

$$\cos(x+y) + y^2 \sin x = (2y\cos x)y' - \cos(x+y) \cdot y'$$

So
$$y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

SOLUTION Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$\boxed{3} \qquad y' = -\frac{x^3}{y^3}$$

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\,(d/dx)(x^3) - x^3\,(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

$$\because \; y' = -\frac{x^3}{y^3}$$

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

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$$= \frac{3x^2y^3 + \dfrac{3x^6}{y}}{y^6} = \frac{\dfrac{3x^2y^4+3x^6}{y}}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4+x^4)}{y^7}$$

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$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$

sq8

5–20 Find dy/dx

7. $x^4 + x^2y^2 + y^3 = 5$

Differentiate both sides with respect to x

$$\frac{d}{dx}\left(x^4 + x^2y^2 + y^3\right) = \frac{d}{dx}(5)$$

$$4x^3 + \left(2xy^2 + 2yy'x^2\right) + 3y^2y' = 0$$

$$4x^3 + 2xy^2 + y'\left(2x^2y + 3y^2\right) = 0$$

$$y'\left(2x^2y + 3y^2\right) = -4x^3 - 2x^2y$$

$$y' = \frac{-4x^3 - 2xy^2}{2x^2y + 3y^2} = \frac{-2x(2x^2 + y^2)}{y(2x^2 + 3y)}$$

5. $x^2 - 4xy + y^2 = 4$ $\quad \dfrac{dy}{dx} = \dfrac{x - 2y}{2x - y}$

9. $\dfrac{x^2}{x+y} = y^2 + 1$

Differentiate both sides with respect to x

$$\frac{2x(x+y) - (1+y')(x^2)}{(x+y)^2} = 2yy' + 0$$

$$\frac{2x^2 + 2xy - x^2 - x^2y'}{(x+y)^2} = 2yy'$$

$$2x^2 + 2xy - x^2 - x^2y' = 2yy'(x+y)^2$$

$$x^2 + 2xy = 2yy'(x+y)^2 + x^2y'$$

$$x^2 + 2xy = y'(2y(x+y)^2 + x^2)$$

$$y' = \frac{x^2 + 2xy}{2y(x+y)^2 + x^2}$$

10. $xe^y = x - y$

$$(1)\ e^y + xe^y y' = 1 - y'$$

$$y' + xe^y y' = 1 - e^y$$

$$y'(1 + xe^y) = 1 - e^y$$

$$y' = \frac{1 - e^y}{1 + xe^x}$$

11. $y \cos x = x^2 + y^2$

$$y' \cos x + y(-\sin x) = 2x + 2yy'$$

$$y' \cos x - 2yy' = 2x + y \sin x$$

$$y'(\cos x - 2y) = 2x + y \sin x$$

$$y' = \frac{2x + y \sin x}{\cos x - 2y}$$

14. $e^y \sin x = x + xy$

$$e^y y' \sin x + e^y \cos x = 1 + (1y) + xy'$$

$$e^y y' \sin x - xy' = 1 + y - e^y \cos x$$

$$y'(e^y \sin x - x) = 1 + y - e^y \cos x$$

$$y' = \frac{1 + y - e^y \cos x}{e^y \sin x - x}$$

4. [10 pts.] Find $\dfrac{dy}{dx}$ by **implicit differentiation**: $e^y \sin x = x + xy$.

We have $e^y \cos x + e^y \sin x \dfrac{dy}{dx} = 1 + x\dfrac{dy}{dx} + y$.

Therefore, we obtain $\dfrac{dy}{dx} = \dfrac{1 + y - e^y \cos x}{e^y \sin x - x}$.

15. $e^{x/y} = x - y$

$$e^{x/y} \cdot \frac{(1)y - (y')(x)}{y^2} = 1 - y'$$

$$e^{x/y} \cdot \frac{y - y'x}{y^2} = 1 - y'$$

ضرب طرفين $* \frac{y^2}{e^{x/y}}$

$$y - y'x = \left(\frac{1 - y'}{e^{x/y}}\right) y^2$$

$$y - y'x = \frac{y^2}{e^{x/y}} - \frac{y'y^2}{e^{x/y}}$$

$$\frac{y'y^2}{e^{x/y}} - y'x = \frac{y^2}{e^{x/y}} - y$$

$$y'\left(\frac{y^2}{e^{x/y}} - x\right) = \frac{y^2}{e^{x/y}} - y$$

$$y' = \frac{\frac{y^2}{e^{x/y}} - y}{\frac{y^2}{e^{x/y}} - x} = \frac{\frac{y^2 - ye^{x/y}}{e^{x/y}}}{\frac{y^2 - xe^{x/y}}{e^{x/y}}} = \frac{y^2 - ye^{x/y}}{y^2 - xe^{x/y}}$$

21. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

Differentiate both sides

$$f'(x) + (2x[f(x)]^3 + 3[f(x)]^2 f'(x)\, x^2 = 0$$

$$f'(x)[1 + 3[f(x)]^2 x^2] = -2x[f(x)]^3$$

$$f'(x) = \frac{-2x[f(x)]^3}{1+3[f(x)]^2 x^2}$$

$$f'(1) = \frac{-2(1)[f(1)]^3}{1+3[f(1)]^2 1^2} = \frac{-2[2]^3}{1+3(2)^2 * 1^2}$$

$$= \frac{-2(8)}{1+12} = \frac{-16}{13}$$

22. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

$$g'(x) + 1 \sin g(x) + x \cos(g(x)) \cdot g'(x) = 2x$$

$$g'(x) + x\cos(g(x)) \cdot g'(x) = 2x - \sin g(x)$$

$$g'(x)\left[1 + x\cos(g(x))\right] = 2x - \sin g(x)$$

$$g'(x) = \frac{2x - \sin(g(x))}{1 + x\cos(g(x))}$$

we need $g(0)$, sub $x = 0$ in main equation

$$g(0) + 0 \sin g(0) = 0, \quad \therefore g(0) = 0$$

$$\therefore g'(0) = \frac{2(0) - \sin(g(0))}{1 + 0\cos(g(0))} = \frac{0-0}{1+0} = 0$$

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1 - x^2}}$$

(b)
$$f'(x) = x\frac{1}{1 + (\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$
$$= \frac{\sqrt{x}}{2(1 + x)} + \arctan\sqrt{x}$$

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49–60 Find the derivative of the function. Simplify wher possible.

49. $y = (\tan^{-1} x)^2$

$$y' = 2(\tan^{-1} x)\left(\frac{1}{1+x^2}\right)$$

$$= \frac{2\tan^{-1} x}{1+x^2}$$

50. $y = \tan^{-1}(x^2)$

$$y' = \frac{1}{1+(x^2)^2}(2x) = \frac{2x}{1+x^4}$$

51. $y = \sin^{-1}(2x + 1)$

$$y' = \frac{1}{\sqrt{1 - (2x+1)^2}} \cdot (2)$$

$$y' = \frac{2}{\sqrt{1 - (4x^2+4x+1)}}$$

$$y' = \frac{2}{\sqrt{1 - 4x^2 - 4x - 1}}$$

$$y' = \frac{2}{\sqrt{-4x^2 - 4x}} = \frac{2}{\sqrt{4(-x^2 - x)}}$$

$$y' = \frac{2}{2\sqrt{-x^2 - x}} = \frac{1}{\sqrt{-x^2 - x}}$$

52. $g(x) = \arccos\sqrt{x}$

$$g'(x) = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}$$

$$g'(x) = -\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

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$$g'(x) = -\frac{1}{2\sqrt{x(1-x)}}$$

$$g'(x) = -\frac{1}{2\sqrt{x-x^2}}$$

54. $y = \tan^{-1}\left(x - \sqrt{1 + x^2}\right)$

$$y' = \frac{1}{1 + \left(x - \sqrt{1+x^2}\right)^2} \cdot \frac{d}{dx}\left(x - \sqrt{1+x^2}\right)$$

$$\frac{d}{dx}\left(x - \sqrt{1+x^2}\right) = 1 - \tfrac{1}{2}\left(1 + x^2\right)^{-1/2}(2x)$$

$$= 1 - \frac{x}{(1+x^2)^{1/2}}$$

$$\therefore\ y' = \frac{1}{1 + \left(x^2 - 2x\sqrt{1+x^2} + 1 + x^2\right)}\left(1 - \frac{x}{(1+x^2)^{1/2}}\right)$$

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$$= \frac{1}{1 + 2x^2 - 2x\sqrt{1+x^2} + 1} \cdot \frac{(1+x^2)^{1/2} - x}{(1+x^2)^{1/2}}$$

$$= \frac{(1+x)^{1/2} - x}{\left(2 + 2x^2 - 2x\sqrt{1+x^2}\right)(1+x^2)^{1/2}}$$

55. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

$$h'(t) = \frac{-1}{1+t^2} + \frac{-1}{1+\left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}$$

$$h'(t) = \frac{-1}{1+t^2} + \frac{-1 \times -1}{\left(1+\frac{1}{t^2}\right)t^2}$$

$$h'(t) = \frac{-1}{1+t^2} + \frac{1}{t^2+1}$$

$$h'(t) = \frac{-1+1}{1+t^2} = \frac{0}{1+t^2} = 0$$

57. $y = x\sin^{-1}x + \sqrt{1-x^2}$

$$y' = \left(1\,(\sin^{-1}(x)) + x\,\frac{1}{\sqrt{1-x^2}}\right) + \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$$

$$y' = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} + \left(-\frac{x}{\sqrt{1-x^2}}\right)$$

$$y' = \sin^{-1}(x)$$

III. If $\sin^{-1}x + e^y = e$, then $\left.\frac{dy}{dx}\right|_{(0,1)} =$

$$\frac{1}{\sqrt{1-x^2}} + e^y y' = 0$$

$$e^y y' = -\frac{1}{\sqrt{1-x^2}} \Rightarrow y' = -\frac{1}{e^y\sqrt{1-x^2}}$$

at $(0,1)$

$$y' = -\frac{1}{e^1\sqrt{1-0}} = \frac{-1}{e}$$

III. If $f(x) + x\sin(f(x)) = x^3 + \frac{\pi}{2}$, then $f'(0)$ is equal to

a) 2.

b) -1.

c) 0.

d) 1.

e) None of the above.

$$f'(x) + 1\sin(f(x)) + x\cos(f(x))\cdot(f'(x)) = 3x^2 + 0$$

$$f'(x) + x\cos(f(x))\cdot f'(x) = 3x^2 - \sin(f(x))$$

$$f'(x)\left(1 + x\cos(f(x))\right) = 3x^2 - \sin(f(x))$$

$$f'(x) = \frac{3x^2 - \sin f(x)}{1 + x\cos(f(x))}$$

We need $f(0)$, sub $x = 0$ in main equation

$$f(0) + 0\sin(f(0)) = 0 + \frac{\pi}{2}$$

$\therefore f(0) = \frac{\pi}{2}$, $\therefore f'(0) = \dfrac{3(0)^2 - \sin f(0)}{1 + 0\cos(f(0))}$

$$\therefore f'(0) = \frac{0 - \sin\frac{\pi}{2}}{1+0} = \frac{-1}{1} = -1$$

IV. If $\sin(x) + y^3 = x^2 + 1$, then $\left.\frac{dy}{dx}\right|_{(0,1)} =$

a) $-1/3$. b) $1/3$. c) 0. d) 1. e) None of the above.

$$\cos x + 3y^2 y' = 2x$$

$$3y^2 y' = 2x - \cos x$$

$$y' = \frac{2x - \cos x}{3y^2}$$

at $(0,1)$

$$y' = \frac{2(0) - \cos 0}{3(1)^2} = \frac{-1}{3}$$

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I. If $f(x) = \sin^{-1}(2x)$, then $f'(x) =$

(A) $\dfrac{1}{\sqrt{1-4x^2}}$.

(B) $\dfrac{-1}{\sqrt{1-4x^2}}$.

(C) $\dfrac{2}{\sqrt{1-4x^2}}$.

(D) $\dfrac{-2}{\sqrt{1-4x^2}}$.

(E) None of the above.

$$y' = \frac{1}{\sqrt{1-(2x)^2}}(2)$$

$$y' = \frac{2}{\sqrt{1-4x^2}}$$

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II. The slope of the tangent line to the graph of $x^2 - 3y^3 = -2$ at $(1, 1)$ is

(A) $1/3$.

(B) $-9/2$.

(C) $\boxed{2/9}$. ✓

(D) $3/2$.

(E) None of the above.

$$2x - 9y^2 y' = 0$$

$$2x = 9yy' \qquad \overset{x\ \ y}{(1, 1)}$$

$$\Rightarrow \frac{2x}{9y} = y' \Rightarrow y' = \frac{2(1)}{9(1)} = \frac{2}{9}$$

Show that or prove

3) $\frac{d}{dx}(\sin^{-1} 2x) = \frac{2}{\sqrt{1-4x^2}}$

Let $y = \sin^{-1}(2x)$

$\therefore \sin y = 2x$

$z = \sqrt{1-4x^2}$

$z = \sqrt{1^2 - (2x)^2} = \sqrt{1-4x^2}$

now we diff with respect to x

$\because \sin y = 2x$

$(\cos y)\, y' = 2$ اشتقيناها اشتقاق ضمني

$\therefore y' = \frac{2}{\cos y} = \frac{2}{\frac{\sqrt{1-4x^2}}{1}}$

$\therefore \frac{d}{dx}(\sin^{-1} 2x) = y' = \frac{2}{\sqrt{1-4x^2}}$

ثمـــرة
Thamrah

Kuwait University

Calculus 1 – Derivatives of Logarithmic

(Section 3.6)

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3.6 Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}(\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

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في بعض المسائل رح نستخدم خواص الln أو log كثر ما نقدر عشان تسهل علينا عملية الاشتقاق

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- $\ln(AB) = \ln A + \ln B$
- $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
- $\ln A^x = x \ln A$

↳ ex: $\ln x^4 = 4\ln x$, $\ln x^{\sin x} = (\sin x)\ln x$

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

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EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Differentiate $f(x) = \log_{10}(2 + \sin x)$.

SOLUTION Using Formula 1 with $b = 10$, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\log_{10}(2 + \sin x) \\ &= \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) \\ &= \frac{\cos x}{(2 + \sin x)\ln 10} \end{aligned}$$

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EXAMPLE 5 Find $\dfrac{d}{dx} \ln \dfrac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1} \frac{\sqrt{x-2} \cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

أسهل

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx} \left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.) ■

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EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 6 is worth remembering:

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$$\boxed{\frac{d}{dx} \ln|x| = \frac{1}{x}}$$

Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x + y) \cancel{=} \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \cancel{=} \log\left(\frac{6}{2}\right) \quad \text{and} \quad (\log_2 x)^3 \cancel{=} 3 \log_2 x$$

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2–22 Differentiate the function.

2. $f(x) = x \ln x - x$

$$f'(x) = 1 \ln(x) + x \frac{1}{x} - 1$$

$$f'(x) = \ln(x) + 1 - 1 = \ln(x)$$

3. $f(x) = \sin(\ln x)$

$$f'(x) = \cos(\ln x)\left(\frac{1}{x}\right) = \frac{\cos(\ln x)}{x}$$

7. $f(x) = \log_{10}(1 + \cos x)$

$$f'(x) = \frac{1}{(1+\cos x)\ln 10} \cdot \frac{d}{dx}(1+\cos x)$$

$$= \frac{1}{(1+\cos x)\ln 10} \cdot -\sin x = \frac{-\sin x}{\ln 10(1+\cos x)}$$

I. Let $f(x) = \log_{10}(x^2+1)$. Then:

(A) $f'(x) = \dfrac{2x}{x^2+1}$.

✓ (B) $f'(x) = \dfrac{2x}{(x^2+1)\ln 10}$.

(C) $f'(x) = \dfrac{1}{x^2+1}$.

(D) $f'(x) = \dfrac{1}{(x^2+1)\ln 10}$.

(E) None of the above.

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$$y = \log_{10}(x^2+1)$$

$$y' = \frac{1}{(x^2+1)\ln 10}(2x) = \frac{2x}{(x^2+1)\ln 10}$$

9. $g(x) = \ln(xe^{-2x})$

$g(x) = \ln x + \ln e^{-2x}$ خواص ال ln

$g(x) = \ln x + (-2x \ln e)$

$g(x) = \ln x - 2x$

$g'(x) = \frac{1}{x} - 2$

11. $F(t) = (\ln t)^2 \sin t$

$F'(t) = 2(\ln t)(\frac{1}{t}) \sin t + (\ln t)^2 \cos t$

$F'(t) = \frac{2 \ln t \sin t}{t} + (\ln t)^2 \cos t$

$F'(t) = \ln t \left(\frac{2 \sin t}{t} + \ln t \cos t \right)$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

$$h'(x) = \frac{1}{x + \sqrt{x^2-1}} \quad \frac{d}{dx}\, x + \sqrt{x^2-1}$$

$$h'(x) = \frac{1}{x + \sqrt{x^2-1}} \left(1 + \frac{1}{\cancel{2}\sqrt{x^2-1}} \cdot (\cancel{2}x)\right)$$

$$h'(x) = \frac{1}{x + \sqrt{x^2-1}} \left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$h'(x) = \frac{1}{x + \sqrt{x^2-1}} \left(\frac{\sqrt{x^2-1} + x}{\sqrt{x^2-1}}\right)$$

$$h'(x) = \frac{\cancel{\sqrt{x^2-1} + x}}{\cancel{\left(x + \sqrt{x^2-1}\right)}\left(\sqrt{x^2-1}\right)}$$

$$h'(x) = \frac{1}{\sqrt{x^2-1}}$$

13. $G(y) = \ln \dfrac{(2y+1)^5}{\sqrt{y^2+1}}$

$$G(y) = \ln(2y+1)^5 - \ln\left(\sqrt{y^2+1}\right)$$

$$G(y) = 5\ln(2y+1) - \ln(y^2+1)^{1/2}$$

$$G'(y) = 5\left(\frac{2}{2y+1}\right) - \frac{1}{2}\ln(y^2+1)$$

$$G'(y) = \frac{10}{2y+1} - \frac{1}{\cancel{2}}\,\frac{\cancel{2}y}{y^2+1}$$

$$G'(y) = \frac{10}{2y+1} - \frac{y}{y^2+1}$$

14. $P(v) = \dfrac{\ln v}{1 - v}$

$$P'(v) = \frac{\left(\frac{1}{v}\right)(1-v) - (-1)(\ln v)}{(1-v)^2}$$

$$P'(v) = \frac{\frac{1}{v} - \frac{v}{v} + \ln v}{(1-v)^2}$$

$$P'(v) = \frac{\frac{1-v}{v} + \ln v}{(1-v)^2} = \frac{\frac{1-v}{v} + \frac{v\ln v}{v}}{(1-v)^2}$$

$$P'(v) = \frac{\frac{1-v+v\ln v}{v}}{(1-v)^2}$$

$$P'(v) = \frac{1-v+v\ln v}{v(1-v)^2}$$

16. $y = \ln|1 + t - t^3|$

$$y' = \frac{1 - 3t^2}{1 + t - t^3}$$

17. $T(z) = 2^z \log_2 z$

$$T'(z) = 2^z \ln 2 \,(\log_2 z) + \left(\frac{1}{z \ln 2}\right) 2^z$$

قانون change formula

سويتها عشان اختصر

$$T'(z) = 2^z \cancel{\ln} 2 \left(\frac{\ln z}{\cancel{\ln} 2}\right) + \frac{2^z}{z \ln 2}$$

$$T'(z) = 2^z \ln z + \frac{2^z}{z \ln 2}$$

$$T'(z) = 2^z \left(\ln z + \frac{1}{z \ln 2}\right)$$

19. $y = \ln(e^{-x} + xe^{-x})$

$$y' = \frac{-e^{-x} + (1e^{-x} + (-e^{-x})x)}{e^{-x} + xe^{-x}}$$

$$y' = \frac{-\cancel{e^{-x}} + \cancel{e^{-x}} - xe^{-x}}{e^{-x} + xe^{-x}}$$

$$y' = \frac{-x\cancel{e^{-x}}}{\cancel{e^{-x}}(1+x)} = \frac{-x}{1+x}$$

20. $H(z) = \ln\sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

$$H(z) = \ln\left(\frac{a^2 - z^2}{a^2 + z^2}\right)^{1/2}$$

$$H(z) = \frac{1}{2}\ln\left(\frac{a^2 - z^2}{a^2 + z^2}\right)$$

$$H(z) = \frac{1}{2}\left[\ln(a^2 - z^2) - \ln(a^2 + z^2)\right]$$

$$H'(z) = \frac{1}{2}\left[\frac{-2z}{a^2 - z^2} - \frac{2z}{a^2 + z^2}\right]$$

$$H'(z) = \frac{1}{2}\left[\frac{-2z(a^2 + z^2) - 2z(a^2 - z^2)}{(a^2 - z^2)(a^2 + z^2)}\right]$$

$$H'(z) = \frac{1}{2}\left[\frac{-2a^2z - 2z^3 - 2a^2z + 2z^3}{(a^2 - z^2)(a^2 + z^2)}\right]$$

$$H'(z) = \frac{1}{\cancel{2}}\left[\frac{-\cancel{4}a^2z}{(a^2 - z^2)(a^2 + z^2)}\right]$$

$$H'(z) = \frac{-2a^2z}{(a^2 - z^2)(a^2 + z^2)}$$

21. $y = \tan[\ln(ax + b)]$

$$y' = \left(\sec^2[\ln(ax+b)]\right)\frac{a}{ax+b}$$

$$y' = \frac{a\sec^2[\ln(ax+b)]}{ax+b}$$

5. [10 pts.] Find y'' if $y = \ln(1 + \ln x)$.

We have $y' = \dfrac{\frac{1}{x}}{1+\ln x} = \dfrac{1}{x(1+\ln x)}$. Now we use quotient rule to obtain $y'' = \dfrac{-(2+\ln x)}{(x(1+\ln x))^2}$.

23–26 Find y' and y''.

23. $y = \sqrt{x} \ln x$

$$\frac{\sqrt{x}}{x} = \frac{x^{1/2}}{x} = x^{1/2 - 1} = x^{-1/2}$$

$$y' = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x}\,\frac{1}{x}$$

$$y' = \frac{1}{2}\,\frac{\ln x}{\sqrt{x}} + x^{-1/2}$$

$$y' = \frac{1}{2} x^{-1/2} \ln x + x^{-1/2}$$

$$\frac{x^{-1/2}}{x} = x^{-1/2-(1)} = x^{-3/2}$$

$$y'' = \frac{1}{2}\left[\left(-\frac{1}{2} x^{-3/2} \ln x + \frac{1}{x}\left(x^{-1/2}\right)\right) + -\frac{1}{2} x^{-3/2}\right.$$

$$y'' = \frac{-\ln x}{4\sqrt{x^3}} + \cancel{\frac{1}{2\sqrt{x^3}}} - \cancel{\frac{1}{2\sqrt{x^3}}}$$

$$y'' = -\frac{\ln x}{4\sqrt{x^3}}$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

ex: $x^{\sin x}$, $x^{\sqrt{x}}$, $\sqrt{x}^{\ln x}$ ....

نستخدم الطريقة هذي لما :

١- دالة أس دالة .

٢- دالة معقدة علينا

EXAMPLE 7 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$ ■

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طريقة حل اشتقاق دالة أس دالة:

$$\frac{d}{dx}\left[f(x)\right]^{g(x)}$$

1) $y = f(x)^{g(x)}$

2) $\ln y = \ln f(x)^{g(x)}$ أخذ ln

3) $\ln y = g(x) \ln f(x)$ خواص ال ln

4) $\frac{1}{y} y' = g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x)$ اشتق الطرفين

5) $y' = y\left(g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x)\right)$ أبسط إذا في مجال للتبسيط

بعدين أعوض بقيمة y

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You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(b^x)' = b^x \ln b]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

Constant base, constant exponent

1. $\frac{d}{dx}(b^n) = 0$ (b and n are constants) $\frac{d}{dx} 2^\pi = 0$, $\frac{d}{dx} e^4 = 0$

Variable base, constant exponent

2. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$ $\frac{d}{dx}(2x+3)^3 = 3(2x+3)^2(2)$

Constant base, variable exponent

3. $\frac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$ $\frac{d}{dx} 3^{\sin x} = 3^{\sin x} \ln(3) \cos x$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$, logaritmic differentiation can be used, as in the next example.

EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

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III. The derivative of $f(x) = x^{\sqrt{x+1}}$ at $x = 1$ is

a) 0.

b) 1.

$\boxed{\text{c)}}$ $\sqrt{2}$.

d) 2.

e) None of the above.

$$y = x^{\sqrt{x+1}}$$

$$\ln y = \ln x^{\sqrt{x+1}} \Rightarrow \ln y = \sqrt{x+1}\ln x$$

now differentiate:-

$$\frac{1}{y}y' = \frac{1}{2\sqrt{x+1}}\ln x + \frac{1}{x}\sqrt{x+1}$$

$$y' = y\left(\frac{\ln x}{2\sqrt{x+1}} + \frac{\sqrt{x+1}}{x}\right)$$

$$y' = x^{\sqrt{x+1}}\left(\frac{\ln x}{2\sqrt{x+1}} + \frac{\sqrt{x+1}}{x}\right)$$

$$\because x = 1, \therefore y' = 1^{\sqrt{1+1}}\left(\frac{\ln 1}{2\sqrt{1+1}} + \frac{\sqrt{1+1}}{1}\right)$$

$$y' = 1^{\sqrt{2}}\left(0 + \sqrt{2}\right) = \sqrt{2}$$

3. [10 pts.] Let $f(x) = (3+4x)^{\cos x}$. Find $f'(x)$.

Let $y = (3+4x)^{\cos x}$. Then, $\ln y = \cos x \ln(3+4x)$.

Differentiating both sides implicitly w.r.t. x, we get:

$$\frac{y'}{y} = (-\sin x)\ln(3+4x) + \cos x\left(\frac{4}{3+4x}\right).$$

Therefore, $\boxed{y' = (3+4x)^{\cos x}\left[\frac{4\cos x}{3+4x} - \sin x \ln(3+4x)\right]}$.

---

1. [10 pts.] Let $y = (x^2+1)^{\cos x}$. Use logarithmic differentiation to find $\frac{dy}{dx}$ at $x = 0$.

---

1. [10 pts.] Using logarithmic differentiation: $\boxed{\ln y = \ln(x^2+1)^{\cos x} = \cos x \ln(x^2+1)}$.

Therefore, $\boxed{\frac{y'}{y} = (-\sin x)\ln(x^2+1) + \cos x\left(\frac{2x}{x^2+1}\right)}$.

That is, $\boxed{y' = \left[(-\sin x)\ln(x^2+1) + \cos x\left(\frac{2x}{x^2+1}\right)\right](x^2+1)^{\cos x}}$. This implies that $\boxed{\left.\frac{dy}{dx}\right|_{x=0} = 0}$.

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2. [10 pts.] Find an equation of the tangent line to the curve $y = x^{\ln(x)}$ at the point $(1, 1)$.

We have $\ln y = \ln x^{\ln x} = \ln x \ln x = (\ln x)^2$. Therefore, $\dfrac{y'}{y} = 2(\ln x)\dfrac{1}{x} = \dfrac{2\ln x}{x}$.

This implies that $\boxed{\dfrac{dy}{dx} = \left(\dfrac{2\ln x}{x}\right) y = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)}$. So $\boxed{\frac{dy}{dx}|_{(1,1)} = 0}$.

Finally, an equation of the tangent line at the point $(1, 1)$ is given by $y - 1 = 0(x - 1) = 0$, i.e., $\boxed{y = 1}$.

3. [10 pts.] Let $y = (x^2 + 1)^x$. Find $\dfrac{dy}{dx}$.

We have $\boxed{\ln y = x\ln(x^2 + 1)}$. Now we differentiate both sides with respect to x to obtain

$$\boxed{\frac{y'}{y} = \ln(x^2 + 1) + x\left(\frac{2x}{x^2 + 1}\right)}.$$

Therefore, we have $\boxed{\dfrac{dy}{dx} = \left(\ln(x^2 + 1) + x\left(\dfrac{2x}{x^2 + 1}\right)\right)(x^2 + 1)^x}$.

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39–50 Use logarithmic differentiation to find the derivative of the function.

39. $y = (x^2 + 2)^2(x^4 + 4)^4$

$$\ln y = \ln[(x^2+2)^2(x^4+4)^4]$$

$$\ln y = \ln(x^2+2)^2 + \ln(x^4+4)^4$$

$$\ln y = 2\ln(x^2+2) + 4\ln(x^4+4)$$

$$\frac{1}{y}\cdot y' = 2\,\frac{2x}{x^2+2} + 4\,\frac{4x^3}{x^4+4}$$

$$\frac{1}{y}\,y' = \frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}$$

$$y' = \left(\frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}\right)y$$

$$y' = \left(\frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}\right)(x^2+2)^2(x^4+4)^4$$

40. $y = \dfrac{e^{-x}\cos^2 x}{x^2 + x + 1}$

$$\ln y = \ln\left(\frac{e^{-x}\cos^2 x}{x^2+x+1}\right)$$

$$\ln y = \ln(e^{-x}\cos^2 x) - \ln(x^2+x+1)$$

$$\ln y = \ln(e^{-x}) + \ln(\cos^2 x) - \ln(x^2+x+1)$$

$$\ln y = -x + 2\ln(\cos x) - \ln(x^2+x+1)$$

$$\frac{1}{y}y' = -1 + 2\,\frac{-\sin x}{\cos x} - \frac{2x+1}{x^2+x+1}$$

$$y' = y\left(-1 - 2\,\frac{\sin x}{\cos x} - \frac{2x+1}{x^2+x+1}\right)$$

$$y' = \frac{e^{-x}\cos^2 x}{x^2+x+1}\left(-1 - 2\tan x - \frac{2x+1}{x^2+x+1}\right)$$

41. $y = \sqrt{\dfrac{x-1}{x^4+1}}$

$$\ln y = \ln\left(\frac{x-1}{x^4+1}\right)^{1/2}$$

$$\ln y = 1/2 \ln\left(\frac{x-1}{x^4+1}\right)$$

$$\ln y = \frac{1}{2}\left[\ln(x-1) - \ln(x^4+1)\right]$$

$$\frac{1}{y}y' = \frac{1}{2}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$y' = y\left[1/2\left(\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right)\right]$$

$$y' = \frac{1}{2}\sqrt{\frac{x-1}{x^4+1}}\left(\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right)$$

46. $y = \sqrt{x}^{\,x}$

$\sqrt{x} = x^{1/2}$

$$\ln y = \ln x^{\frac{1}{2}x}$$

$$\ln y = \frac{x}{2} \ln x$$

$$\frac{1}{y} y' = \frac{1}{2} \ln x + \frac{\cancel{x}}{2} \frac{1}{\cancel{x}}$$

$$\frac{1}{y} y' = \frac{1}{2} \ln(x) + \frac{1}{2}$$

$$\frac{1}{y} y' = \frac{1}{2} (\ln(x) + 1)$$

$$y' = y \left[\frac{1}{2} (\ln x + 1) \right]$$

$$y' = \sqrt{x}^{\,x} \left[\frac{1}{2} (\ln(x) + 1) \right]$$

$$y' = \frac{\sqrt{x}^{\,x}}{2} (\ln(x) + 1)$$

47. $y = (\cos x)^x$

$$\ln y = \ln(\cos x)^x$$

$$\ln y = x \ln(\cos x)$$

$$\frac{1}{y} y' = 1 \ln(\cos x) + x \frac{-\sin x}{\cos x}$$

$$y' = y\left(\ln(\cos x) - x \tan x\right)$$

$$y' = (\cos x)^x \left(\ln(\cos x) - x \tan x\right)$$

6. [10 pts.] Find $\dfrac{dy}{dx}$ if $x^y = y^x$.

$$\ln x^{y} = \ln y^{x} \Rightarrow y \ln x = x \ln y$$

$$y' \ln x + \frac{1}{x} y = 1 \ln y + x \frac{1}{y} y'$$

نشتق الطرفين

$$y' \ln(x) - \frac{x}{y} y' = \ln y - \frac{y}{x}$$

$$y' \left(\ln(x) - \frac{x}{y} \right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

3. [10 pts.] Let $y = (\sqrt{x})^{\sin x}$. Find $\dfrac{dy}{dx}$.

$$\boxed{\ln y = \ln(\sqrt{x})^{\sin x} = \sin x \ln \sqrt{x} = \frac{1}{2} \sin x \ln x}.$$

Differentiating implicitly with respect to x gives:

$$\boxed{\frac{y'}{y} = \frac{1}{2}\left(\frac{\sin x}{x} + \cos x \ln x\right).}$$

Therefore, $\boxed{\dfrac{dy}{dx} = \dfrac{(\sqrt{x})^{\sin x}}{2}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)}.$

III. If $f(x) = x^3 e^{x^2-1}$, them $f'(1)$.

✓ a) 5.

b) 0.

c) 1.

d) 2.

e) None of the above.

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Calculus 1 – Related Rates

(Section 3.9)

For Contact and Support:

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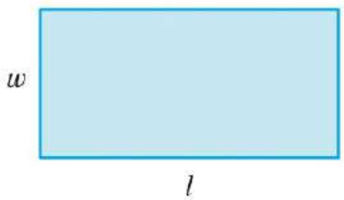
S=Surface Area = A = Area. P = Perimeter = Circumference = C. Volume = V

Rectangle

$A = lw$

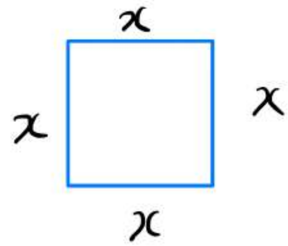
$P = 2l + 2w$



Square

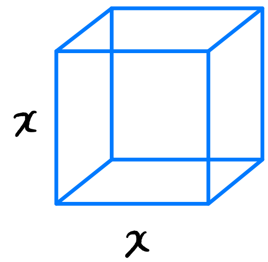
$A = x^2$

$P = 4x$



Cube

$V = x^3$

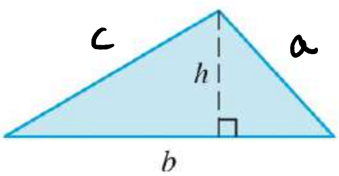


$S = 6x^2$

Triangle

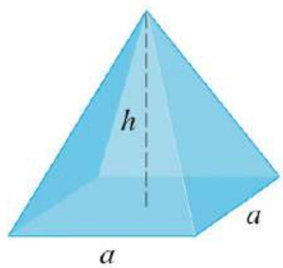
$A = \frac{1}{2}bh$

$P = a+b+c$



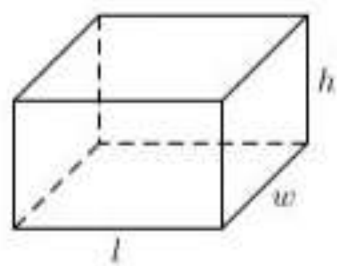
Pyramid

$V = \frac{1}{3}ha^2$



RECTANGULAR SOLID

l = length, w = width,
h = height
Volume: $V = lwh$
Surface Area:
$S = 2lw + 2lh + 2wh$

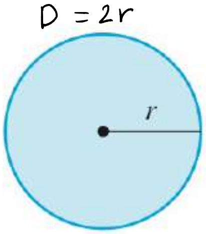


Circle

$A = \pi r^2$

$C = 2\pi r$

$D = 2r$

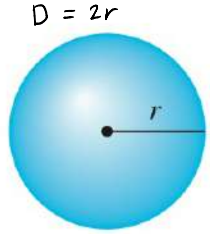


Sphere

$V = \frac{4}{3}\pi r^3$

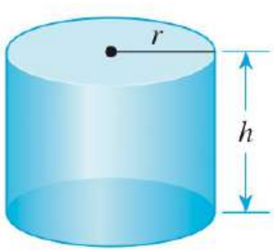
$A = 4\pi r^2$

$D = 2r$



Cylinder

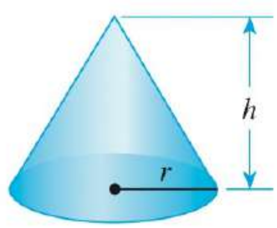
$V = \pi r^2 h$



$S = 2\pi rh + 2\pi r^2$

Cone

$V = \frac{1}{3}\pi r^2 h$



$S = \pi r\sqrt{r^2+h^2}$

Name	unit	الترجمة
Side	m, ft, cm	ضلع
Length	m, ft, cm --	طول
Height	m, ft, cm ...	ارتفاع
Base	m, ft, cm..	قاعدة
Radius	m, ft, cm ..	نصف القطر
Diameter	m, ft, cm ...	قطر
Area	m^2, ft^2, cm^2 ...	مساحة
Surface Area	m^2, ft^2, cm^2 ...	مساحة السطح
Volume	m^3, ft^3, cm^3 ...	الحجم
Rate	تغير الشيء بالنسبة للوقت	معدل
Increasing	+	في ازدياد
Decreasing	—	في تناقص

ex: Area increasing by $3 m^2/s$

$\therefore \frac{dA}{dt} = 3 m^2/s$

إذا شفنا بالسؤال مسافة أو ضلع أو حجم أو نصف القطر
أو ارتفاع أو طول إلخ.. مقسوم على الزمن... يعني هذي
المشتقة لهذا الشي

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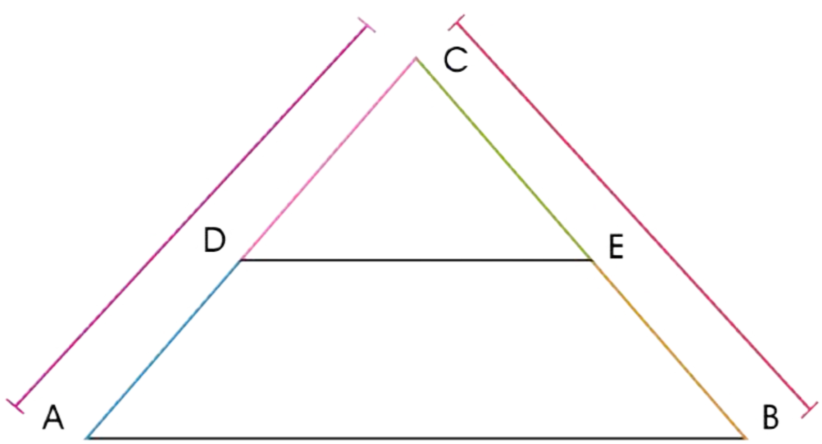
PART II: Related Rates

يعني
المشتقة

Related rates problems can be identified by their request for finding ***how quickly some quantity is changing*** when you are given how quickly another variable is changing. There exist a few classic *types* of related rates problems with which you should familiarize yourself.

1. The Falling Ladder (and other Pythagorean Problems)
2. The Leaky Container
3. The Lamppost and the Shadow
4. The Change in Angle Problem

Triangle similarity theorem



$$\frac{DC}{DA} = \frac{EC}{EB} \quad \text{or} \quad \frac{DC}{AC} = \frac{EC}{BC}$$

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STEPS:

1. As you read the problem pull out essential information & make a diagram if possible.

اكتب المعطيات كلها والمطلوب من السؤال

2. Write down any known rate of change & the rate of change you are looking for, e.g.

$$\frac{dV}{dt} = 3 \quad \& \quad \frac{dr}{dt} = ?$$

3. Be careful with signs...if the amount is decreasing, the rate of change is negative.

4. Pay attention to whether quantities are fixed or varying. For example, if a ladder is 12 meters long you can just call it 12. And if a radius is changing a changing rate, just call it r. You will plug in values for varying quantities at the end.

القانون المناسب الي يربط بين المعطيات والمطلوب

6. Set up an equation involving the appropriate quantities.

7. Differentiate with respect to t using implicit differentiation.

8. Plug in known items (you may need to find some quantities using geometry).

9. Solve for the item you are looking for, most often this will be a rate of change.

10. Express your final answer in a full sentence with units that answers the question asked.

أي كمية يعطيني إياها قاعده تتناقص مع الوقت
رح أحط ماينس جدامه
مثال :
السرعة تتناقص ، تقلص الحجم ، الخ...

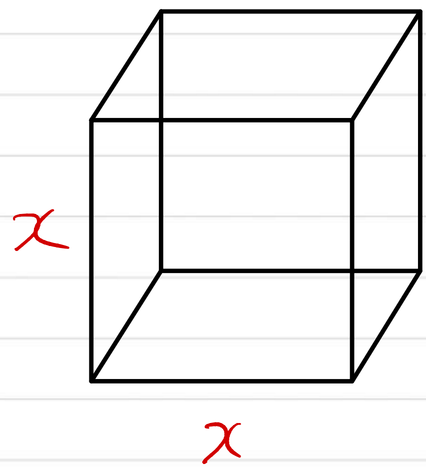
$\frac{d}{dt} = -$

@Precalculusq8

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

Volume of the cube

$$\therefore V(x) = x^3$$

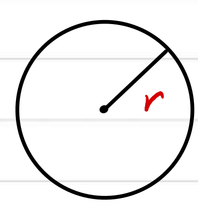


$$\therefore \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

Area of the circle :-



$$A(r) = \pi r^2 \quad , \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

b) $\frac{dr}{dt} = 1 \text{ m/s}$, $\frac{dA}{dt} = ?$, $r = 30$ m

$$\therefore \frac{dA}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi \text{ m}^2/\text{s}$$

3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

given:- square, $\frac{dx}{dt} = 6$ cm/s

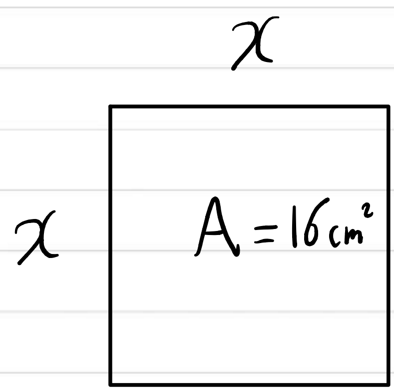
$A = 16$ cm^2

unknown: $\frac{dA}{dt} = ??$

Formula: $A(x) = x^2$

Diff:-

$$\frac{dA}{dt} = 2x \frac{dx}{dt}$$



we need x:-

$\because A = 16$ cm^2 & $A = x^2$

$\therefore 16$ cm$^2 = x^2 \Rightarrow x = 4$ cm

$\ominus$
ما في طول بالماينس

$$\therefore \frac{dA}{dt} = 2(4\,cm)(6\,cm/s) = 48\,cm^2/s$$

3.9 Related Rates

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Given: $\dfrac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Unknown: $\dfrac{dr}{dt}$ when $r = 25$ cm

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

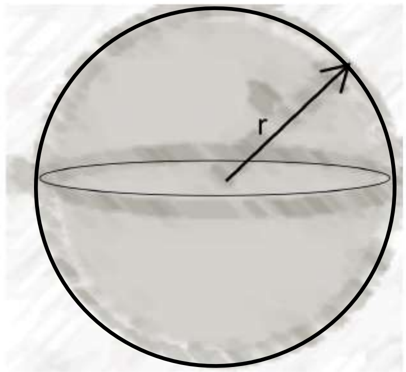
$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s. ■

@Precalculusq8

Example 2: "The Leaky Container"

Gas is escaping from a spherical balloon at the rate of 2 cubic feet per minute. How fast is the surface area shrinking when the radius of the balloon is 12 feet?



SOLUTION:

First, we identify the related rates, that is, the two values that are changing together - the change of volume and the change of the surface area (ΔV and ΔSA respectively) and state the formula for each:

Therefore, beginning with $V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$, we take the derivative of each to obtain the change of rate for each:

So we have: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ (1) and $\frac{d(SA)}{dt} = 8\pi r \frac{dr}{dt}$ (2)

We are given $\frac{dV}{dt}$ and we are looking for $\frac{d(SA)}{dt}$. If we knew the value of $\frac{dr}{dt}$, then we would be done.

So how do we find $\frac{dr}{dt}$? We look at what we are given and what we now need to know. Using equation (1), and the facat that we are given values for the change of volume and the radius, we find that $\frac{dr}{dt} = \frac{1}{288\pi}$. Now, the known information into equation (2), we obtain

$$\frac{d(SA)}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(12)\left(\frac{1}{288\pi}\right) = \frac{1}{3} \text{ ft/min}$$

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EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

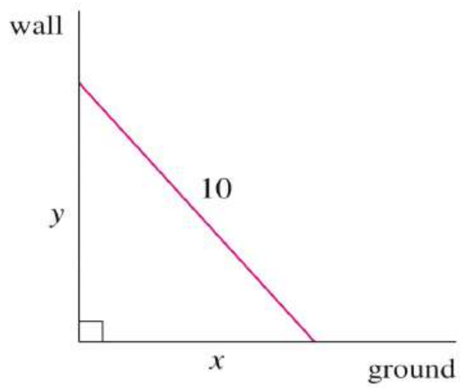


FIGURE 1

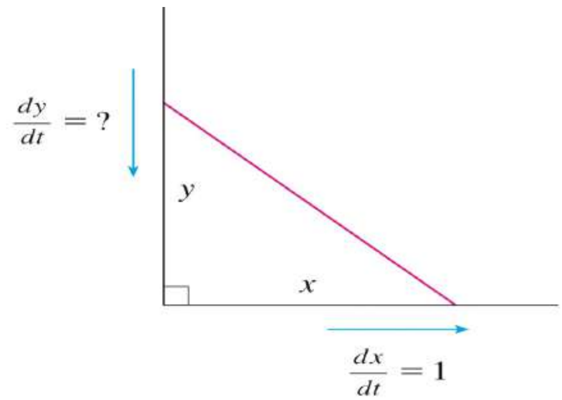


FIGURE 2

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt}$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$, we have

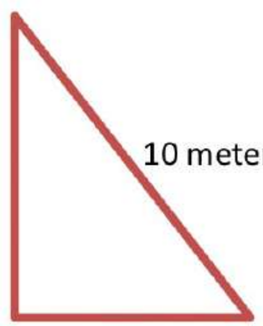
$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}\text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s. ■

Example 1: "The Falling Ladder"

A ladder is sliding down along a vertical wall. If the ladder is 10 meters long and the top is slipping at the constant rate of 10 m/s, how fast is the bottom of the ladder moving along the ground when the bottom is 6 meters from the wall?

SOLUTION:



The relevant equation here is the Pythagorean Theorem:

$a^2 + b^2 = c^2$ Note that the base is x and the height is y $\therefore\ x^2 + y^2 = 10^2$ is our equation.

Implicitly differentiating this yields

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ Plug in all known values.

Pythagorean Theorem.
$6^2 + y^2 = 10^2$ yields y = 8.

عشان نطلع قيمة y ... رح نطلعها باستخدام نظرية فيثاغورث

$$2(6)\frac{dx}{dt} + 2(8)(-10) = 0$$

Hence, $\frac{dx}{dt} = \frac{160}{12} = 13\frac{1}{3}$ m/s

EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

We are given that $dx/dt = -50$ mi/h and $dy/dt = -60$ mi/h. (The derivatives are negative because x and y are decreasing.) We are asked to find dz/dt. The equation that

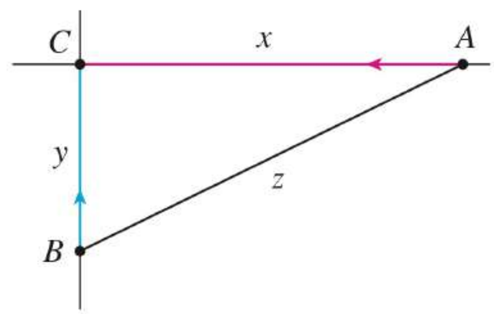


FIGURE 4

relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3$ mi and $y = 0.4$ mi, the Pythagorean Theorem gives $z = 0.5$ mi, so

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{0.5}[0.3(-50) + 0.4(-60)]\\ &= -78 \text{ mi/h}\end{aligned}$$

The cars are approaching each other at a rate of 78 mi/h. ■

EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

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$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$

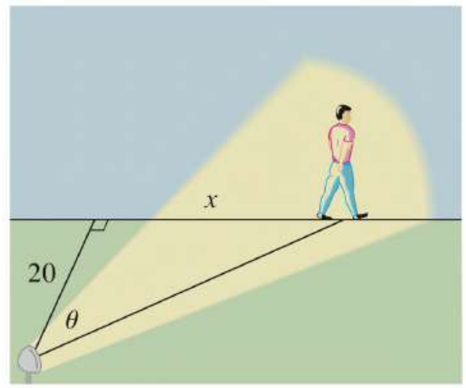


FIGURE 5

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\,\frac{dx}{dt}$$

$$= \frac{1}{20}\cos^2\theta\,(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. ■

@Precalculusq8

4. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

given: rectangle, $\frac{dL}{dt} = 8 \text{ cm/s}$

$\frac{dw}{dt} = 3 \text{ cm/s}$, $L = 20 \text{ cm}$, $w = 10 \text{ cm}$

unknown: $\frac{dA}{dt} = ??$

Formula: $A = Lw$

Diff:-

$$\frac{dA}{dt} = \frac{dL}{dt} w + L \frac{dw}{dt}$$

$$\frac{dA}{dt} = (8 \text{ cm/s})(10 \text{ cm}) + (20 \text{ cm})(3 \text{ cm/s})$$

$$\frac{dA}{dt} = 80 \text{ cm}^2/\text{s} + 60 \text{ cm}^2/\text{s}$$

$$= 140 \text{ cm}^2/\text{s}$$

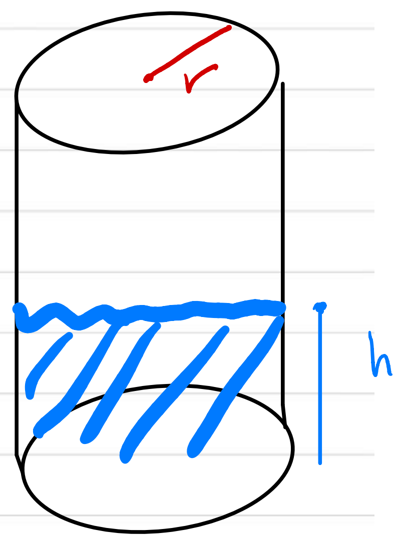
5. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

given: cylindrical, $r = 5m$, $\frac{dV}{dt} = 3 m^3/min$

unknown: $\frac{dh}{dt} = ??$

formula:-

$V = \pi r^2 h = \pi (5m)^2 h = 25\pi h$



Diff:-

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

$$3 = 25\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{25\pi} \ m/min$$

6. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

---

given = sphere , $\frac{dr}{dt} = 4\ mm/s$

$D = 2r = 80\,mm \therefore\ r = \frac{80\,mm}{2} = 40\ mm$

unknown : $\frac{dV}{dt} = ??$

formula :-

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)\frac{dr}{dt}$$

$$\frac{dv}{dt} = 4\pi(40\,mm)^2(4\,mm/s)$$

$$\frac{dv}{dt} = 4\pi(1600\,mm^2)(4\,mm/s)$$

$$= 16\,mm/s \cdot 1600\,mm^2 = 25600\pi\ mm^3/s$$

$$\begin{array}{r} 1600 \\ \times\ 16 \\ \hline 9600 \\ 16000+ \\ \hline 25600 \end{array}$$

14. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

---

given: snowball "spherical", $\frac{dA}{dt} = -1 \text{ cm}^2/\text{min}$

$D = 2r = 10 \text{ cm} \quad \therefore r = 5 \text{ cm}$

unknown: $\frac{dD}{dt} = \frac{d(2r)}{dt} = 2\frac{dr}{dt}$

formula:-

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$-1 = 8\pi 5 \frac{dr}{dt}$$

$$\Rightarrow \frac{-1}{40\pi} = \frac{dr}{dt}$$

$$\therefore 2\frac{dr}{dt} = 2\left(-\frac{1}{40\pi}\right) = -\frac{1}{20\pi}$$

9. Suppose $y = \sqrt{2x + 1}$, where x and y are functions of t.
(a) If $dx/dt = 3$, find dy/dt when $x = 4$.
(b) If $dy/dt = 5$, find dx/dt when $x = 12$.

$$\frac{dy}{dt} = \frac{1}{\cancel{2}\sqrt{2x+1}} \cancel{2} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{2x+1}} \frac{dx}{dt} \longrightarrow (1)$$

$\because \frac{dx}{dt} = 3$, $x = 4$

$$\therefore \frac{dx}{dt} = \frac{1}{\sqrt{2(4)+1}} (3) = \frac{3}{\sqrt{9}} = \frac{3}{3}$$

$$= 1$$

---

when $\frac{dy}{dt} = 5$ & $x = 12$, sub in (1)

$$5 = \frac{1}{\sqrt{2(12)+1}} \cdot \left(\frac{dx}{dt}\right) \Rightarrow 5 = \frac{1}{\sqrt{25}} \cdot \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = 5 * 5 = 25$$

10. Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t.

(a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.

(b) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.

---

a) $\dfrac{dx}{dt} = -\dfrac{\sqrt{5}}{4}$

b) $\dfrac{4\sqrt{5}}{5}$

11. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.

Diff. both side with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0$$

$$2\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) = 0$$

$$\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) = 0$$

$$2(5) + 2(4) + (1)\frac{dz}{dt} = 0$$

$$10 + 8 = -\frac{dz}{dt}$$

$$\frac{dz}{dt} = -18$$

3. [10 pts.] The height of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the height is 10 cm and the area is 100 cm^2?

Let A, b and h be the area, base and height of the triangle at time t.

We are given that $h' = 1$ cm/min and $A' = 2$ cm^2/min. We need to find b' when $A = 100$ cm^2 and $h = 10$ cm.

We have $\boxed{A = \frac{1}{2}(bh)}$. Differentiating wrt time t we get: $\boxed{A' = \frac{1}{2}(b'h + h'b)}$.

Therefore, $\boxed{b' = \dfrac{2A' - h'b}{h}}$.

When $A = 100$ and $h = 10$ we get $\boxed{b = \dfrac{2A}{h} = 2(100)/10 = 20 \text{ cm}}$.

Thus, $\boxed{b' = \dfrac{2A' - h'b}{h} = (4 - 20)/10 = -1.6 \text{ cm/min}}$.

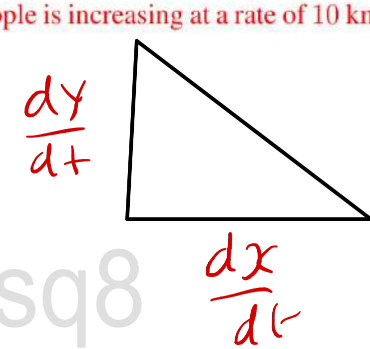
The base of the rectangle is decreasing at a rate of 1.6 cm/min.

2. [10 pts.] Two people start running from the same point P. One runs east at 6 km/h and the other runs north at 8 km/h. How fast is the distance between the two people changing after 0.5 hour?

2. [10 pts.] At a given time t, let x be the distance from one person (the one running east) to P, let y be the distance from the other person (the one running north) to P, and let z be the distance between the two people, where x, y, and z are measured in km. We are given that $x' = \dfrac{dx}{dt} = 6$ km/h and $y' = \dfrac{dy}{dt} = 8$ km/h. We are asked to find $z' = \dfrac{dz}{dt}$. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$\boxed{z^2 = x^2 + y^2}.$$

Differentiating each side with respect to t, we get $\boxed{2zz' = 2xx' + 2yy'}$. Thus, $\boxed{z' = \frac{1}{z}(xx' + yy')}$. When $x = 3$ km and $y = 4$ km (after 0.5 hour), the Pythagorean Theorem gives $z = 5$ km, so $\boxed{z' = \frac{1}{5}(3(6) + 4(8)) = 10 \text{ km/h}}$. Thus, the distance between the two people is increasing at a rate of 10 km/h.



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3. [10 pts.] A rectangle has a fixed area of 24 cm^2. The length is decreasing at a rate of 2 cm/min. At what rate the width is increasing when the length is 8 cm.

$A = 24\, cm^2$, $\frac{dL}{dt} = -2\, cm/min$, $L = 8\, cm$

$\frac{dw}{dt} = ??$

$L = \text{Length}$

$w = \text{width}$

$\because A = Lw \longrightarrow (1)$

$\because \frac{dA}{dt}^{?} = w^{?} \frac{dL}{dt} + L \frac{dw}{dt}^{?} \longrightarrow (2)$

$*$ to get "w"

$24 = 8w \Rightarrow w = 3$

$\because$ fixed Area = the area does not change with time

$\therefore \frac{dA}{dt} = 0$, now sub $w = 3$, $\frac{dA}{dt} = 0$ in equ (2)

$$0 = 3(-2) + 8\left(\frac{dw}{dt}\right)$$

$$6 = 8\frac{dw}{dt} \qquad \therefore \frac{dw}{dt} = \frac{6}{8} = \frac{3}{4}\, cm/min$$

3. [10 pts.] A cylindrical tank of diameter 6 m is being filled with water at a rate of 2 m^3/hr, how fast is the water level rising?

3. [10 pts] We have $\boxed{V = \pi r^2 h}$. This implies that $\boxed{\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}}$. Therefore, $\boxed{\frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt}}$.

Hence, $\boxed{\frac{dh}{dt} = \frac{2}{9\pi} \text{ m/hr}}$.

3. [10 pts.] A ladder 10 m long rests on horizontal ground and leans against a vertical wall. If the top of the ladder is being pulled up the wall at a rate of 1 m/min, how fast is the bottom of the ladder approaching the wall when the bottom of the ladder is 8 m away from the wall?

We have $\boxed{x^2 + y^2 = 100, \text{ so } 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0}$. $\boxed{\text{Now when } x = 8, \text{ we get } y = 6}$. This implies that $\boxed{\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt} = \frac{-3}{4}(1) = -0.75 \text{ m/min}}$.

5. [10 pts.] The radius of a circle is decreasing at a rate of 2 cm/min. At what rate is the area of the circle changing when the area is 25π cm^2?

The area of a circle is: $\boxed{A = \pi r^2}$. Therefore, $\boxed{\frac{dA}{dt} = 2\pi r\frac{dr}{dt}}$.

When $A = 25\pi$ cm^2, $r = 5$ cm. Hence, $\boxed{\left.\frac{dA}{dt}\right|_{r=5} = (2\pi)(5)(-2) = -20\pi \text{ cm}^2/\text{min}}$.

$A = \pi r^2 = 25\pi$

$\therefore r^2 = 25 \Rightarrow r = 5$

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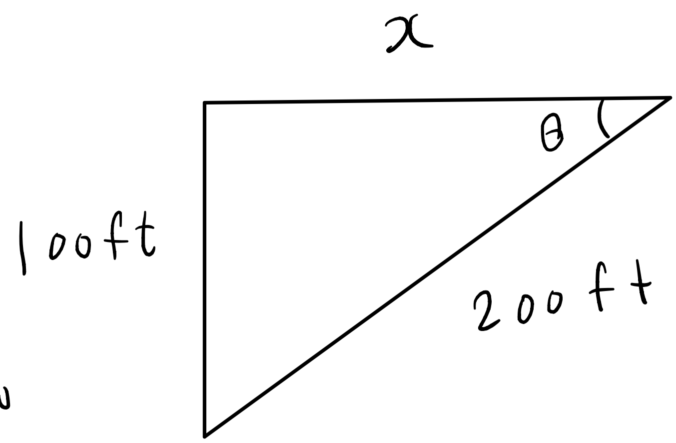
6. [10 pts.] Air is being bumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon is increasing when the radius is 25 cm?

We have $V = \frac{4}{3}\pi r^3$. Therefore, we obtain $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. In particular, $100 = 4\pi(25)^2\frac{dr}{dt}$. This implies that $\frac{dr}{dt} = \frac{1}{25\pi} cm/s$.

طيارة ورقية

30. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



يصير

$$\tan\theta = \frac{100}{x}$$

شمعنى cot؟

$$\cot\theta = \frac{x}{100}$$

$$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dx}{dt}$$

$$-\frac{1}{\sin^2(\frac{\pi}{6})}\frac{d\theta}{dt} = \frac{1}{100} * 8$$

$$-\frac{1}{(\frac{1}{2})^2}\frac{d\theta}{dt} = \frac{8}{100} \Rightarrow \frac{d\theta}{dt} = \frac{8}{100} * -\frac{1}{4} = \frac{-2}{100} \text{ rad/s}$$

0.02 rad/s

We can get "θ" by:

$$\sin^{-1}\theta = \frac{10\cancel{0}}{20\cancel{0}}$$

$$\theta = \frac{\pi}{6}$$

@Precalculusq8

Thamrah

Kuwait University

Calculus 1 – Liner Approximation and Differential

(Section 3.10)

For Contact and Support:

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أنواع الأسئلة في سكشن 3.10

1) Find the linearization

$$L(x) = f(a) + f'(a)(x - a)$$

2) Use a linear approximation (or differentials) to estimate

$$f(x) \approx f(a) + f'(a)(x - a)$$

يعني تقريبا

ex: $\sqrt{0.9}$, $x = 0.9$ $\therefore a = 1$

$e^{0.1}$, $x = 0.1$ $\therefore a = 0$

$\sqrt{99.8}$, $x = 99.8$ $\therefore a = 100$

إذا مو عطيك قيمة الـa بالسؤال .. أنت رح تييب قيمة الـa الي هي رح تكون العدد الاقرب للسؤال

شلون تختار a

١- تكون قيمة قريبة من الـx

٢- تكون عارف قيمة الـf(a)

3) Find the differential of each function.

$$dy = f'(x)\,dx \qquad \Delta y = f(x + \Delta x) - f(x)$$

3.10 Linear Approximations and Differentials

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

2
$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \tfrac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (1) is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

In particular, we have

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1–4 Find the linearization $L(x)$ of the function at a.

1. $f(x) = x^3 - x^2 + 3, \quad a = -2$

$L(x) = f(a) + f'(a)(x-a)$

$a = -2$

$f(a) = f(-2) = (-2)^3 - (-2)^2 + 3 = -8 - 4 + 3 = -9$

we need to find $f'(x)$ then $f'(-2)$

$f'(x) = 3x^2 - 2x$

$f'(-2) = 3(-2)^2 - 2(-2) = 12 + 4 = 16$

$L(x) = (-9) + (16)(x-(-2))$

$= -9 + 16x + 32$

$L(x) = 16x + 23$

2. $f(x) = \sin x, \quad a = \pi/6$

$$L(x) = f(a) + f'(a)(x-a)$$

$$a = \frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$f'(x) = \cos x, \quad f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$

3. $f(x) = \sqrt{x}, \quad a = 4$

$$L(x) = 2 + \tfrac{1}{4}(x-4)$$

$$= 2 + \tfrac{1}{4}x - 1 = \tfrac{1}{4}x + 1$$

11–14 Find the differential of each function

11. (a) $y = xe^{-4x}$

$$y' = 1e^{-4x} + (-4)(e^{-4x})x$$

$$y' = e^{-4x} - 4x\, e^{-4x}$$

$$\frac{dy}{dx} = e^{-4x}(1 - 4x)$$

$$\boxed{y' = \frac{dy}{dx}}$$

$$dy = e^{-4x}(1-4x)\,dx$$

(b) $y = \sqrt{1 - t^4}$

$$dy = -\frac{2t^3}{\sqrt{1-t^4}}\,dt$$

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11–14 Find the differential of each function

12. (a) $y = \dfrac{1+2u}{1+3u}$ (b) $y = \theta^2 \sin 2\theta$

$$\frac{dy}{du} = \frac{(2)(1+3u) - [(3)(1+2u)]}{(1+3u)^2}$$

$$\frac{dy}{du} = \frac{2+6u-3-6u}{(1+3u)^2}$$

$$\frac{dy}{du} = \frac{-1}{(1+3u)^2}$$

$$dy = \frac{-1}{(1+3u)^2}\,du$$

(b) $y = \theta^2 \sin 2\theta$

$$\frac{dy}{d\theta} = 2\theta \sin 2\theta + \theta^2 \cos(2\theta)\,(2)$$

$$dy = [2\theta(\sin 2\theta + \theta\cos(2\theta))]\,d\theta$$

11–14 Find the differential of each function

13. (a) $y = \tan \sqrt{t}$

$$\frac{dy}{dt} = \sec^2(\sqrt{t}) \cdot \left(\frac{1}{2\sqrt{t}}\right)$$

$$dy = \frac{\sec^2(\sqrt{t})}{2\sqrt{t}}\, dt$$

(b) $y = \dfrac{1 - v^2}{1 + v^2}$

$$dy = \frac{-4v}{(1+v^2)^2}\, dv$$

11–14 Find the differential of each function

14. (a) $y = \ln(\sin\theta)$ (b) $y = \dfrac{e^x}{1 - e^x}$

$$\frac{dy}{d\theta} = \frac{\cos\theta}{\sin\theta}$$

$$dy = \cot\theta \; d\theta$$

(b) $y = \dfrac{e^x}{1 - e^x}$

$$\frac{dy}{dx} = \frac{e^x(1-e^x) - (-e^x)e^x}{(1-e^x)^2}$$

$$\frac{dy}{dx} = \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2}$$

$$dy = \frac{e^x}{(1-e^x)^2} \; dx$$

15–18 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

18. $y = \dfrac{x+1}{x-1}, \quad x = 2, \quad dx = 0.05$

---

a) $$\frac{dy}{dx} = \frac{1(x-1) - (1)(x+1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{x-1-x-1}{(x-1)^2}$$

$$dy = \frac{-2}{(x-1)^2}\,dx$$

b) $$dy = \frac{-2}{(2-1)^2}(0.05)$$

$$dy = \frac{-2}{1}(0.05)$$

$$dy = -2(0.05) = -0.1$$

19–22 Compute Δy and dy for the given values of x and $dx = \Delta x$.

21. $y = \sqrt{x-2}, \quad x = 3, \quad \Delta x = 0.8$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$f(x+\Delta x) = \sqrt{x + \Delta x - 2}$$

$$f(x) = \sqrt{x-2}$$

sub $x = 3$, $\Delta x = 0.8$, we get

$$\Delta y = \sqrt{3 + 0.8 - 2} - \sqrt{3-2}$$

$$\Delta y = \sqrt{1.8} - \sqrt{1} = \sqrt{1.8} - 1$$

$$dy = \frac{1}{2\sqrt{x-2}}\,dx = \frac{1}{2\sqrt{3-2}} * 0.8$$

$$= \frac{1}{\cancel{2}} \cdot 0.\cancel{8}^{4} = 0.4$$

23–28 Use a linear approximation (or differentials) to estimate the given number.

23. $(1.999)^4$

$f(x) \simeq f(a) + f'(a)(x-a)$

Let $x = 1.999$

$a = 2$

$\therefore (x-a) = 1.999 - 2 = -0.001$

$f(x) = x^4$ $\quad f'(x) = 4x^3$

$f(2) = 2^4 = 16$, $f'(2) = 4(2)^3 = 32$

$\therefore (1.999)^4 \simeq 16 + (32)(-0.001)$

$\therefore (1.999)^4 \simeq 16 - (0.032) = 15.968$

$$\begin{array}{r} \overset{15}{\cancel{16}}.\overset{9}{\cancel{0}}\overset{9}{\cancel{0}}\overset{10}{\cancel{0}} \\ -\ 0.032 \\ \hline 15.968 \end{array}$$

23–28 Use a linear approximation (or differentials) to estimate the given number.

27. $e^{0.1}$

$f(x) \simeq f(a) + f'(a)(x-a)$

Let $x = 0.1$

$a = 0$

$\therefore (x-a) = 0.1 - 0 = 0.1$

$f(x) = e^{x}$ $\quad f'(x) = e^{x}$

$f(0) = e^{0} = 1$, $f'(0) = e^{0} = 1$

$\therefore e^{0.1} \simeq 1 + 1(0.1) = 1 + 0.1$

$e^{0.1} \simeq 1.1$

23–28 Use a linear approximation (or differentials) to estimate the given number.

28. $\cos 29^\circ$

$$f(x) \simeq f(a) + f'(a)(x-a)$$

$$x = 29^\circ \quad , \quad a = 30^\circ$$

$$(x-a) = 29^\circ - 30^\circ = -1^\circ * \frac{\pi}{180} = \frac{-\pi}{180}$$

$$f(x) = \cos x \; , \; f(30^\circ) = \cos 30^\circ = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$f'(x) = -\sin x \; , \; f(30^\circ) = -\sin 30^\circ = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

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$$\therefore \cos 29^\circ \simeq \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\left(\frac{-\pi}{180}\right)$$

$$\therefore \cos 29^\circ \simeq \frac{\sqrt{3}}{2} + \frac{\pi}{360}$$

1) find the linear approximation

$$\sin(46^\circ)$$

$$f(x) \simeq f(a) + f'(a)(x-a)$$

$$x = 46^\circ \quad , \quad a = 45^\circ$$

$$(x-a) = 46^\circ - 45^\circ = 1^\circ * \frac{\pi}{180} = \frac{\pi}{180}$$

$$f(x) = \sin x \,, \quad f(45^\circ) = \sin 45^\circ = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(x) = \cos x \,, \quad f(45^\circ) = \cos 45^\circ = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \sin 46^\circ \simeq \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\pi}{180}\right)$$

$$\therefore \sin 46^\circ \simeq \frac{1}{\sqrt{2}} + \left(\frac{\pi}{180\sqrt{2}}\right)$$

2) estimate $\sqrt{99.8}$

$$f(x) \simeq f(a) + f'(a)(x-a)$$

$x = 99.8$, $a = 100$

$(x-a) = 99.8 - 100 = -0.2$

$f(x) = \sqrt{x}$, $f(100) = \sqrt{100} = 10$

$f'(x) = \dfrac{1}{2\sqrt{x}}$, $f'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}$

$\therefore \sqrt{99.8} \simeq 10 + \left(\frac{1}{20}\right)(-0.2)$

$\therefore \sqrt{99.8} \simeq 10 + \left(\frac{1}{\cancel{20}_{10}}\right)\left(-\frac{\cancel{2}}{10}\right)$

$\therefore \sqrt{99.8} \simeq 10 - \frac{1}{100} = \frac{1000}{100} - \frac{1}{100}$

$\simeq \frac{999}{100} \simeq 9.99$

Use linear approximation to estimate

$$\ln(1.013)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$x = 1.013 \quad , \quad a = 1$$

$$(x-a) = 1.013 - 1 = 0.013$$

$$f(x) = \ln x \quad , \quad f(1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x} \quad , \quad f'(1) = \frac{1}{1} = 1$$

$$\therefore \ln(1.013) \approx 0 + 1\,(0.013)$$

$$\therefore \ln(1.013) \approx 0.013$$

7. [10 pts.] Use **linear approximation** to estimate $2e^{0.1} + (0.1)^2$.

We let $y = f(x) = 2e^x + x^2$. Also, we let $a = 0$ and $\Delta x = 0.1$.

It is clear that $f'(x) = 2e^x + 2x$. According to differentials or linear approximation,

we obtain $f(0.1) = 2e^{0.1} + (0.1)^2 \approx f(0) + f'(0)\Delta x = 2 + 2(0.1) = 2.2$.

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4. [10 pts.] Let $f(x) = 3x^2 + 4x - \ln(\cos x)$.

(a) Find the linearization of f at $x = 0$.

(b) Estimate $f(0.01)$.

$$L(x) = f(a) + f'(a)\,(x-a)$$

$$f(x) = 3x^2 + 4x - \ln(\cos x)$$

$$f'(x) = 6x + 4 - \frac{-\sin x}{\cos x}$$

$$f'(x) = 6x + 4 + \frac{\sin x}{\cos x}$$ or $\tan x$

$$f(a) = f(0) = 3(0)^2 + 4(0) - \ln\cos 0$$

$$= 0 + 0 - \ln 1 = 0$$

$$f'(a) = f'(0) = 6(0) + 4 + \tan 0$$

$$= 0 + 4 + 0 = 4$$

$$\therefore L(x) = 0 + (4)(x-0)$$

$$= 0 + 4x = 4x$$

b) $f(0.01) \simeq 4(0.01) = 0.04$

6. [10 pts.] Let $f(x) = \dfrac{x-1}{x^2-3}$

(a) Find the linearization of f at $x = 1$.

(b) Use part (a) to estimate $f(1.02)$.

We have $\boxed{f'(x) = \dfrac{(x^2-3)-(2x)(x-1)}{(x^2-3)^2} = \dfrac{-x^2+2x-3}{(x^2-3)^2}}$. So $\boxed{f'(1) = \dfrac{-1}{2}}$.

Therefore, $\boxed{L(x) = 0 + \dfrac{-1}{2}(x-1) = -\dfrac{1}{2}(x-1)}$.

Now $\boxed{f(1.02) \approx -\dfrac{1}{2}(1.02-1) = -0.01}$.

---

4. [10 pts.] Use linear approximation to estimate $f(2.1)$, where $f(x) = \sqrt{x^3+1}$.

$\boxed{f(x) \approx f(2) + f'(2)(x-2)}$ when x is near 2.

We have $\boxed{f(2) = 3}$, $\boxed{f'(x) = \dfrac{3x^2}{2\sqrt{x^3+1}}}$ and $\boxed{f'(2) = 2}$.

Thus, $\boxed{f(2.1) \approx 3 + 2(2.1-2) = 3.2}$.

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5. [10 pts.] Find the linearization of the function $f(x) = x \ln x$ at $a = 1$ and use it to approximate $f(1.1)$.

5. [10 pts.] We have: $\boxed{L(x) = f(1) + f'(1)(x-1) = 0 + f'(1)(x-1)}$.

Since $f'(x) = x(1/x) + \ln x = 1 + \ln x$ and hence $f'(1) = 1$, we get $\boxed{L(x) = x - 1}$.

Thus, $\boxed{f(x) \approx x - 1}$, when x is near 1.

Hence, $\boxed{f(1.1) \approx 1.1 - 1 = 0.1}$.

3. [10 pts.] Let $f(x) = \tan^{-1}(x) - \dfrac{\pi}{4}$. Use linear approximation to estimate $f(1.05)$.

3. [10 pts.] $\boxed{f(x) \approx f(1) + f'(1)(x-1)}$, when x is near 1.

We have $\boxed{f'(x) = \dfrac{1}{x^2+1}}$. Since $\boxed{f(1) = \tan^{-1}(1) - \dfrac{\pi}{4} = 0}$ and $\boxed{f'(1) = 1/2}$,

therefore, $\boxed{f(x) \approx \dfrac{1}{2}(x-1)}$, when x is near 1. Thus, $\boxed{f(1.05) \approx \dfrac{1}{2}(1.05 - 1) = 0.025}$.

4. [10 pts.] Let $f(x) = e^x \cos x$. Use linear approximation to estimate $f(0.1)$.

4. [10 pts] We have $\boxed{f'(x) = e^x \cos x - e^x \sin x}$. Also, we have $\boxed{f(0) = 1}$ and $\boxed{f'(0) = 1}$.

Therefore, $\boxed{f(0.1) \simeq f(0) + f'(0)(0.1 - 0) = 1 + 0.1 = 1.1}$.

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Thamrah

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Calculus 1 – Hyperbolic Functions

(Section 3.11)

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3.11 Hyperbolic Functions

مهم

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

انتبه هذي مو المشتقة ..

هذيل بينهم يساوي!!!!!

مهم

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

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EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$
$$= \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

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$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

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1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\left(\cosh\sqrt{x}\right) = \sinh\sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

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1–6 Find the numerical value of each expression.

1. (a) sinh 0

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\therefore \sinh 0 = \frac{e^{0} - e^{-0}}{2} = \frac{1-1}{2} = 0$$

(b) cosh 0

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\cosh 0 = \frac{e^{0} + e^{-0}}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$$

(a) sech 0 $= \dfrac{1}{\cosh 0} = \dfrac{1}{\frac{e^{0}+e^{-0}}{2}} = \dfrac{2}{e^{0}+e^{-0}}$

$$= \frac{2}{1+1} = \frac{2}{2} = 1$$

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1–6 Find the numerical value of each expression.

2. (a) tanh 0

$$\tanh(x) = \frac{\sinh x}{\cosh x} = \frac{\frac{e^{x} - e^{-x}}{2}}{\frac{e^{x} + e^{-x}}{2}}$$

$$\therefore \tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\therefore \tanh 0 = \frac{e^{0} - e^{-0}}{e^{0} + e^{-0}} = \frac{1-1}{1+1} = 0$$

(b) tanh 1

$$\tanh(1) = \frac{e^{1} - e^{-1}}{e^{1} + e^{-1}} = \frac{e - \frac{1}{e}}{e + \frac{1}{e}}$$

نوحد مقامات

$$= \frac{\frac{e^{2}-1}{\cancel{e}}}{\frac{e^{2}+1}{\cancel{e}}} = \frac{e^{2}-1}{e^{2}+1}$$

حتى لو سويت factor حق البسط مرح تقدر تختصر مع المقام لانه المقام e تربيع ما تشابه الfactor الي رح تسويه

$$(e^{2}-1) = (e+1)(e-1)$$

1–6 Find the numerical value of each expression.

3. (a) cosh(ln 5) (b) cosh 5

$$\cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2} = \frac{e^{\ln 5} + e^{\ln 5^{-1}}}{2}$$

$$= \frac{5 + \frac{1}{5}}{2} = \frac{\frac{25+1}{5}}{2}$$

$$= \frac{26}{10}$$

(b) cosh 5

$$\cosh 5 = \frac{e^{5} + e^{-5}}{2} = \frac{e^{5} + \frac{1}{e^{5}}}{2}$$

$$= \frac{\frac{e^{10}+1}{e^{5}}}{2} = \frac{e^{10}+1}{2e^{5}}$$

1–6 Find the numerical value of each expression.

4. (a) sinh 4 (b) sinh(ln 4)

a) $\sinh 4 = \dfrac{e^{4} - e^{-4}}{2}$

$$\sinh 4 = \frac{e^{4} - \frac{1}{e^{4}}}{2} = \frac{\frac{e^{8} - 1}{e^{4}}}{2}$$

$$\sinh 4 = \frac{e^{8} - 1}{2e^{4}}$$

b) $\sinh(\ln 4) = \dfrac{15}{8}$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2}$$

$$= \frac{e^{-x} - e^{x}}{2} = -\left(\frac{-e^{-x} + e^{x}}{2}\right)$$

بس رتبتها

$$= -\left(\frac{e^{x} - e^{-x}}{2}\right) = -\sinh(x)$$

8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh(x)$$

7–19 Prove the identity.

9. $\cosh x + \sinh x = e^x$

$$\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

$$= \frac{e^x + e^{-x} + e^x - e^{-x}}{2}$$

$$= \frac{2e^x}{2} = e^x$$

10. $\cosh x - \sinh x = e^{-x}$

$$\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}$$

$$\frac{e^x + e^{-x} - e^x + e^{-x}}{2} = \frac{2e^{-x}}{2} = e^{-x}$$

7–19 Prove the identity.

11. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$$\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)+\left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)$$

$$\frac{e^{x+y}\ e^{x-y}-e^{-x+y}-e^{-x-y}}{4}+\cdots$$

$$\frac{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}}{4}$$

$$=\frac{2e^{x+y}-2e^{-x-y}}{4}=\frac{\cancel{2}(e^{x+y}-e^{x-y})}{\cancel{4}\ 2}$$

$$=\frac{e^{x+y}-e^{-(x+y)}}{2}=\sinh(x+y)$$

30–45 Find the derivative. Simplify where possible.

30. $f(x) = e^x \cosh x$

$$f'(x) = e^x \cosh x + e^x \sinh x$$

$$= e^x (\cosh x + \sinh x)$$

$\because \sinh x + \cosh x = e^x$

$\therefore \; = e^x (e^x) = e^{x+x} = e^{2x}$

31. $f(x) = \tanh \sqrt{x}$

$$f'(x) = \sec^2 h \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec^2 h(\sqrt{x})}{2\sqrt{x}}$$

30–45 Find the derivative. Simplify where possible.

32. $g(x) = \sinh^2 x$

$\sin^2 h x = (\sinh x)^2$

$g'(x) = 2\sinh(x) \cdot \cosh x$

$\because \sinh 2x = 2\sinh x \cosh x$

$g'(x) = \sinh 2x$

33. $h(x) = \sinh(x^2)$

$h'(x) = \cosh(x^2) \cdot 2x$

$= 2x\cosh(x^2)$

35. $G(t) = \sinh(\ln t)$

$$G'(t) = \cosh(\ln t) \cdot \frac{1}{t}$$

$$G'(t) = \frac{e^{\ln t} + e^{-\ln t}}{2} \cdot \frac{1}{t}$$

$$G'(t) = \frac{t + \frac{1}{t}}{2} \cdot \frac{1}{t}$$

$$= \frac{\frac{t^2+1}{t}}{2} \cdot \frac{1}{t}$$

$$= \frac{t^2+1}{2t} \cdot \frac{1}{t}$$

$$= \frac{t^2+1}{2t^2}$$

30–45 Find the derivative. Simplify where possible.

37. $y = e^{\cosh 3x}$

$$y' = e^{\cosh 3x} \cdot \sinh(3x) \cdot 3$$

$$y' = 3 \sinh(3x)\, e^{\cosh 3x}$$

III. $\lim\limits_{x\to\infty} \dfrac{\sinh(x)}{8e^x} =$

(A) ∞.

(B) $\frac{1}{4}$.

(C) 0.

(D) $\boxed{\frac{1}{16}}$.

(E) None of the above.

---

cannot use sandwich theorem on sinhx coshx

$$\lim_{x\to\infty} \frac{\frac{e^x-e^{-x}}{2}}{8e^x} = \lim_{x\to\infty} \frac{e^x-e^{-x}}{16e^x} = \frac{\infty-0}{\infty} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{e^x}{e^x}-\frac{e^{-x}}{e^x}}{\frac{16e^x}{e^x}} = \frac{1-e^{-x-x}}{16} = \frac{1-e^{-2x}}{16}$$

$$\therefore \lim_{x\to\infty} \frac{1-e^{-2x}}{16} = \frac{1-e^{-\infty}}{16} = \frac{1-0}{16} = \frac{1}{16}$$

IV. $\lim_{x \to 0} \frac{\cosh x - 1}{x} =$

a) $\boxed{0}$. ✓

b) 1.

c) e.

d) None of the above.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\sinh(0) = \lim_{x \to 0} \frac{\cosh x - \cosh 0}{x - 0}$$

$$0 = \lim_{x \to 0} \frac{\cosh x - 1}{x}$$

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1. [10 pts.] Let $f(x) = x\sinh(x^2 - 1)$. Find $f'(1)$.

Answer. We have $f'(x) = \sinh(x^2-1) + x\left(2x\cosh(x^2-1)\right)$.

Therefore, $f'(1) = 2\cosh 0 = 2$.

2. [10 pts.] Find the slope of the tangent line at the point $(0,0)$ to the curve with equation

$$e^{xy} + \sin(x^2) = \cosh(x) + y.$$

Differentiating implicitly wrt x we get: $\boxed{e^{xy}(y + xy') + 2x\cos(x^2) = \sinh(x) + y'}$.

At the point $(0,0)$ $\boxed{y' = 0}$.

IV. The derivative of $\cosh^2 x - \sinh^2 x$ is

(A) $\boxed{0}$.

(B) $4\cosh x \sinh x$.

(C) $-2\cosh x \sinh x$.

(D) $2\cosh x \sinh x$.

(E) None of the above.

$2\cosh x\,(\sinh x) - 2\sinh x\,(\cosh x)$

$= 0$

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II. The value of $\lim\limits_{x\to\infty} \tanh x$ is

a) does not exist.

b) 0.

c) $-\pi^2$.

$\boxed{\text{d)}}$ 1.

e) None of the above.

$$\tanh(x) = \frac{\sinh x}{\cosh x} = \frac{\frac{e^{x}-e^{-x}}{2}}{\frac{e^{x}+e^{-x}}{2}}$$

$$\therefore \tanh(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\lim_{x\to\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{\frac{e^{x}}{e^{x}}-\frac{e^{-x}}{e^{x}}}{\frac{e^{x}}{e^{x}}+\frac{e^{-x}}{e^{x}}}$$

$$\lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1-e^{-\infty}}{1+e^{-\infty}} = \frac{1-0}{1+0} = 1$$

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I. If $x = \ln\left(\frac{5}{2}\right)$, then $\cosh(x) =$

a) $\frac{29}{20}$.

b) $\frac{21}{20}$.

c) $\frac{20}{21}$.

d) 0.

e) None of the above.

$$\cosh\left(\ln \frac{5}{2}\right) = \frac{e^{\ln \frac{5}{2}} + e^{-\ln \frac{5}{2}}}{2}$$

$$= \frac{e^{\ln \frac{5}{2}} + e^{\ln\left(\frac{5}{2}\right)^{-1}}}{2} = \frac{\frac{5}{2} + \frac{2}{5}}{2}$$

$$= \frac{\frac{25+4}{10}}{2} = \frac{\frac{29}{10}}{2} = \frac{29}{20}$$

Kuwait University

Calculus 1 – Maximum and Minimum Values

(Section 4.1)

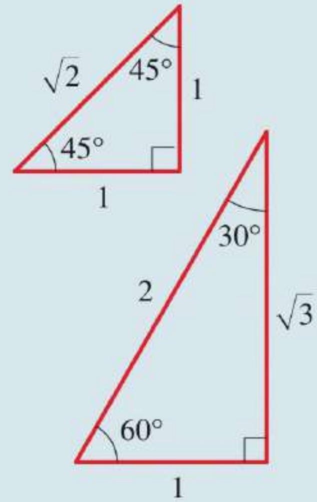
For Contact and Support:

YouTube: Precalculusq8 Twitter: Precalculusq8

SPECIAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

The following values of the trigonometric functions are obtained from the special triangles.

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	0	1	0	—	1	—
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	—	1	—	0



Remark:-

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \boxed{\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}}$$

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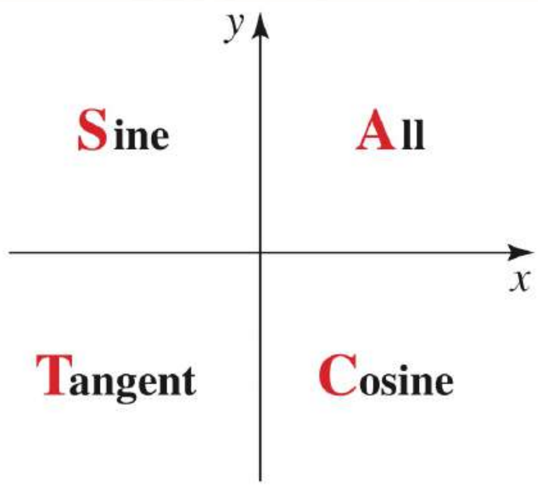
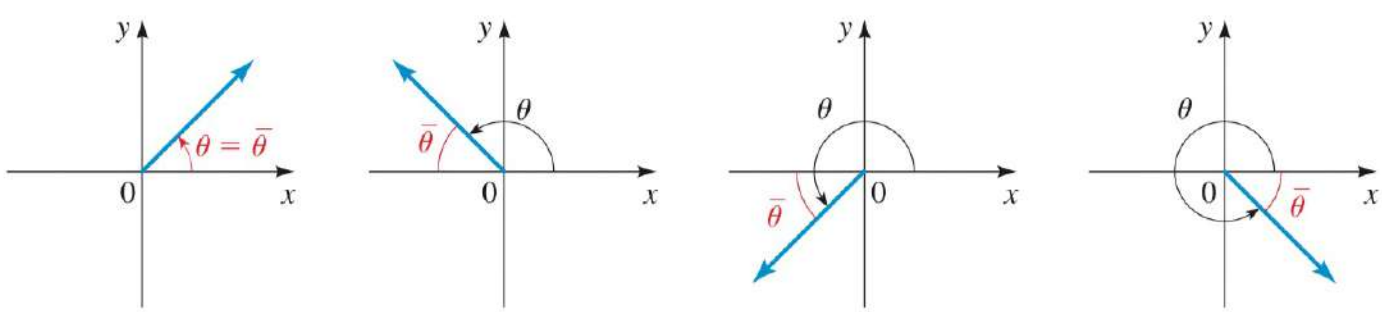
SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

Remark:-



You can remember this as "All Students Take Calculus."



$\bar{\theta} = \theta$

$\bar{\theta} = \pi - \theta$ $\quad$ $\theta = \pi - \bar{\theta}$

$\bar{\theta} = \theta - \pi$ $\quad$ $\theta = \pi + \bar{\theta}$

$\bar{\theta} = 2\pi - \theta$ $\quad$ $\theta = 2\pi - \bar{\theta}$

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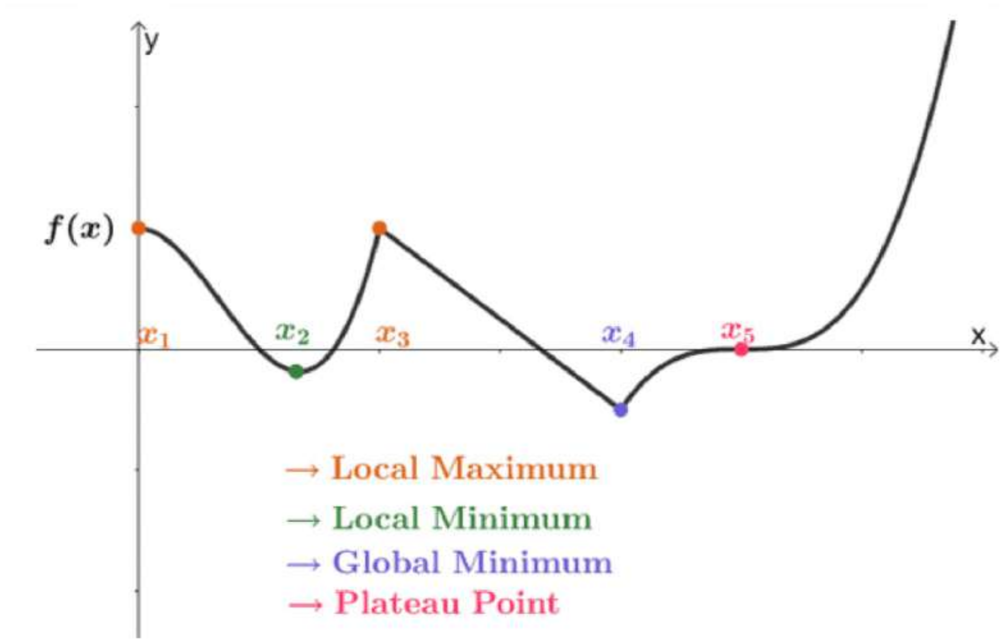
What are critical numbers?

Critical numbers or critical points are values of x where the **first derivative of a function is either equal to zero or undefined**. Let's say we have $x = c$, the critical numbers of the function, $f(x)$, will satisfy either of the following:

$$f'(c) = 0$$

$$f'(c) = \text{ DNE (Does Not Exist)}$$

This means that $x = c$ is a critical number when a **tangent line passing through $x = c$ is either a horizontal or vertical line.**



How to find critical numbers?

We can use the definition of critical numbers to determine the critical numbers – by finding the values of x where $f'(x) = 0$ or $f'(x) = \text{ DNE}$. These are some guide points to help you find a function's critical numbers:

- Use the derivative rules to find $f'(x)$'s expression.
- Find the values of x where $f'(x) = 0$.
- Include the values of x where $f'(x)$ is undefined.

Only include critical numbers that are within the function's domain. This means that values of x where $f'(x)$ is undefined will only count when the point is within the domain of $f(x)$.

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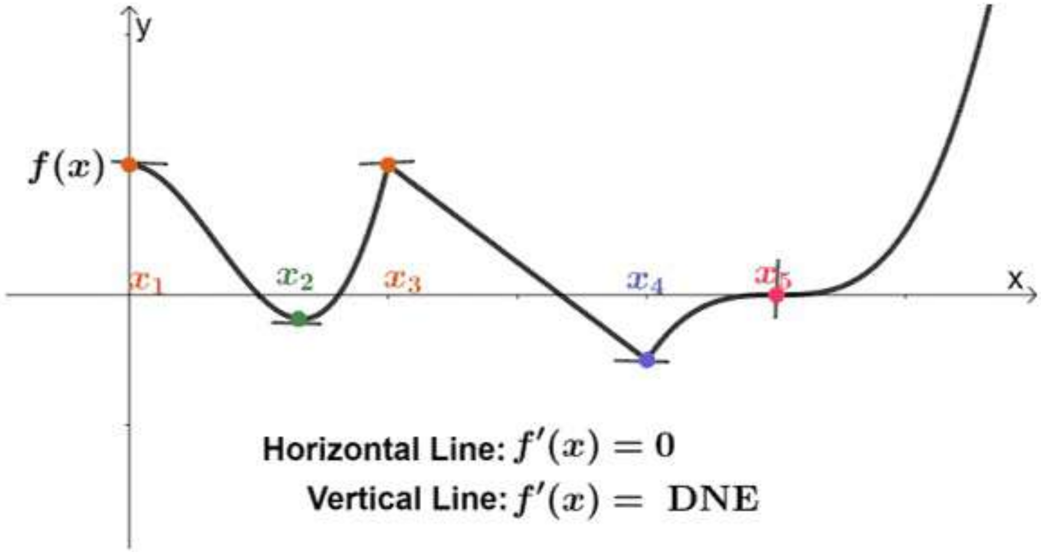


- When the tangent line at $x = c$ is vertical, $f'(c) = 0$ and we have critical point at $x = c$.

- Similarly, when the tangent line passing through $x = c$ is horizontal, $f'(c)$ is undefined and $x = c$ is also a critical point.

Example 1

قراءة

What are the critical numbers of the function, $f(x) = 2x^3 – 8x^2 + 2x – 1$?

$$f(x) = 2x^3 – 8x^2 + 2x – 1$$

D : ℝ

يعني الدالة معرفة بكل
مكان فما أقدر أطلع
الcritical من هالطريقة

$$f'(x) = 6x^2 – 16x + 2,$$ Since $f'(x)$ is Defined everywhere

We can only find the critical number by $f'(x)=0$

$$f'(x) = 0$$

$$6x^2 – 16x + 2 = 0$$

$$3x^2 – 8x + 1 = 0$$

قسمت على ٢ المعادلة كاملة لأنه
أقدر اصغر الأعداد

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 – 4(3)(1)}}{2(3)} = \frac{4 \pm \sqrt{13}}{3}$$

This means that $f(x)$ has critical numbers at $x = \dfrac{4 \pm \sqrt{13}}{3}$.

Example 2

What are the critical numbers of the function, $g(x) = \dfrac{x^2 - 6}{x - 4}$?

Domain of $g(x) = \mathbb{R} / \{4\}$

$$g'(x) = \frac{2x(x-4) - (x^2 - 6)}{(x-4)^2} = \frac{2x^2 - 8x - x^2 + 6}{(x-4)^2}$$

$$= \frac{x^2 - 8x + 6}{(x-4)^2}$$

We can find the critical numbers of $g(x)$ by equating $g'(x)$ to zero or by finding undefined values within its domain. For $g'(x)$ to be undefined, the denominator must be zero.

$$(x-4)^2 = 0$$

$$x = 4 \quad \notin \mathbb{R} / \{4\}$$

It may be tempting to include $x = 4$ as one of $g(x)$'s critical numbers, but keep in mind $x = 4$ is not part of $g(x)$'s domain. Hence, we disregard it.

$$g'(x) = 0$$

$$x^2 - 8x + 6 = 0 \qquad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(6)}}{2(1)} = \frac{8 \pm \sqrt{64 - 24}}{2}$$

$$= \frac{8 \pm 2\sqrt{10}}{2} = 4 \pm \sqrt{10}$$

This means that $g(x)$ has critical numbers at $x = 4 - \sqrt{10}$ and $x = 4 + \sqrt{10}$.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

عشان إذا تبي تعرف الcritical number حق أي function :

1) $f'(x) = 0$ ١- نشوف المشتقة ونساوييها بالصفر

٢- نشوف المشتقة لما تكون غير معرفة (يعني أساوي المقام بصفر)

2) $f'(x) = DNE$

ملاحظة رقم 2 أحتاج إني أوحّد المقامات إذا كانت مو موحده

EXAMPLE 7 Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

Domain of $f(x) = \mathbb{R}$

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0. ■

29–44 Find the critical numbers of the function.

39. $F(x) = x^{4/5}(x-4)^2$

The domain of f is $\mathbb{R}$

$$f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5}\,2(x-4)$$

$$= \frac{4(x-4)^2}{5x^{1/5}} + 2x^{4/5}(x-4)$$

$$= \frac{4(x-4)^2}{5x^{1/5}} + \frac{2x^{4/5}(x-4)(5x^{1/5})}{5x^{1/5}}$$

$$\frac{4(x-4)^2}{5x^{1/5}} + \frac{10x(x-4)}{5x^{1/5}} = \frac{4(x-4)^2+10x(x-4)}{5x^{1/5}}$$

$$\frac{2(x-4)[2(x-4)+5x]}{5x^{1/5}} = \frac{2(x-4)[2x-8+5x]}{5x^{1/5}}$$

$$\frac{2(x-4)(7x-8)}{5x^{1/5}}$$

$f'(x) = 0$, when

$2(x-4)(7x-8) = 0$

$\therefore x = 4,\ x = \frac{8}{7}$

$f'(x) = \text{DNE}$, when

$5x^{1/5} = 0 \quad \therefore x = 0$

$x = 0,\ x = 4,\ x = \frac{8}{7}$ are critical number

29–44 Find the critical numbers of the function.

41. $f(\theta) = 2\cos\theta + \sin^2\theta$

1) Domain of $f(\theta) = \mathbb{R}$

2) $f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$

$f'(\theta) = 2\sin\theta\,(\cos\theta - 1)$

$2\sin\theta = 0$	$\cos\theta - 1 = 0$
$\sin\theta = 0$	$\cos\theta = 1$
$\therefore\ \theta = 0 + n\pi$	$\therefore\ \theta = 0 + 2n\pi$

$\therefore\ \theta = n\pi$ is the critical number

29–44 Find the critical numbers of the function.

43. $f(x) = x^2 e^{-3x}$

Domain of $f(x) = \mathbb{R}$

$$f'(x) = 2x e^{-3x} + (-3 e^{-3x}) x^2$$

$$= x e^{-3x} (2 - 3x)$$

2) to find the critical numbers we look for $f' = 0$ and undefined

مقام = صفر

$f' = 0 \Rightarrow x e^{-3x} (2 - 3x) = 0$

$\therefore x e^{-3x} = 0$, when $x = 0$

$\therefore (2 - 3x) = 0 \Rightarrow$ when $x = \frac{2}{3}$

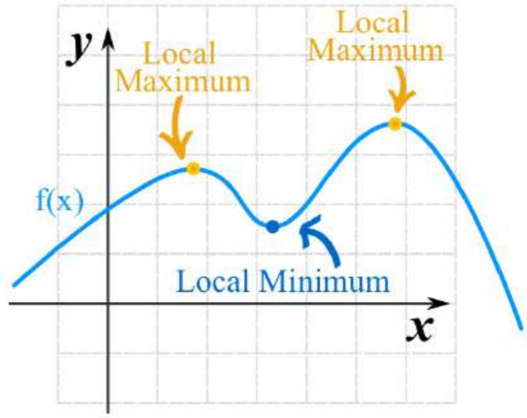
$\because f'$ is defined on $\mathbb{R}$, $\therefore$ The critical numbers

$x = 0$, $x = \frac{3}{2}$

Local Maximum and Minimum

Functions can have "hills and valleys": places where they reach a minimum or maximum value.

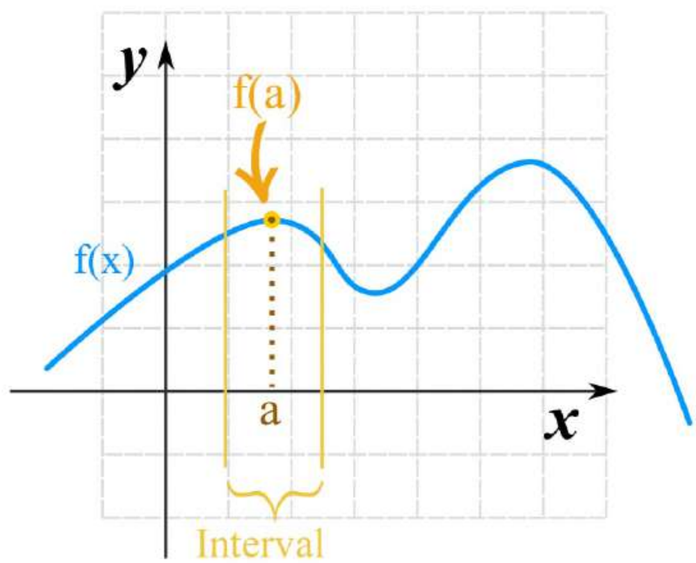
It may not be the minimum or maximum for the **whole function**, but **locally** it is.



We can see where they are,
but how do we define them?

Local Maximum

First we need to choose an interval:



Then we can say that a local **maximum** is the point where:

$f(a) \geq f(x)$ for all x in the interval

In other words, there is no height greater than f(a).

Note: a should be **inside** the interval, not at one end or the other.

Local Minimum

Likewise, a local **minimum** is:

$$f(a) \le f(x) \text{ for all } x \text{ in the interval}$$

The plural of Maximum is **Maxima**

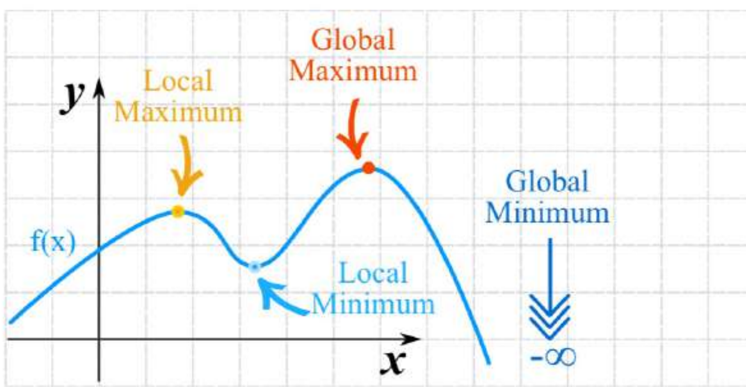
The plural of Minimum is **Minima**

Maxima and Minima are collectively called **Extrema**

Global (or Absolute) Maximum and Minimum

The maximum or minimum over the **entire function** is called an "Absolute" or "Global" maximum or minimum.

There is only one global maximum (and one global minimum) but there can be more than one local maximum or minimum.



Assuming this function continues downwards to left or right:

- The Global Maximum is about 3.7
- The Global Minimum is −Infinity

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The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

---

1) function is continous

أكتب أسم الدالة و إن
الدالة continuous
على الفترة المطلوبة

2) find the critical numbers $\in$ Domain

local max
local min
في الفترة المفتوحة دائماً

تأكد الcritical values الي
طلعتهم ينتمون للdomain
الدالة الاصلية

3) find endpoints

أطراف الفترة وعوضها بال f(x) مع
الcritcal numbers

4) The largest value is the absolute maximum

بعد ما تعوض في الفترة
المعطاة مع قيم
الcritical شوف أكبر
نتيجة تطلع لك وقول عنها
abs maximum

The smallest value is the absolute minimum

بعد ما تعوض
شوف أصغر
نتيجة تطلع لك
وسمها abs
minimum

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Example . Determine the absolute maximu

and minimum of $f(x) = -\frac{1}{3}x^3 + 4x$ on the interval $[0\,,\,3]$. Also, verify that the function satisfies the hypothesis of the Extreme Value Theorem.

Solution:

Verification of the Extreme Value Theorem: f is continuous since it is a polynomial, and the domain is a closed interval, so the conditions of the Extreme Value Theorem are met. (**this part gets some marks**).

Now, use the Closed Interval Method:

(**the discussion below gets the remaining part of the grade**)

Next Step: Find the derivative:

$$f'(x) = -x^2 + 4$$

Next Step: Search for the points in the domain at which the derivative does not exist: there are no such points.

Next Step: Search for the points in the domain at which the derivative is zero:

$$f'(x) = -x^2 + 4 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2 \quad \Rightarrow \quad \text{only 2 is in the domain}$$

Next Step: According to the previous two steps, the only critical number is $x = 2$.

Next Step: Calculate the value of f at the the critical numbers and the endpoints and then compare these values:

$$\begin{cases} f(2) &= -\frac{1}{3}2^3 + 4(2) = -\frac{8}{3} + 8 = \frac{16}{3} \\ \\ f(0) &= 0 \\ \\ f(3) &= -\frac{1}{3}3^3 + 4(3) = -9 + 12 = 3 \end{cases}$$

Therefore, f has the absolute maximum of $\frac{16}{3}$ and has the absolute minimum of 0.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

SOLUTION Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of f is sketched in Figure 15.

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47–62 Find the absolute maximum and absolute minimum values of f on the given interval.

47. $f(x) = 12 + 4x - x^2, \quad [0, 5]$

$\because$ f is continous on $[0,5]$ (polynomial)

1) critical numbers

a) $f'(x) = 4 - 2x$

$f'(x) = 0$, $4 - 2x = 0$

$\therefore$ $2x = 4 \Rightarrow x = 2$

b) $f'(x)$ exists for all x, $\therefore$ The only critical number is $x = 2$

2) The values of f at the end points

$f(0) = 12 + 4(0) - (0)^2 = 12$

$f(5) = 12 + 4(5) - (5)^2 = 7$

$f(2) = 12 + 4(2) - (2)^2 = 16$

The absolute maximum value of f, $f(2) = 16$

The absolute minimum value of f, $f(5) = 7$

Find the absolute maximum and minimum values of the function

53. $f(x) = x + \dfrac{1}{x}, \quad [0.2, 4]$

$\because$ f is continous on $[0.2, 4]$ (Poly, Rotional)

1) critical number

a) $f'(x) = 1 - \dfrac{1}{x^2}$

$f'(x) = 0$, $1 - \dfrac{1}{x^2} = 0$

$\therefore \dfrac{1}{x^2} = 1 \Rightarrow x^2 = 1 \Rightarrow x = 1, \quad x = -1$

But $x = -1 \notin [0.2, 4]$

b) $f'(x) = DNE$, when $x^2 = 0$

$x = 0 \notin [0.2, 4]$

$\therefore$ The critical is $x = 1$

2) The values of f at the end points

$\frac{1}{\frac{2}{10}} = \frac{10}{2} = 5$

$f(0.2) = 0.2 + \dfrac{1}{0.2} = 0.2 + \dfrac{1}{\frac{2}{10}} = 0.2 + 5 = 5.2$

$f(4) = 4 + \dfrac{1}{4} = \dfrac{16}{4} + \dfrac{1}{4} = \dfrac{17}{4}$

$f(1) = 1 + \dfrac{1}{1} = 2$

The absolute maximum value of f, $f(0.2) = 5.2$

The absolute minimum value of f, $f(1) = 2$

Find the absolute maximum and minimum values of the function

59. $f(x) = x^{-2} \ln x, \quad \left[\frac{1}{2}, 4\right]$

$\because f$ is continous on $[\frac{1}{2}, 4]$

1) critical number

a) $f'(x) = -2x^{-3} \ln x + x^{-2} \frac{1}{x} = \frac{-2\ln x + 1}{x^3}$

$f'(x) = 0$, $\frac{-2\ln x + 1}{x^3} = 0$

$\therefore -2\ln x + 1 = 0 \Rightarrow -2\ln x = -1$, $\ln x = \frac{1}{2}$

$x = e^{1/2} \in [\frac{1}{2}, 4]$

b) $f'(x) = DNE$, when $x^3 = 0$

$x = 0 \notin [\frac{1}{2}, 4]$

$\therefore$ The critical is $x = e^{1/2}$

2) The values of f at the end points

قراءة

$f(\frac{1}{2}) = (\frac{1}{2})^{-2} (\ln \frac{1}{2}) = 4 \ln \frac{1}{2} = 4 \ln 2^{-1} = -4 \ln 2$

$f(4) = (4)^{-2} (\ln 4) = \frac{1}{16} (\ln 4)$

$f(\sqrt{e}) = (\frac{1}{\sqrt{e}})^{-2} (\ln e^{1/2}) = (\frac{1}{e})(\frac{1}{2} \ln e) = \frac{1}{2e}$

The absolute maximum value of f,

The absolute minimum value of f,

ما أدري لانه لازم آلة
حاسبة .. المهم الفكرة
والخطوات

47–62 Find the absolute maximum and absolute minimum values of f on the given interval.

62. $f(x) = x - 2\tan^{-1}x, \quad [0, 4]$

---

f is continous on $[0,4]$

1) critical number

a) $f'(x) = 1 - 2\dfrac{1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

$$= \frac{x^2+1-2}{x^2+1} = \frac{x^2-1}{x^2+1}$$

$f'(x) = 0$

$x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = -1,\ x = 1$

But $x = -1 \notin [0,4]$, The critical value is $x = 1$

b) $f'(x)$ exists for all x, $\therefore$ The only critical number is $x = 1$ قراءة

2) $f(0) = 0 - 2\tan^{-1}0 = 0$

$f(4) = 4 - 2\tan^{-1}4$

$f(1) = 1 - 2\tan^{-1}1 = 1 - 2\dfrac{\pi}{4} = 1 - \dfrac{\pi}{2}$

The absolute maximum value of f.., ما أدري لانه لازم آلة حاسبة

The absolute minimum value of f, المهم الفكرة والخطوات

Find the absolute maximum and minimum values of the function

$$f(x) = \sin x - \cos x \qquad \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$f(x)$ is continuous on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

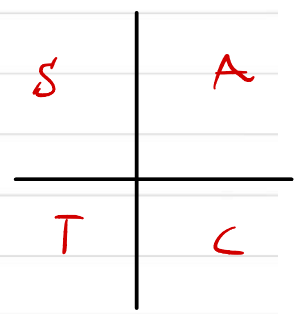
$$f'(x) = \cos x + \sin x$$

$$f'(x) = 0 \quad , \quad \cos x + \sin x = 0$$

$$\sin x = -\cos x \qquad \div \cos x$$

$$\tan x = -1$$

$$x = \frac{3\pi}{4} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$



$$x = \frac{7\pi}{4} \notin \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$$

$$f\left(\frac{3\pi}{2}\right) = \sin\frac{3\pi}{2} - \cos\frac{3\pi}{2} = -1 - 0 = -1$$

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} \text{ or } \sqrt{2}$$

The absolute maximum value of f, $f\left(\frac{3\pi}{4}\right) = \frac{2}{\sqrt{2}}$

نفس الشيء

$$\frac{2}{\sqrt{2}} * \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

The absolute minimum value of f, $f\left(\frac{3\pi}{2}\right) = -1$

5. [10 pts.] Find the absolute maximum and the absolute minimum values, if any, of $f(x) = \sin(2x) - 2x$ on the interval $[0, 2\pi]$.

$f(x)$ is continous on $[0, 2\pi]$

$f'(x) = \cos(2x)(2) - 2$

$= 2\cos(2x) - 2 = 2[\cos(2x) - 1]$

To find critical values we set $f'(x) = 0$

$2[\cos 2x - 1] = 0$

$\therefore 2 \neq 0$, $\cos(2x) - 1 = 0$

$\cos(2x) = 1$

$2x = 0$
$x = 0 \in (0, 2\pi)$ | $2x = 2\pi \Rightarrow x = \pi \in (0, 2\pi)$

$\therefore$ The critical value is $x = 0$ and $x = \pi$

$f(0) = \sin(2(0)) - 2(0) = 0$

$f(2\pi) = \sin(2(2\pi)) - 2(2\pi) = 0 - 4\pi = -4\pi$

$f(\pi) = \sin(2(\pi)) - 2(\pi) = 0 - 2\pi = -2\pi$

The absolute maximum value of f, $f(0) = 0$

The absolute minimum value of f, $f(2\pi) = -4\pi$

8. [10 pts.] Find the **absolute maximum and absolute minimum values** of $f(x) = \cos^2 x + \sin x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

f(x) is continous on $[0, \frac{\pi}{2}]$ (Trigonometric)

We have $f'(x) = -2\cos x \sin x + \cos x = \cos x(-2\sin x + 1) = 0$

on the given interval only at $x = \frac{\pi}{6}$ or $x = \frac{\pi}{2}$.

Therefore, the only critical numbers of f on the given interval are $x = \frac{\pi}{6}, \frac{\pi}{2}$.

Now we have $f(0) = 1$, $f\left(\frac{\pi}{6}\right) = \frac{5}{4}$ and $f\left(\frac{\pi}{2}\right) = 1$.

that the absolute maximum value of f is $\frac{5}{4}$ and the absolute minimum value is 1.

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5. [10 pts.] Find the absolute maximum and minimum values of the function $f(x) = 3x - 5x^{3/5}$ on the interval $[-1, 32]$.

We have $\boxed{f'(x) = 3 - 3x^{\frac{-2}{5}} = 3\left(\frac{x^{\frac{2}{5}} - 1}{x^{\frac{2}{5}}}\right)}$.

Therefore, the critical numbers of f in $(-1, 32)$ are $\boxed{x = 0, 1}$.

Evaluating f at the end points and the critical numbers, we get:

$\boxed{f(-1) = 2,\ f(32) = 56,\ f(0) = 0,\ \text{and } f(1) = -2}$.

Therefore, $\boxed{\text{the absolute maximum value of } f \text{ is } 56}$ and $\boxed{\text{the absolute minimum value of } f \text{ is } -2}$.

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II. Let $f(x) = x^2 + 1$. On the interval $[-1, 1]$:

(A) $\boxed{\text{1 is the absolute minimum value of } f \text{ and 2 is the absolute maximum value of } f}$.

(B) $f(0.5)$ is a local minimum value of f.

(C) f has no absolute extreme values.

(D) $f(0.5)$ is a local maximum value of f.

(E) None of the above.

4. [10 pts.] Find the absolute maximum and absolute minimum values of $f(x) = x^2 e^{-2x}$ on the interval $[-1, 1]$.

- We have $\boxed{f(-1) = e^2 \text{ and } f(1) = e^{-2}}$.
- Since $\boxed{f'(x) = 2xe^{-2x} + x^2(-2e^{-2x}) = 2xe^{-2x}(1-x)}$, the function has only one critical number in $(-1, 1)$, that is $\boxed{x = 0}$. The value of f at $x = 0$ is $\boxed{f(0) = 0}$.
- Comparing $f(-1)$, $f(1)$ and $f(0)$, we see that the absolute maximum value of f is $\boxed{f(-1) = e^2}$ and the absolute minimum value of f is $\boxed{f(0) = 0}$.

III. The function f defined by $f(x) = \dfrac{1}{x}$ has a critical point at:

a) $x = 0$. b) $x = 1$. c) $x = -1$. d) $x = 2$. e) $\boxed{\text{None of the above}}$.

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7. [10 pts.] Find the absolute maximum and absolute minimum values of

$$f(x) = xe^{-x} \text{ on } [0, 2].$$

We have $\boxed{f'(x) = e^{-x}(1 - x)}$, so f has one critical number $x = 1$.

Now $\boxed{f(0) = 0,\ f(1) = \frac{1}{e}}$ and $\boxed{f(2) = \frac{2}{e^2}}$.

Thus, f has $\boxed{\text{absolute maximum value } \frac{1}{e} \text{ and absolute minimum value } 0}$.

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Thamrah

Kuwait University

Calculus 1 – The Mean Value Theorem

(Section 4.2)

For Contact and Support:

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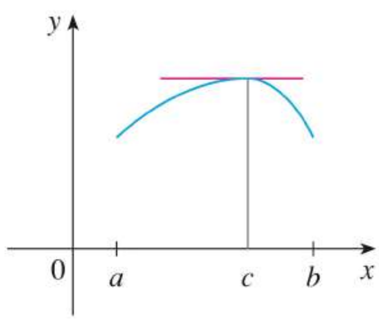
حفظ نص النظرية !!!

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



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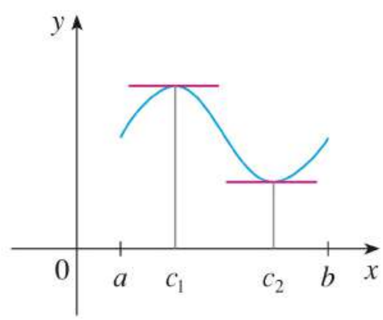
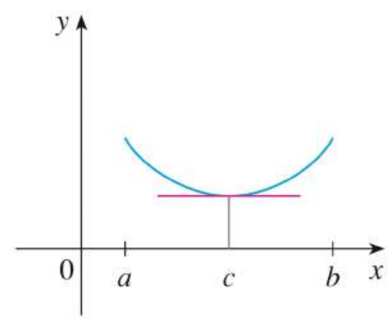


(c)



(d)

حفظ نص النظرية !!!

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

كيف

or, equivalently,

2
$$f(b) - f(a) = f'(c)(b - a)$$

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Remark

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

$-1 \leq \sin x \leq 1$

$-1 \leq \cos x \leq 1$

Contradict = تناقض
Applicable = قابلة للتطبيق
Hypothesis = فرضيات
Root = حل للمعادلة

---

or critical number

- To prove the function has exactly one root :

section 2.5

1- IVT theorem (to show that a root exist)

مناقشة تناقض

2- Rolles theorem and argue by contradiction and we suppose it had two roots a and b & $f(a) = f(b) = 0$, function cont. & diff. Then by Rolles theorem, There is number c between a and b such that

$f'(c) = 0$ - - - - -

الخطوة الثانية يعني رح نفرض أن هناك حلّين للمعادلة ، ورح نثبت أن فرضيتنا غلط باستخدام النظرية Rolles ورح يصير تناقض آخر شي إن ما يصير لها حلين .. ورح نشوف في حل المسائل

- To prove the function at most one real solution

- only using Rolles theorem argue and contradiction

يعني جنه أهو يدري أن في root
بس يبي يتأكد إن مافي غير root
واحد .. فعشان جذي ماله داعي
أثبت عن طريق نظرية IVT أن في
at least root يعني على الاقل
في root

Remark:-

> **4 Theorem** If f is differentiable at a, then f is continuous at a.

> **10 The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

تذكر طريقة حل
I.V.T

53. $x^4 + x - 3 = 0, \quad (1, 2)$

Let $f(x) = x^4 + x - 3 = 0$

لازم تتأكد أن المعادلة
صفرية .. إذا ما كانت
صفرية ... خلها صفرية

Domain of f is $\mathbb{R}$ and continuous

$f(1) = (1)^4 + 1 - 3 = -1 < 0$

$f(2) = (2)^4 + 2 - 3 = 15 > 0$

f is cont. on $(1, 2)$ Then

by I.V.T, There exists a c in $(1, 2)$

such that $f(c) = 0$ $\therefore$ c is a root

5–8 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

5. $f(x) = 2x^2 - 4x + 5, \quad [-1, 3]$

أذكر السبب لما تقول continous أو differentiable!!!

1) $f(x)$ is continuous on $[-1,3]$ (polynomial)

$f'(x) = 4x - 4$

2) $f'(x)$ is differentiable on $(-1,3)$

3) $f(-1) = 2(-1)^2 - 4(-1) + 5 = 11$

$f(3) = 2(3)^2 - 4(3) + 5 = 11$

By ①, ②, ③, f satisfies the conditions of Rolle's theorem $[-1,3]$

Then there exists $c \in (-1,3)$ such that $f'(c) = 0$ $\quad \therefore f'(c) = 4c - 4 = 0$

$\therefore 4c = 4 \quad \therefore c = 1 \in [-1,3]$

9. [10 pts.] Verify that the function $f(x) = x + \frac{1}{x}$ satisfies the hypotheses of **Rolle's Theorem** on the interval $\left[\frac{1}{2}, 2\right]$. Then find all numbers c, if any, that satisfy the conclusion of Rolle's Theorem.

The function is continuous on the interval $\left[\frac{1}{2}, 2\right]$ since it is an algebraic function with domain that includes this interval. Also, the function is differentiable on the interval $\left(\frac{1}{2}, 2\right)$ for the same reason. Now $f\left(\frac{1}{2}\right) = 2 + \frac{1}{2} = \frac{1}{2} + 2 = f(2)$. Thus, by Rolle's Theorem there exits $c \in \left(\frac{1}{2}, 2\right)$ such that $f'(c) = 0$ which implies that $1 - \frac{1}{c^2} = 0$. Therefore, $c = 1$.

6. [5+5=10 pts.] Let $f(x) = \ln(x^2 + 1)$.

(a) Verify that f satisfies the conditions of Rolle's Theorem on $[-1, 1]$.

(b) Find all $c \in (-1, 1)$ that satisfy the conclusion of Rolle's Theorem.

(a) [5 pts] The function is a composition of a polynomial and a logarithmic function with domain all of the real numbers. Therefore, $\boxed{\text{the function is continuous on } [-1,1] \text{ and differentiable on } (-1,1)}$, $\boxed{f'(x) = \frac{2x}{x^2+1}}$. Also, we have $\boxed{f(-1) = \ln 2 = f(1)}$. Therefore, f satisfies the conditions of Rolle's Theorem on $[-1, 1]$.

(b) [5 pts] $\boxed{f'(c) = \frac{2c}{c^2+1} = 0}$. This gives $\boxed{c = 0 \in (-1, 1)}$, so 0 satisfies the conclusion of Rolle's Theorem.

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9. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

$f(x) = 1 - (\sqrt[3]{x})^2$ "power function"

$f(-1) = 1 - x^{2/3} = 1 - (\sqrt[3]{(-1)})^2 = 0$

$f(1) = 1 - x^{2/3} = 1 - (\sqrt[3]{1})^2 = 0$

$\therefore f(-1) = f(1)$

$f'(c) = 0$

$f'(c) = -\frac{2}{3} c^{-1/3} = \frac{-2}{3c^{1/3}}$ & $c \neq 0$

$\because 0 \in (-1, 1)$

$\therefore$ f is not differentiable on $(-1, 1)$

$\therefore$ Rolle's theorem is not applicable

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

1) Let's prove it has a root

تذكر: الصفر والموجب
واحد من عندي يابيهم..
المهم واحد يعطيني
موجب ...وواحد يعطيني
سالب عشان نظرية IVT

$f(x) = x^3 + x - 1$

$f(0) = 0^3 + 0 - 1 = -1 < 0$

$f(1) = (1)^3 + 1 - 1 = 1 > 0$

$\because$ f is cont. on $[0, 1]$ "poly"

and since $f(0) < 0 < f(1)$ then by IVT

There exists c in $(0, 1)$ such that

$f(c) = 0$, $\therefore$ c it has at least one real root

2) Let's prove it has only one root

contradiction: Let's have two real roots

$x = a$, $x = b$ $\therefore f(a) = 0$, $f(b) = 0$

1) f is cont on $[a, b]$ "poly", $f'(x) = 3x^2 + 1$

2) f is diff. on (a, b) $\xrightarrow{\text{cont.}}$

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3) $f(a) = f(b) = 0$

By ①, ②, ③ Then by Rolle's theorem

There is number c between a and b

Such that $f'(c) = 0$

$f'(c) = 3c^2 + 1 = 0$ impossible

$\therefore \underbrace{3c^2}_{+ve} + \underbrace{1}_{+1} \geq 1$

since $f'(c) \neq 0$, our assumption is wrong

$\therefore$ eqution has exactly one real root

19–20 Show that the equation has exactly one real root.

20. $x^3 + e^x = 0$

---

1) prove it has root

$f(x) = x^3 + e^x$

$f(0) = (0)^3 + e^0 = 1 > 0$

$f(-2) = (-2)^3 + e^{-2} = -8 + \frac{1}{e^2} < 0$

$\because$ f is cont. on $[-2, 0]$ "poly, exponential"

and since $f(-2) < 0 < f(0)$ then by IVT

There exists c in $(-2, 0)$ such that

$f(c) = 0$, $\therefore$ c it has at least one real root

---

2) Let's prove it has only one root

contradiction: Let's have two real roots

$x = a$, $x = b$ $\therefore$ $f(a) = 0$, $f(b) = 0$

1) f is cont on $[a, b]$ "poly", $f'(x) = 3x^2 + e^x$

2) f is diff. on (a, b) $\xrightarrow{\text{cont}}$

3) $f(a) = f(b) = 0$

By ①, ②, ③ Then by Rolle's theorem

There is number c between a and b

Such that $f'(c) = 0$

$f'(c) = 3c^2 + e^c = 0$ impossible

عدد موجب

+ve = postive value

$\therefore \underbrace{3c^2}_{+ve} + \underbrace{e^c}_{e^x \geq 0} > 0$

since $f'(c) \neq 0$, our assumption is wrong

$\therefore$ eqution has exactly one real root

19–20 Show that the equation has exactly one real root.

19. $2x + \cos x = 0$

تذكير: عشان نستخدم النظرية IVT:-

١- لازم المعادلة صفرية
٢- لازم يكون عندي دالة
٣- لازم فترة

$f(x) = 2x + \cos x$

$f(0) = 2(0) + \cos 0 = 1 > 0$

$f(-1) = 2(-1) + \cos(-1) = -2 + \cos(-1) < 0$

أدري قيمه

$-1 \le \cos x \le 1$

$\because$ f is cont. on $[-1, 0]$ "poly, Trigonometric"

and since $f(-1) < 0 < f(0)$ then by IVT

There exists c in $(-1, 0)$ such that

$f(c) = 0$, $\therefore$ c it has at least one real root

2) Let's prove it has only one root

contradiction: Let's have two real roots

$x = a$, $x = b$ $\therefore f(a) = 0$, $f(b) = 0$

1) f is cont on $[a, b]$ "Poly, Trig" $f'(x) = 2 - \sin x$

2) f is diff. on (a, b)

3) $f(a) = f(b) = 0$

By ①, ②, ③ Then by Rolle's theorem

There is number c between a and b

Such that $f'(c) = 0$

$f'(c) = 2 - \sin c = 0$ impossible

because we know $-1 \leq \sin x \leq 1$

since $f'(c) \neq 0$, our assumption is wrong

$\therefore$ eqution has exactly one real root

23. (a) Show that a polynomial of degree 3 has at most three real roots.

Let $f(x) = px^3 + qx^2 + rx + s$

$\because$ $f(x)$ is poly. Domain of f is $\mathbb{R}$

$$f'(x) = 3px^2 + 2qx + r$$

$\therefore$ $f(x)$ is continous and differentiable everywhere $= \mathbb{R}$

Suppose $f(x)$ has four real roots

بس عشان أقول إنهم قيم مختلفة

$$p < q < r < s$$

and $f(p) = f(q) = f(r) = f(s) = 0$

$\because$ $f'(x) = 3px^2 + 2qx + r$ ← 2^{nd} Degree

$\because$ $f'(x)$ is a quadratic function and she can have no more than two real roots

There is contradiction with Rolles Theorem,

There must be Three number m, n, k and $m \in (a,b)$, $n \in (a,b)$, $k \in (c,d)$

$f'(m) = f'(n) = f'(m) = 0$ $\therefore$ The function $f(x)$ has at most three real roots.

21. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

1) Let's prove it has only one root

contradiction: Let's have two real roots

$x = a$, $x = b$ $\therefore f(a) = 0$, $f(b) = 0$

1) f is cont on $[a, b]$ "poly."

$\because f'(x) = 3x^2 - 15$

2) f is differentiable on (a, b)

3) $f(a) = f(b) = 0$

By ①, ②, ③ Then by Rolle's theorem

There is number c between a and b

Such that $f'(c) = 0$

$f'(c) = 3c^2 - 15 = 0 \Rightarrow 3c^2 = 15$

$c^2 = \frac{15}{3} \Rightarrow c^2 = 5 \quad \therefore c = \pm\sqrt{5}$

But $\pm\sqrt{5} \notin [-2, 2]$

$\therefore$ The given equation cannot have two real solution

Q4. [10 pts.] Show that, if $a > 0$, then the equation: $x^3 + ax - 1 = 0$ has at most one real root.

Let $g(x) = x^3 + ax - 1$. To show that g can not have two real roots. We use Rolle's Theorem and argue by contradiction. Suppose that g had two real roots c_1 and c_2 (where $c_1 < c_2$). Then $g(c_1) = g(c_2) = 0$ and, since g is a polynomial, it is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2). Then, by Rolle's Theorem, there is a number $c \in (c_1, c_2)$ such that $g'(c) = 0$. But $g'(x) = 3x^2 + a > 0$ for all x. This gives a contradiction. Therefore, the given equation cannot have two real solutions.

8. [10 pts.] Verify that $f(x) = 2x^2 + 3x + 4$ satisfies the hypotheses of Rolle's Theorem on the interval $\left[-2, \frac{1}{2}\right]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

It is clear that f is <u>continuous</u> on $\left[-2, \frac{1}{2}\right]$ and <u>differentiable</u> on $\left(-2, \frac{1}{2}\right)$ (f is a polynomial).

Also, $\boxed{f(-2) = 6 = f(1/2)}$.

Therefore, by Rolles's Theorem there exists a $c \in \left(-2, \frac{1}{2}\right)$ s.t. $\boxed{f'(c) = 4c + 3 = 0}$. Thus, $\boxed{c = \frac{-3}{4} \in \left(-2, \frac{1}{2}\right)}$.

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The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2
$$f(b) - f(a) = f'(c)(b - a)$$

3
$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1. Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB. (Imagine a line far away that stays parallel to AB while moving toward AB until it touches the graph for the first time.)

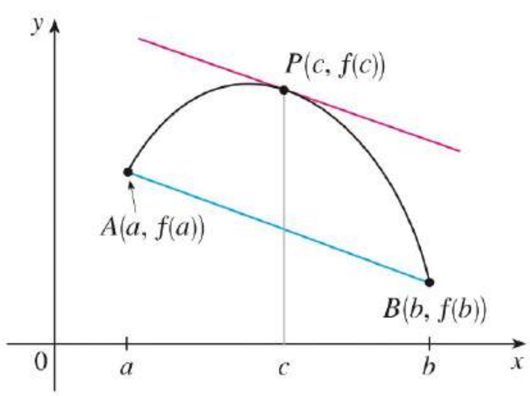

FIGURE 3

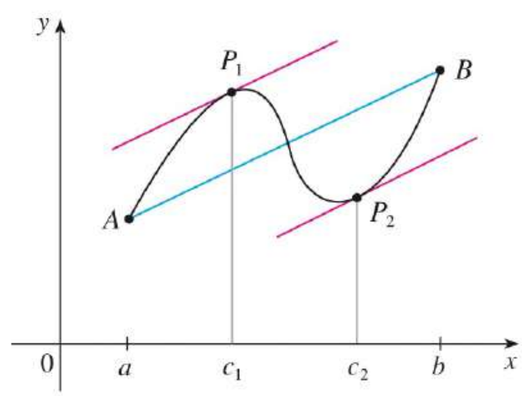

FIGURE 4

EXAMPLE 3 To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$.

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

SOLUTION We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

so

$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leqslant 5$ for all x, so in particular we know that $f'(c) \leqslant 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leqslant 10$, so

$$f(2) = -3 + 2f'(c) \leqslant -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7. ■

@Precalculusq8

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

11. $f(x) = 2x^2 - 3x + 1, \quad [0, 2]$

Domain of $f(x)$ is $\mathbb{R}$ "polynomial"

1) $f(x)$ is continuous on $[0,2]$

$f'(x) = 4x - 3$

2) $f(x)$ is differentiable on $(0,2)$

by MVT, there exists at least

$c \in (0,2)$ such that $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

$f'(c) = 4c - 3$

$f(2) = 2(2)^2 - 3(2) + 1 = 3$

$f(0) = 2(0)^2 - 3(0) + 1 = 1$

$4c - 3 = \dfrac{3 - 1}{2 - 0} \Rightarrow 4c - 3 = 1$

$\Rightarrow 4c = 4 \Rightarrow c = 1 \in [0,2]$

@Precalculusqa

17. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

$$f(x) = (x-3)^{-2} = \frac{1}{(x-3)^2}$$

$$f(4) = \frac{1}{(4-3)^2} = 1, \quad f(1) = \frac{1}{(1-3)^2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$f'(x) = -2(x-3)^{-3} = \frac{-2}{(x-3)^3}$$

$$f(4) - f(1) = 1 - \frac{1}{4} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

$$f(4) - f(1) = f'(c)(4-1)$$

$$\frac{3}{4} = \frac{-2}{(c-3)^3}(3) \Rightarrow 3(c-3)^3 = -24$$

$$(c-3)^3 = -8 \Rightarrow c - 3 = -2 \Rightarrow c = 1 \notin (1,4)$$

$f(x) = \dfrac{1}{(x-3)^2}$, Domain of $f(x)$ is $\mathbb{R} / \{3\}$, f is not continuous on 3

$\because 3 \in (1,4) \therefore$ MVT is not applicable

25. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?

We have $f'(x) \geq 2$ for $1 \leq x \leq 4$

$\therefore$ $f(x)$ is differentiable on $(1, 4)$

$\because$ $f(x)$ is differentiable, $\therefore$ f is continuous on $[1, 4]$ "Theorem"

$\therefore$ By MVT, there exists at least $c \in (1, 4)$ such that $f'(c) = \dfrac{f(4) - f(1)}{4 - 1}$

$f'(c) = \dfrac{f(4) - 10}{3} \geq 2 \Rightarrow f(4) - 10 \geq 6$

$f(4) \geq 16$

$f(4)$ can be at least 16

26. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

We have $3 \leqslant f'(x) \leqslant 5$

for $18 \leqslant f(8) - f(2) \leqslant 30$

$\therefore f(x)$ is differentiable on $(2,8)$

$\because f(x)$ is differentiable, $\therefore f$ is continous on $[2,8]$ "Theorem"

$\therefore$ By MVT, there exists at least

$c \in (2,8)$ such that $f'(c) = \dfrac{f(8)-f(2)}{8-2}$

$$f'(c) = \frac{f(8)-f(2)}{6} \qquad \because 3 \leqslant f'(x) \leqslant 5$$

$$\therefore 3 \leqslant \frac{f(8)-f(2)}{6} \leqslant 5$$

$$\therefore 18 \leqslant f(8) - f(2) \leqslant 30$$

27. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leqslant 2$ for all x?

We have $f'(x) \leq 2$

$\therefore$ $f(x)$ is differentiable on $(0, 2)$

$\because$ $f(x)$ is differentiable, $\therefore$ f is continous on $[0, 2]$ "Theorem"

$\therefore$ By MVT, there exists at least $c \in (0, 2)$ such that $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

$$f'(c) = \frac{4-(-1)}{2-0} = \frac{5}{2} \notin (0, 2)$$

$\because$ $f'(x) < 2$

$\therefore$ $\frac{5}{2} < 2$ which is wrong

Because $2 < \frac{5}{2}$ $\therefore$ f does not exist

4. [10 pts.] Verify that $f(x) = \sqrt{4 - x^2}$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 0]$. Then, find all numbers c that satisfy the conclusion of the Theorem.

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4. [10 pts.] f is continuous on its domain $[-2, 2]$. Thus, it is continuous on $[-2, 0] \subset [-2, 2]$. Moreover, $\boxed{f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}}$ and hence f is differentiable on $(-2, 0)$.

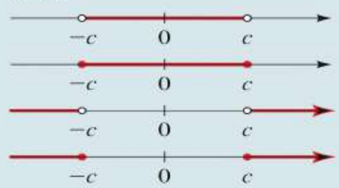
Applying MVT, there exists at least $c \in (-2, 0)$ such that $\boxed{f'(c) = \dfrac{f(0) - f(-2)}{0 - (-2)}}$, that is,

$$\frac{-c}{\sqrt{4 - c^2}} = 1 \Longrightarrow -c = \sqrt{4 - c^2} \Longrightarrow c^2 = 4 - c^2 \text{ with } c \in (-2, 0).$$

Solving the equation $c^2 = 4 - c^2$, we get $c = \pm\sqrt{2}$. Recalling that $c \in (-2, 0)$, Thus, $\boxed{c = -\sqrt{2}}$ is the only real number that satisfy the conclusion of MVT.

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \le c$	$-c \le x \le c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	



الفترة

5. [10 pts.] Given a function $f(x)$ such that $f(6) = 5$ and $f'(x) \leqslant -5$ for all $x \in [6, 10]$. How large can $f(10)$ possibly be?

الفترة

We are given that $f'(x) \leqslant -5$ for all $6 \leqslant x \leqslant 10$, so certainly f is continuous on $[6, 10]$ and differentiable on $(6, 10)$. Therefore, using the MVT on $[6, 10]$, there exists a number $c \in (6, 10)$ such that $f(10) - f(6) = (10 - 6)f'(c)$ i.e., $\boxed{f(10) = f(6) + 4f'(c) = 5 + 4f'(c)}$.

Now since $f'(c) \leqslant -5$, $\boxed{f(10) = 5 + 4f'(c) \leqslant 5 + 4(-5) = -15}$.

Hence the largest possible value of $f(10)$ is -15.

@Precalculusq8

31. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

We have $f(x) = \sin x$ Domain of f is $\mathbb{R}$

$f'(x) = \cos x$

$f(x)$ is continous everywhere "Trigonometric"

$f'(x)$ is differentiable everywhere

$\therefore$ By MVT, there exists at least

$c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$\cos(c) = \dfrac{\sin b - \sin a}{b - a}$ ناخذ مطلق الطرفين

$|\cos c| = \left| \dfrac{\sin b - \sin a}{b - a} \right|$, we know $0 \leq \boxed{|\cos c| \leq 1}$

$\therefore \left| \dfrac{\sin b - \sin a}{b - a} \right| \leqslant 1$

$\therefore |\sin b - \sin a| \leqslant |b - a|$

Let $f'(x) = \dfrac{1}{3+2x^2} \quad \forall\ x \in \mathbb{R}$

and $f(1) = 0$, Show that

$$\frac{1}{11} < f(2) < \frac{1}{5}$$

Using MVT :-

Since $f'(x) = \dfrac{1}{3+2x^2}$ exists

$\therefore f$ is diff. on $(1,2)$

$\therefore f$ is cont on $[1,2]$ "theorem"

Then by MVT there is number $c \in (1,2)$

Such that $f'(c) = \dfrac{f(2)-f(1)}{2-1} = \dfrac{f(2)-0}{1}$

$$\frac{1}{3+2c^2} = f(2) \quad , \quad c \in (1,2) \quad \therefore 1 < c < 2$$

$1 < c^2 < 4 \Rightarrow 2 < 2c^2 < 8 \quad \Rightarrow 5 < 3+2c^2 < 11$

$$\Rightarrow \frac{1}{5} > \underbrace{\frac{1}{3+2c^2}}_{= f(2)} > \frac{1}{11}$$

Use MVT to show that for $x > 0$

$$\tan^{-1} x < x$$

Let $f(x) = \tan^{-1} x$

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إلى ماله نهاية لانها
نظرية

$$f'(x) = \frac{1}{x^2+1}$$

$[0, x]$
$(0, x)$

$[0, x]$

$f(x)$ is continous and differentiable on $(0, x)$

(inverse Trigonometric function)

$\therefore$ By MVT, there exists at least

$c \in (0, x)$ such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$

$$\frac{1}{c^2+1} = \frac{\tan^{-1} x - 0}{x} = \frac{\tan^{-1} x}{x}$$

لان كسر

$\because \dfrac{1}{c^2+1} < 1$, $\therefore \dfrac{\tan^{-1} x}{x} < 1$

$$\tan^{-1} x < x$$